Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 794203, 12 pages
doi:10.1155/2011/794203

Research Article

Algorithms Construction for Variational Inequalities

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Received 4 October 2010; Accepted 19 February 2011

Academic Editor: Yeol J. Cho

We devote this paper to solving the variational inequality of finding x^* with property $x^* \in \mathrm{Fix}(T)$ such that $\langle (A-\gamma f)x^*, x-x^* \rangle \geq 0$ for all $x \in \mathrm{Fix}(T)$. Note that this hierarchical problem is associated with some convex programming problems. For solving the above VI, we suggest two algorithms: Implicit Algorithmml: $x_t = TPc[I - t(A-\gamma f)]x_t$ for all $t \in (0,1)$ and Explicit Algorithm: $x_{n+1} = \beta_n x_n + (1-\beta_n)TPc[1-\alpha_n(A-\gamma f)]x_n$ for all $n \geq 0$. It is shown that these two algorithms converge strongly to the unique solution of the above VI. As special cases, we prove that the proposed algorithms strongly converge to the minimum norm fixed point of T.

1. Introduction

Variational inequalities are being used as a mathematical programming tool in modeling a wide class of problems arising in several branches of pure and applied sciences. Several numerical techniques for solving variational inequalities and the related optimization problem have been considered by some authors. See, for example, [1–16].

Our main purpose in this paper is to consider the following variational inequality:

$$\text{Find } x^* \in \mathrm{Fix}(T) \text{ such that } \left\langle (A-\gamma f)x^*, x-x^* \right\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.1}$$

where T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H, $A : C \to H$ is a strongly positive linear bounded operator, and $f : C \to H$ is a ρ-contraction.

At this point, we wish to point out this hierarchical problem associated with some convex programming problems. The reader can refer to [17–21] and the references therein.

For solving VI (1.1), we suggest two algorithms which converge to the unique solution of VI (1.1). As special cases, we prove that the proposed algorithms strongly converge to the minimum norm fixed point of T.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, and let C be a nonempty closed convex subset of H. Let $f : C \to H$ be a *ρ-contraction*; that is, there exists a constant $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho\|x - y\|, \quad \forall x, y \in C. \tag{2.1}$$

A mapping A is said to be *strongly positive* on H if there exists a constant $\tilde{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \tilde{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2.2}$$

Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.3}$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $\mathrm{Fix}(T)$ the set of fixed points of T; that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

Remark 2.1. If $A : C \to H$ is a strongly positive linear bounded operator and $f : C \to H$ is a ρ-contraction, then for $0 < \gamma < \tilde{\gamma}/\rho$, the mapping $A - \gamma f$ is strongly monotone. In fact, we have

$$\begin{aligned} \langle (A - \gamma f)x - (A - \gamma f)y, x - y \rangle &= \langle A(x - y), x - y \rangle - \gamma\langle f(x) - f(y), x - y \rangle \\ &\geq \tilde{\gamma}\|x - y\|^2 - \gamma\rho\|x - y\|^2 \\ &\geq 0. \end{aligned} \tag{2.4}$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C}\|x - y\| =: d(x, C). \tag{2.5}$$

The following properties of projections are useful and pertinent to our purposes.

Lemma 2.2. *Given $x \in H$ and $z \in C$,*

(a) $z = P_C x$ *if and only if there holds the relation*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C, \tag{2.6}$$

(b) *$z = P_C x$ if and only if there holds the relation*

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C, \tag{2.7}$$

(c) *there holds the relation*

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.8}$$

Consequently, P_C is nonexpansive and monotone.

In the sequel, we will make use of the following for our main results.

Lemma 2.3 (Demiclosedness Principle for Nonexpansive Mappings, [22]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Lemma 2.4 (see [14]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $F : C \to H$ is monotone and weakly continuous along segments, that is, $F(x + ty) \to F(x)$ weakly as $t \to 0$. Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C \tag{2.9}$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{2.10}$$

Lemma 2.5 (see [23]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1. \tag{2.11}$$

Suppose that $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.6 (see [24]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad \forall n \geq 0, \tag{2.12}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(a) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(b) $\limsup_{n \to \infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

3. Main Results

In this section, we first consider an implicit algorithm and prove its strong convergence for solving variational inequality (1.1).

