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Research Article

An Iteration Method for Common Solution of a System of Equilibrium Problems in Hilbert Spaces

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The strong convergence theorem is proved for finding a common solution for a system of equilibrium problems: find $u^* \in S := \cap_{i=1}^N \mathrm{EP}(F_i), \mathrm{EP}(F_i) := \{z \in C : F_i(z,v) \geq 0\ \forall v \in C\}, i = 1,\ldots,N$, where C is a closed convex subset of a Hilbert space H and F_i are N bifunctions from $C \times C$ into $\mathbf{R}$ given exactly or approximatively. As an application, finding a common solution for a system of variational inequality problems is given.

1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and let $F_i (i = 1,\ldots,N)$ be N bifunctions from $C \times C$ into R. In this paper, we consider the system of equilibrium problems:

$$\text{find } u^* \in S := \cap_{i=1}^N \mathrm{EP}(F_i), \qquad \mathrm{EP}(F_i) := \{z \in C : F_i(z,v) \geq 0\ \forall v \in C\}, \quad i = 1,\ldots,N. \tag{1.1}$$

We assume that $S \neq \emptyset$ and the bifunctions F_i satisfy the following conditions.

Condition 1. The bifunction F satisfies the following conditions:

(A1) $F(u,u) = 0$ for all $u \in C$.

(A2) $F(u,v) + F(v,u) \leq 0$ for all $(u,v) \in C \times C$.

(A3) For every $u \in C$, $F(u, \cdot) : C \to \mathbf{R}$ is lower semicontinuous and convex.

(A4) $\overline{\lim}_{t \to +0} F((1-t)u + tz, v) \leq F(u, v)$ for all $(u, z, v) \in C \times C \times C$.

Definition 1.1. A mapping A of C into H is called monotone if

$$\langle A(x) - A(y), x - y \rangle \geq 0, \tag{1.2}$$

for all $x, y \in C$.

Now, we consider the variational inequality problem: find $u^* \in C$ such that

$$\langle A(u^*), x - u^* \rangle \geq 0, \tag{1.3}$$

for all $x \in C$. We denote $\mathrm{VI}(C, A)$ the set of solutions of the variational inequality problem.

Definition 1.2. A mapping T of C into H is called k-strictly pseudocontractive in the terminology of Browder and Petryshyn [1], if there exists a number $k \in [0, 1)$ such that

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|(I - T)(x) - (I - T)(y)\|^2, \tag{1.4}$$

where I is the identity operator in H.

The above inequality is equivalent to

$$\langle A(x) - A(y), x - y \rangle \geq \lambda \|A(x) - A(y)\|^2, \tag{1.5}$$

where the operator $A := I - T$ is $\lambda = (1 - k)/2$-inverse strongly monotone (hence monotone) and Lipschitz continuous with the Lipschitz constant $2/(1 - k)$. Clearly, when $k = 0$, T is nonexpansive, that is,

$$\|T(x) - T(y)\| \leq \|x - y\| \tag{1.6}$$

for all $x, y \in D(T)$, the domain of T. It means that the class of k-strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings. Denote by $F(T)$ the set of fixed points of the operator T in C, that is,

$$F(T) = \{x \in C : x = T(x)\}. \tag{1.7}$$

If $N = 1$, then (1.1) is a single equilibrium problem [2, 3] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems.

For finding approximative solutions of (1.1), there exist several approaches: the regularization approach in [4–7], the gap-function approach in [8–10], and iterative procedure approach in [11–15].

If $N > 1$, then (1.1) is a problem of finding a common solution for a system of equilibrium problems which is studied firstly in [5] (cf. [16]) under the condition that F_i $(i = 1, \ldots, N)$ are bounded, Fréchet differentiable with respect to v and $\nabla_v F_i(u, u)$ are Lipschitz continuous, that is,

$$\|\nabla_v F_i(x, x) - \nabla_v F_i(y, y)\| \leq L\|x - y\| \quad \forall x, y \in C, \ i = 1, 2, \ldots, N, \tag{1.8}$$

where L is a positive constant.

With the case that

$$F_i(u, v) = \langle (I - T_i)(u), v - u \rangle, \tag{1.9}$$

and T_i $(i = 2, \ldots, N)$ are $N - 1$ strictly pseudocontractive mappings, (1.1) is a problem of finding a solution of an equilibrium problem which is also a common fixed point for a system of a finite family of strictly pseudocontractive mappings [17–19].