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a strongly positive linear bounded operator and $f : C \to H$ be a ρ-contraction. Let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Let $\gamma > 0$ be a constant satisfying $(\tilde{\gamma} - 1)/\rho < \gamma < \tilde{\gamma}/\rho$. For each $t \in (0,1)$, let the net $\{x_t\}$ be defined by*

$$x_t = TP_C[I - t(A - \gamma f)]x_t, \quad \forall t \in (0,1). \tag{3.1}$$

Then the net $\{x_t\}$ converges in norm, as $t \to 0^+$, to $x^* \in \mathrm{Fix}(T)$ which is the unique solution of VI (1.1).

Proof. First, we note that the net $\{x_t\}$ defined by (3.1) is well-defined. As a matter of fact, we have, for sufficiently small t,

$$\begin{aligned}
&\|TP_C[I - t(A - \gamma f)]x - TP_C[I - t(A - \gamma f)]y\| \\
&\quad \leq \|[I - t(A - \gamma f)]x - [I - t(A - \gamma f)]y\| \\
&\quad \leq t\gamma\|f(x) - f(y)\| + \|I - tA\|\|x - y\| \\
&\quad \leq t\gamma\rho\|x - y\| + (1 - t\tilde{\gamma})\|x - y\| \\
&\quad = [1 - (\tilde{\gamma} - \gamma\rho)t]\|x - y\|, \quad \forall x, y \in C,
\end{aligned} \tag{3.2}$$

which implies that the mapping $x \mapsto TP_C[I - t(A - \gamma f)]x$ is a contractive from C into C. Using the Banach contraction principle, there exists a unique point $x_t \in C$ satisfying the following fixed point equation:

$$x = TP_C[I - t(A - \gamma f)]x, \tag{3.3}$$

this is,

$$x_t = TP_C[I - t(A - \gamma f)]x_t, \tag{3.4}$$

which is exactly (3.1).

Next, we show that the net $\{x_t\}$ is bounded. Take an $x^* \in \mathrm{Fix}(T)$ to derive that

$$\begin{aligned}
\|x_t - x^*\| &= \|TP_C[I - t(A - \gamma f)]x_t - TP_C x^*\| \\
&\leq \|[I - t(A - \gamma f)]x_t - x^*\| \\
&\leq t\gamma\|f(x_t) - f(x^*)\| + t\|\gamma f(x^*) - Ax^*\| + \|(I - tA)(x_t - x^*)\| \\
&\leq (1 - \tilde{\gamma}t)\|x_t - x^*\| + t\gamma\rho\|x_t - x^*\| + t\|\gamma f(x^*) - Ax^*\|.
\end{aligned} \tag{3.5}$$

This implies that

$$\|x_t - x^*\| \leq \frac{1}{\tilde{\gamma} - \gamma\rho} \|\gamma f(x^*) - Ax^*\|. \tag{3.6}$$

It follows that $\{x_t\}$ is bounded, so are the nets $\{f(x_t)\}$ and $\{Ax_t\}$.

From (3.1), we get

$$\begin{aligned} \|x_t - Tx_t\| &= \|TP_C[I - t(A - \gamma f)]x_t - TP_C x_t\| \\ &\leq t\|(A - \gamma f)x_t\| \\ &\longrightarrow 0. \end{aligned} \tag{3.7}$$

Set $y_t = P_C[I - t(A - \gamma f)]x_t$ for all $t \in (0,1)$. It follows that

$$\|y_t - x_t\| \leq t\|(A - \gamma f)x_t\| \longrightarrow 0. \tag{3.8}$$

At the same time, we note that

$$\|x_t - x^*\| \leq \|y_t - x^*\|. \tag{3.9}$$

From (3.1) and the property of the metric projection, we have

$$\begin{aligned} \|y_t - x^*\|^2 &= \langle P_C[I - t(A - \gamma f)]x_t - [I - t(A - \gamma f)]x_t, y_t - x^*\rangle \\ &\quad + \langle [I - t(A - \gamma f)]x_t - x^*, y_t - x^*\rangle \\ &\leq \langle [I - t(A - \gamma f)]x_t - x^*, y_t - x^*\rangle \\ &= t\langle \gamma f(x_t) - Ax^*, y_t - x^*\rangle + \langle (I - tA)(x_t - x^*), y_t - x^*\rangle \\ &\leq (1 - t\tilde{\gamma})\|x_t - x^*\|\|y_t - x^*\| + t\langle \gamma f(x_t) - Ax^*, y_t - x^*\rangle \\ &\leq (1 - t\tilde{\gamma})\|y_t - x^*\|^2 + t\langle \gamma f(x_t) - Ax^*, y_t - x^*\rangle. \end{aligned} \tag{3.10}$$

It follows that

$$\begin{aligned} \|y_t - x^*\|^2 &\leq \frac{1}{\tilde{\gamma}}\langle \gamma f(x_t) - Ax^*, y_t - x^*\rangle \\ &= \frac{1}{\tilde{\gamma}}[\gamma\langle f(x_t) - f(x^*), y_t - x^*\rangle + \langle \gamma f(x^*) - Ax^*, y_t - x^*\rangle] \\ &\leq \frac{1}{\tilde{\gamma}}\left[\gamma\rho\|y_t - x^*\|^2 + \langle (A - \gamma f)x^*, x^* - y_t\rangle\right]. \end{aligned} \tag{3.11}$$

That is,

$$\|y_t - x^*\|^2 \leq \frac{1}{\tilde{\gamma} - \gamma\rho}\langle (A - \gamma f)x^*, x^* - y_t \rangle. \tag{3.12}$$

Therefore,

$$\|x_t - x^*\|^2 \leq \|y_t - x^*\| \leq \frac{1}{\tilde{\gamma} - \gamma\rho}\langle (A - \gamma f)x^*, x^* - y_t \rangle. \tag{3.13}$$

In particular,

$$\|x_n - x^*\|^2 \leq \frac{1}{\tilde{\gamma} - \gamma\rho}\langle (A - \gamma f)x^*, x^* - y_n \rangle. \tag{3.14}$$

Next, we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Assume $\{t_n\} \subset (0,1)$ is such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $y_n := y_{t_n}$. From (3.7), we have

$$\|x_n - Tx_n\| \longrightarrow 0. \tag{3.15}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $\tilde{x} \in C$ and hence y_n also converges weakly to $\tilde{x}$. Noticing (3.15), we can use Lemma 2.3 to get $\tilde{x} \in \mathrm{Fix}(T)$. Therefore, we can substitute $\tilde{x}$ for x^* in (3.14) to get

$$\|x_n - \tilde{x}\|^2 \leq \frac{1}{\tilde{\gamma} - \gamma\rho}\langle (A - \gamma f)\tilde{x}, \tilde{x} - y_n \rangle. \tag{3.16}$$

Consequently, the weak convergence of $\{y_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$ strongly. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t \to 0^+$.

Now, we return to (3.14) and take the limit as $n \to \infty$ to get

$$\|\tilde{x} - x^*\|^2 \leq \frac{1}{\tilde{\gamma} - \gamma\rho}\langle (A - \gamma f)x^*, x^* - \tilde{x} \rangle, \quad \forall x^* \in \mathrm{Fix}(T). \tag{3.17}$$

Hence $\tilde{x}$ solves the following VI:

$$\langle (A - \gamma f)x^*, x^* - \tilde{x} \rangle \geq 0, \quad \forall x^* \in \mathrm{Fix}(T) \tag{3.18}$$

or the equivalent dual VI (see Remark 2.1 and Lemma 2.4)

$$\langle (A - \gamma f)\tilde{x}, x^* - \tilde{x} \rangle \geq 0, \quad \forall x^* \in \mathrm{Fix}(T). \tag{3.19}$$