In addition, when $F_1(u, v) = \langle A_1(u), v - u \rangle$ where A_1 is a monotone operator, (1.1) is a problem of finding an element which is a solution of a variational inequality problem and a common fixed point for a finite family of strictly pseudocontractive mappings and investigated intensively in [20–32]. If all F_i have the form (1.9), then (1.1) is a problem of finding a common fixed point for a finite family of strictly pseudocontractive mappings T_i from C into H [14, 33–35].

In this paper, we present an iteration method for solving (1.1), where the iteration sequence $\{x_n\}$ is defined by

$$\begin{gathered} x_0 = x \in H, \\ u_n^i \in C : F_i\left(u_n^i, v\right) + \left\langle u_n^i - x_n, v - u_n^i \right\rangle \geq 0, \quad \forall v \in C, \ i = 1, \ldots, N, \\ x_{n+1} = x_n - \beta_n \left[\sum_{i=1}^{n} \left(x_n - u_n^i \right) + \alpha_n x_n \right], \end{gathered} \tag{1.10}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences of positive numbers satisfying some conditions.

As an application, we find a common solution for a system of N variational inequality problems with monotone mappings.

2. Main Results

The strong and weak convergence of any sequence are denoted by $\to$ and $\rightharpoonup$, respectively. We formulate the following facts which are necessary in the proof of our main results.

Lemma 2.1 (see [5]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let F be a bifunction of $C \times C$ into R satisfying the Condition 1. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, v) + \frac{1}{r}\langle z - x, v - z \rangle \geq 0, \quad \forall v \in C. \tag{2.1}$$

Lemma 2.2 (see [5]). *Assume that $F : C \times C \to \mathbf{R}$ satisfies the Condition 1. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, v) + \frac{1}{r}\langle z - x, v - z\rangle \geq 0,\ \forall v \in C \right\}. \tag{2.2}$$

Then, the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle; \tag{2.3}$$

(iii) *$F(T_r) = EP(F)$;*

(iv) *$EP(F)$ is closed and convex.*

Lemma 2.3. *Let $F^h(u, v)$ be a bifunction approximating the bifunction $F(u, v)$ in the sense*

$$\left| F^h(u, v) - F(u, v) \right| \leq hg(\|u\|)\|u - v\| \quad \forall u, v \in C,\ h > 0, \tag{2.4}$$

where $g(t)$ is a real positive function. Then, for each $r > 0$ and $x \in H$, we have

$$\|T_r^h(x) - T_r(x)\| \leq rhg(\|T_r(x)\|), \tag{2.5}$$

where

$$T_r^h(x) = \left\{ \tilde{z} \in C : F^h(\tilde{z}, v) + \frac{1}{r}\langle \tilde{z} - x, v - \tilde{z}\rangle \geq 0\ \forall v \in C \right\}. \tag{2.6}$$

Proof. Let x be an arbitrary element of H. By replacing v by $\tilde{z}$ in (2.2) and by z in (2.6), we obtain

$$F(z, \tilde{z}) + F^h(\tilde{z}, z) \geq \frac{1}{r}[\langle x - z, \tilde{z} - z\rangle + \langle \tilde{z} - x, \tilde{z} - z\rangle]. \tag{2.7}$$

Therefore, by virtue of (A2) in Condition 1, we can write

$$F(z, \tilde{z}) - F^h(z, \tilde{z}) \geq \frac{1}{r}\|\tilde{z} - z\|^2. \tag{2.8}$$

Consequently,

$$\|\tilde{z} - z\| \leq rhg(\|T_r(x)\|). \tag{2.9}$$

The proof is completed. □

Lemma 2.4 (see [36]). *Let* $\{a_n\}, \{b_n\}$, *and* $\{c_n\}$ *be sequences of positive numbers satisfying the conditions:*

(i) $a_{n+1} \le (1 - b_n)a_n + c_n$, $b_n < 1$,

(ii) $\sum_{n=0}^{\infty} b_n = +\infty$, $\lim_{n\to+\infty}(c_n/b_n) = 0$.

Then, $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.5 (see [37]). *Assume that* T *is a nonexpansive mapping of a closed convex subset* C *of a Hilbert space* H. *Then* $I - T$ *is demiclosed at zero; that is whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I - T)(x_n)\}$ *strongly converges to zero, it follows* $(I - T)(x) = 0$.

Lemma 2.6 (see [17]). *Let* A *be a* λ*-inverse strongly monotone mapping from* C *into* H *such that* $S_A \ne \emptyset$, *where* $S_A = \{x \in C : A(x) = 0\}$. *Then,* $S_A = \mathrm{VI}(C, A)$.

Now, consider the firmly nonexpansive mappings T_i defined by

$$T_i(x) = \{z \in C : F_i(z, v) + \langle z - x, v - z\rangle \ge 0,\ \forall v \in C\}, \quad i = 1, \ldots, N. \tag{2.10}$$

By virtue of Lemma 2.2, we can see T_i is nonexpansive. Consequently, $A_i := I - T_i$ is $(1/2)$-inverse strongly monotone and Lipschitz continuous with the Lipschitz constant $L_i = 2$, $i = 1, \ldots, N$.

We construct a Tikhonov regularization solution y_n for (1.1) by solving the following operator equation: find $y_n \in H$ such that

$$\sum_{i=1}^{N} A_i(y_n) + \alpha_n y_n = 0, \tag{2.11}$$

where the positive regularization parameter $\alpha_n \to 0$ as $n \to +\infty$. We have the following result.

Theorem 2.7. (i) *For each* $\alpha_n > 0$, *problem* (2.11) *has a unique solution* y_n.

(ii) $\lim_{n\to+\infty} y_n = u^*$, $u^* \in S$, $\|u^*\| \le \|y\|$, *for all* $y \in S$.

(iii) $\|y_n - y_m\| \le (|\alpha_n - \alpha_m|/\alpha_n)\|u^*\|$.

Proof. (i) Since the mapping $\sum_{i=1}^{n} A_i$ is a monotone and Lipschitz continuous mapping defined on H, it is maximal monotone. Therefore, (2.11) has a unique solution for each $\alpha_n > 0$ ([38]).

(ii) For each $y \in S$, on the base of Lemma 2.2, we have that $A_i(y) = 0$, $i = 1, \ldots, N$. Thus, from (2.11) it follows that

$$\sum_{i=1}^{N} \langle A_i(y_n) - A_i(y), y_n - y\rangle + \alpha_n \langle y_n, y_n - y\rangle = 0. \tag{2.12}$$

Since every A_i is monotone, from the last equality, we obtain

$$\langle y_n, y_n - y\rangle \le 0. \tag{2.13}$$

Hence,

$$\|y_n\| \le \|y\|, \quad \forall y \in S. \tag{2.14}$$

It means that the sequence $\{y_n\}$ is bounded. Let $\{y_{n_k}\}$ be a subsequence of the sequence $\{y_n\}$ such that $y_{n_k} \rightharpoonup \tilde{y}$ as $k \to \infty$.

Again, let y be an arbitrary element of S. From the $(1/2)$-inverse strongly monotone property of A_l, and $A_l(y) = 0$, $l = 1, \dots, N$, it implies that

$$\begin{aligned} \frac{1}{2}\|y_{n_k} - T_l(y_{n_k})\|^2 &\le \langle A_l(y_{n_k}), y_{n_k} - y\rangle \\ &\le \sum_{i=1}^{N}\langle A_i(y_{n_k}), y_{n_k} - y\rangle \\ &\le -\alpha_{n_k}\langle y_{n_k}, y_{n_k} - y\rangle \\ &= -\alpha_{n_k}\langle y_{n_k} - y, y_{n_k} - y\rangle - \alpha_{n_k}\langle y, y_{n_k} - y\rangle \\ &\le -\alpha_{n_k}\langle y, y_{n_k} - y\rangle \\ &\le \alpha_{n_k} 2\|y\|^2, \end{aligned} \tag{2.15}$$

that is,

$$\|y_{n_k} - T_l(y_{n_k})\| \le 2\|y\|\sqrt{\alpha_{n_k}}. \tag{2.16}$$

Therefore,

$$\lim_{k\to\infty} \|A_l(y_{n_k})\| = 0. \tag{2.17}$$

By Lemma 2.5, $A_l(\tilde{y}) = 0$, that is, $\tilde{y} \in F(T_l)$, $l = 1, \dots, N$. It means that $\tilde{y} \in S$. Because S is a closed convex subset in Hilbert space, it has a unique minimal element u^* in norm. From (2.14) and the weak convergence of $\{y_{n_k}\}$ to $\tilde{y} = u^*$, it also follows that $\|y_{n_k}\| \to \|u^*\|$, as $k \to \infty$. Moreover, the sequence $\{y_n\}$ converges strongly to u^* as $n \to \infty$.