From the strong monotonicity of $A - \gamma f$, it follows the uniqueness of a solution of the above VI (see [11, Theorem 3.2]), $\tilde{x} = P_{\mathrm{Fix}(T)}(I - A + \gamma f)\tilde{x}$. That is, $\tilde{x}$ is the unique fixed point in $\mathrm{Fix}(T)$

of the contraction $P_{\mathrm{Fix}(T)}(I - A + \gamma f)$. Clearly this is sufficient to conclude that the entire net $\{x_t\}$ converges in norm to $\tilde{x}$ as $t \to 0^+$. This completes the proof. □

Next, we suggest an explicit algorithm and prove its strong convergence.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a strongly positive linear bounded operator and $f : C \to H$ be a ρ-contraction. Let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. Let $\gamma > 0$ be a constant satisfying $(\tilde{\gamma} - 1)/\rho < \gamma < \tilde{\gamma}/\rho$. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T P_C [I - \alpha_n (A - \gamma f)] x_n, \quad \forall n \geq 0, \tag{3.20}$$

where the sequences $\{\alpha_n\} \subset [0,1]$ and $\{\beta_n\} \subset [0,1]$ satisfy the following control conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\lim_{n\to\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(T)$ which is the unique solution of the variational inequality VI (1.1).

Proof. First we show that $\{x_n\}$ is bounded. Set $y_n = T P_C u_n$ and $u_n = [I - \alpha_n (A - \gamma f)] x_n$ for all $n \geq 0$. For any $p \in \mathrm{Fix}(T)$, we have

$$\begin{aligned} \|y_n - p\| &= \|T P_C u_n - T P_C p\| \\ &\leq \left\| [I - \alpha_n (A - \gamma f)] x_n - p \right\| \\ &\leq \alpha_n \|\gamma f(x_n) - \gamma f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \|I - \alpha_n A\| \|x_n - p\| \\ &\leq \alpha_n \gamma \rho \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| + (1 - \alpha_n \tilde{\gamma}) \|x_n - p\| \\ &= [1 - (\tilde{\gamma} - \gamma\rho)\alpha_n] \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|. \end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) [1 - (\tilde{\gamma} - \gamma\rho)\alpha_n] \|x_n - p\| \\ &\quad + \alpha_n (1 - \beta_n) \|\gamma f(p) - Ap\| \\ &= [1 - (\tilde{\gamma} - \gamma\rho)\alpha_n (1 - \beta_n)] \|x_n - p\| \\ &\quad + (\tilde{\gamma} - \gamma\rho)\alpha_n (1 - \beta_n) \frac{\|\gamma f(p) - Ap\|}{\tilde{\gamma} - \gamma\rho}, \end{aligned} \tag{3.22}$$

which implies that

$$\|x_n - p\| \leq \max\left\{ \|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\tilde{\gamma} - \gamma\rho} \right\}, \quad \forall n \geq 0. \tag{3.23}$$

Hence $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{u_n\}$, $\{Ax_n\}$, and $\{f(x_n)\}$.

From (3.20), we observe that

$$\begin{aligned}
\left\|y_{n+1}-y_n\right\| &= \|TP_Cu_{n+1}-TP_Cu_n\| \\
&\le \left\|\left[I-\alpha_{n+1}(A-\gamma f)\right]x_{n+1}-\left[I-\alpha_n(A-\gamma f)\right]x_n\right\| \\
&= \big\|\alpha_{n+1}\gamma\left(f(x_{n+1})-f(x_n)\right)+(\alpha_{n+1}-\alpha_n)\gamma f(x_n) \\
&\qquad +(I-\alpha_{n+1}A)(x_{n+1}-x_n)+(\alpha_n-\alpha_{n+1})Ax_n\big\| \\
&\le \alpha_{n+1}\gamma\left\|f(x_{n+1})-f(x_n)\right\|+\left(1-\alpha_{n+1}\tilde{\gamma}\right)\|x_{n+1}-x_n\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\left(\left\|\gamma f(x_n)\right\|+\|Ax_n\|\right) \\
&\le \alpha_{n+1}\gamma\rho\|x_{n+1}-x_n\|+\left(1-\alpha_{n+1}\tilde{\gamma}\right)\|x_{n+1}-x_n\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\left(\left\|\gamma f(x_n)\right\|+\|Ax_n\|\right) \\
&= \left[1-(\tilde{\gamma}-\gamma\rho)\alpha_{n+1}\right]\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\left(\left\|\gamma f(x_n)\right\|+\|Ax_n\|\right).
\end{aligned} \tag{3.24}$$