(iii) From (2.11), (2.14), and the monotone property of A_i, it follows

$$\alpha_n\langle y_n, y_n - y_m\rangle - \alpha_m\langle y_m, y_n - y_m\rangle \le 0 \tag{2.18}$$

or

$$\|y_n - y_m\| \le \frac{|\alpha_n - \alpha_m|}{\alpha_n}\|y_m\| \le \frac{|\alpha_n - \alpha_m|}{\alpha_n}\|u^*\|, \tag{2.19}$$

for each $\alpha_n, \alpha_m > 0$. The proof is completed. □

Theorem 2.8. *Suppose that α_n, β_n satisfy the following conditions:*

$$\alpha_n, \beta_n > 0 \ (\alpha_n \leq 1), \quad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2 \beta_n} = 0,$$
$$\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty, \quad \overline{\lim_{n\to\infty}} \beta_n \frac{(2N + \alpha_n)^2}{\alpha_n} < 1. \tag{2.20}$$

Then,

$$\lim_{n\to\infty} x_n = u^* \in S, \tag{2.21}$$

where x_n is defined by (1.10).

Proof. Let y_n be a solution of (2.11). Set $\Delta_n = \|x_n - y_n\|$. Then,

$$\Delta_{n+1} = \|x_{n+1} - y_{n+1}\| \leq \|x_{n+1} - y_n\| + \|y_{n+1} - y_n\|,$$
$$\|x_{n+1} - y_n\| = \left\| x_n - y_n - \beta_n \left[\sum_{i=0}^{N} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n) \right] \right\|. \tag{2.22}$$

From the monotone and Lipschitz continuous properties of A_i, $i = 1, \ldots, N$, (2.11), and $u_n^i = T_i(x_n)$, we can write

$$\begin{aligned}
&\left\| x_n - y_n - \beta_n \left[\sum_{i=1}^{N} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n) \right] \right\|^2 \\
&= \|x_n - y_n\|^2 + \beta_n^2 \left\| \left[\sum_{i=1}^{N} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n) \right] \right\|^2 \\
&\quad - 2\beta_n \left\langle \sum_{i=1}^{N} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n), x_n - y_n \right\rangle \\
&\leq \|x_n - y_n\|^2 \left[1 - 2\beta_n \alpha_n + \beta_n^2 (2N + \alpha_n)^2 \right].
\end{aligned} \tag{2.23}$$

Hence,

$$\|x_{n+1} - y_n\| \leq \Delta_n \left[1 - 2\beta_n \alpha_n + \beta_n^2 (2N + \alpha_n)^2 \right]^{1/2}. \tag{2.24}$$

Therefore,

$$\begin{aligned}
\Delta_{n+1} &\leq \Delta_n \left[1 - 2\beta_n \alpha_n + \beta_n^2 (2N + \alpha_n)^2 \right]^{1/2} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} \|u^*\| \\
&\leq \Delta_n (1 - \alpha_n \beta_n)^{1/2} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} \|u^*\|.
\end{aligned} \tag{2.25}$$

Thus, applying the inequality

$$(a+b)^2 \le (1+\varepsilon)\left(a^2 + \frac{b^2}{\varepsilon}\right) \quad (\varepsilon > 0),\ \varepsilon = \frac{\alpha_n\beta_n}{2}, \tag{2.26}$$

we obtain

$$\begin{aligned} 0 \le \Delta_{n+1}^2 &\le \Delta_n^2(1-\alpha_n\beta_n)\left(1+\frac{1}{2}\alpha_n\beta_n\right) + \left(\frac{\alpha_n-\alpha_{n+1}}{\alpha_n}\|u^*\|\right)^2 \frac{2}{\alpha_n\beta_n}\left(1+\frac{1}{2}\alpha_n\beta_n\right) \\ &\le a_n^2\left(1-\frac{1}{2}\alpha_n\beta_n - \frac{1}{2}(\alpha_n\beta_n)^2\right) + \left(\frac{\alpha_n-\alpha_{n+1}}{\alpha_n^2\beta_n}\|u^*\|\right)^2 2\alpha_n\beta_n\left(1+\frac{1}{2}\alpha_n\beta_n\right). \end{aligned} \tag{2.27}$$

Set

$$\begin{aligned} b_n &= \alpha_n\beta_n\left(\frac{1}{2}+\frac{1}{2}\alpha_n\beta_n\right), \\ c_n &= \left(\frac{\alpha_n-\alpha_{n+1}}{\alpha_n^2\beta_n}\|u^*\|\right)^2 2\alpha_n\beta_n\left(1+\frac{1}{2}\alpha_n\beta_n\right). \end{aligned} \tag{2.28}$$