It follows that

$$\left\|y_{n+1}-y_n\right\|-\|x_{n+1}-x_n\| \le (\tilde{\gamma}-\gamma\rho)\alpha_{n+1}\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\left(\left\|\gamma f(x_n)\right\|+\|Ax_n\|\right), \tag{3.25}$$

which implies, from (C1) and the boundedness of $\{x_n\}$, $\{f(x_n)\}$ and $\{Ax_n\}$, that

$$\limsup_{n\to\infty}\left(\left\|y_{n+1}-y_n\right\|-\|x_{n+1}-x_n\|\right)\le 0. \tag{3.26}$$

Hence, by Lemma 2.5, we have

$$\lim_{n\to\infty}\left\|y_n-x_n\right\|=0. \tag{3.27}$$

Consequently, it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\left\|y_n-x_n\right\|=0. \tag{3.28}$$

On the other hand, we have

$$\begin{aligned}
\|x_n-Tx_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-Tx_n\| \\
&= \|x_{n+1}-x_n\|+\left\|\beta(x_n-Tx_n)+(1-\beta_n)(y_n-Tx_n)\right\| \\
&\le \|x_{n+1}-x_n\|+\beta_n\|x_n-Tx_n\|+(1-\beta_n)\left\|y_n-TP_Cx_n\right\| \\
&\le \|x_{n+1}-x_n\|+\beta_n\|x_n-Tx_n\|+(1-\beta_n)\alpha_n\left\|(A-\gamma f)x_n\right\|,
\end{aligned} \tag{3.29}$$

that is,

$$\|x_n - Tx_n\| \leq \frac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \alpha_n\left\|(A-\gamma f)x_n\right\|. \tag{3.30}$$

This together with (C1), (C3), and (3.28) implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.31}$$

Next, we show that, for any $x^* \in F(T)$,

$$\limsup_{n\to\infty}\left\langle u_n - x^*, \gamma f(x^*) - Ax^*\right\rangle \leq 0. \tag{3.32}$$

Now we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle x_n - x^*, \gamma f(x^*) - Ax^*\right\rangle = \lim_{k\to\infty}\left\langle x_{n_k} - x^*, \gamma f(x^*) - Ax^*\right\rangle. \tag{3.33}$$

Since $\{x_n\}$ is bounded, we may assume that $x_{n_k} \to z$ weakly. Note that $z \in \mathrm{Fix}(T)$ by virtue of Lemma 2.3 and (3.31). Therefore,

$$\limsup_{n\to\infty}\left\langle x_n - x^*, \gamma f(x^*) - Ax^*\right\rangle = \left\langle z - x^*, \gamma f(x^*) - Ax^*\right\rangle \leq 0. \tag{3.34}$$

We notice that

$$\|u_n - x_n\| \leq \alpha_n\left\|(A-\gamma f)x_n\right\| \longrightarrow 0. \tag{3.35}$$

Hence, we get

$$\limsup_{n\to\infty}\left\langle u_n - x^*, \gamma f(x^*) - Ax^*\right\rangle \leq 0. \tag{3.36}$$

Finally, we prove that $\{x_n\}$ converges to the point x^*. We observe that

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \alpha_n\left\|(A-\gamma f)x_n\right\|. \tag{3.37}$$

Therefore, from (3.20), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|y_n-x^*\right\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|u_n-x^*\|^2\\
&= \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|\alpha_n(\gamma f(x_n)-Ax^*)+(I-\alpha_nA)(x_n-x^*)\right\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\\
&\quad\times\left[(1-\alpha_n\widetilde{\gamma})^2\|x_n-x^*\|^2+2\alpha_n\langle\gamma f(x_n)-Ax^*,u_n-x^*\rangle\right]\\
&= \left[1-2\alpha_n\widetilde{\gamma}+(1-\beta_n)\alpha_n^2\widetilde{\gamma}^2\right]\|x_n-x^*\|^2\\
&\quad+2\alpha_n\langle\gamma f(x_n)-\gamma f(x^*),u_n-x^*\rangle+2\alpha_n\langle\gamma f(x^*)-Ax^*,u_n-x^*\rangle\\
&\le \left[1-2\alpha_n\widetilde{\gamma}+(1-\beta_n)\alpha_n^2\widetilde{\gamma}^2\right]\|x_n-x^*\|^2\\
&\quad+2\alpha_n\gamma\rho\|x_n-x^*\|\|u_n-x^*\|+2\alpha_n\langle\gamma f(x^*)-Ax^*,u_n-x^*\rangle\\
&\le [1-2\alpha_n(\widetilde{\gamma}-\gamma\rho)]\|x_n-x^*\|^2+(1-\beta_n)\alpha_n^2\widetilde{\gamma}^2\|x_n-x^*\|^2\\
&\quad+2\alpha_n^2\gamma\rho\|x_n-x^*\|\|(A-\gamma f)x_n\|+2\alpha_n\langle\gamma f(x^*)-Ax^*,u_n-x^*\rangle.
\end{aligned}
\tag{3.38}
$$

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{Ax_n\}$ are all bounded, we can choose a constant $M>0$ such that

$$
\sup_n\frac{1}{\widetilde{\gamma}-\gamma\rho}\left\{\frac{(1-\beta_n)\widetilde{\gamma}^2}{2}\|x_n-x^*\|^2+\gamma\rho\|x_n-x^*\|\|(A-\gamma f)x_n\|\right\}\le M. \tag{3.39}
$$

It follows that

$$
\|x_{n+1}-x^*\|^2\le[1-2(\widetilde{\gamma}-\rho\gamma)\alpha_n]\|x_n-x^*\|^2+2(\widetilde{\gamma}-\rho\gamma)\alpha_n\delta_n, \tag{3.40}
$$

where

$$
\delta_n=\alpha_nM+\frac{1}{\widetilde{\gamma}-\gamma\rho}\langle\gamma f(x^*)-Ax^*,u_n-x^*\rangle. \tag{3.41}
$$

By (C1) and (3.36), we get

$$
\limsup_{n\to\infty}\beta_n\le 0. \tag{3.42}
$$

Now, applying Lemma 2.6 and (3.40), we conclude that $x_n \to x^*$. This completes the proof. □

From Theorems 3.1 and 3.2, we can deduce easily the following corollaries.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. For each $t \in (0,1)$, let the net $\{x_t\}$ be defined by*

$$x_t = TP_C(1-t)x_t, \quad \forall t \in (0,1). \tag{3.43}$$

Then the net $\{x_t\}$ defined by (3.43) *converges in norm, as $t \to 0^+$, to the minimum norm element $x^* \in \mathrm{Fix}(T)$.*

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. For $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta_n x_n + \left(1-\beta_n\right) TP_C(1-\alpha_n)x_n, \quad \forall n \geq 0, \tag{3.44}$$

where the sequences $\{\alpha_n\} \subset [0,1]$ and $\{\beta_n\} \subset [0,1]$ satisfy the following control conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\lim_{n\to\infty}\alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ generated by (3.44) *converges strongly to the minimum norm element $x^* \in \mathrm{Fix}(T)$.*

Acknowledgments

The authors thank the referees for their comments and suggestions which improved the presentation of this paper. The first author was supported in part by Colleges and Universities, Science and Technology Development Foundation (20091003) of Tianjin and NSFC 11071279. The second author was supported in part by NSC 99-2221-E-230-006.

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