It is not difficult to check that b_n and c_n satisfy the conditions in Lemma 2.4 for sufficiently large n. Hence, $\lim_{n\to+\infty}\Delta_n^2 = 0$. Since $\lim_{n\to\infty} y_n = u^*$, we have

$$\lim_{n\to\infty} x_n = u^* \in S. \tag{2.29}$$

□

Now, let $F_i^n(u,v) := F_i^{h_n}(u,v)$ be bifunctions approximating the bifunctions $F_i(u,v)$ in the sense (2.4) where $h_n \to 0$, as $n \to \infty$, and $g(t)$ is a real positive and bounded (the image of any bounded set is bounded) function. Then, the sequence of iterations $\{\tilde{x}_n\}$ is defined by

$$\begin{aligned} \tilde{x}_0 &= x \in H, \\ \tilde{u}_n^i \in C : F_i^n\left(\tilde{u}_n^i, v\right) + \left\langle \tilde{u}_n^i - \tilde{x}_n, v - \tilde{u}_n^i\right\rangle &\ge 0 \quad \forall v \in C,\ i = 1,\ldots,N, \\ \tilde{x}_{n+1} &= \tilde{x}_n - \beta_n\left[\sum_{i=1}^{n}\left(\tilde{x}_n - \tilde{u}_n^i\right) + \alpha_n\tilde{x}_n\right], \end{aligned} \tag{2.30}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences of positive numbers satisfying some conditions.

We have the following result.

Theorem 2.9. *Suppose that α_n, β_n, and h_n satisfy the conditions in Theorem 2.8 and*

$$\lim_{n\to\infty} \frac{h_n + h_{n+1}}{\alpha_n^2\beta_n} = 0. \tag{2.31}$$

Then, we have

$$\lim_{n\to\infty} \widetilde{x}_n = u^* \in S, \tag{2.32}$$

where $\widetilde{x}_n$ *is defined by* (2.30).

Proof. Let $\widetilde{y}_n$ be a solution of the following equation:

$$\sum_{i=1}^{N} A_i^n(\widetilde{y}_n) + \alpha_n \widetilde{y}_n = 0, \qquad A_i^n = I - T_i^n, \tag{2.33}$$

where each T_i^n is defined by

$$T_i^n(x) = \{z \in C : F_i^n(z,v) + \langle z - x, v - z\rangle \geq 0,\ \forall v \in C\}, \quad i = 1,\ldots,N. \tag{2.34}$$

Since

$$\|\widetilde{x}_n - u^*\| \leq \|\widetilde{x}_n - \widetilde{y}_n\| + \|\widetilde{y}_n - y_n\| + \|y_n - u^*\|, \tag{2.35}$$

and $\lim_{n\to\infty} y_n = u^*$, in order to prove that $\lim_{n\to\infty}\widetilde{x}_n = u^*$, it is necessary to prove that

$$\lim_{n\to\infty} \|\widetilde{x}_n - \widetilde{y}_n\| = \lim_{n\to\infty} \|\widetilde{y}_n - y_n\| = 0. \tag{2.36}$$

For this purpose, first we estimate the value $\|\widetilde{y}_n - y_n\|$. On the basis of Lemma 2.3, we have

$$\|A_i(x) - A_i^n(x)\| = \|T_i(x) - T_i^n(x)\| \leq h_n g(\|T_i(x)\|). \tag{2.37}$$

Therefore, from (2.11), (2.33), and the monotone property of A_i^n it implies that

$$\begin{aligned} \|y_n - \widetilde{y}_n\|^2 &= \frac{1}{\alpha_n}\sum_{i=1}^{N}\langle A_i^n(\widetilde{y}_n) - A_i(y_n), y_n - \widetilde{y}_n\rangle \\ &\leq \frac{1}{\alpha_n}\sum_{i=1}^{N}\langle A_i^n(y_n) - A_i(y_n), y_n - \widetilde{y}_n\rangle. \end{aligned} \tag{2.38}$$

Consequently, we have

$$\begin{aligned} \|y_n - \widetilde{y}_n\| &\leq \frac{1}{\alpha_n}\sum_{i=1}^{N}\|A_i^n(y_n) - A_i(y_n)\| \\ &\leq N\frac{h_n}{\alpha_n} g(\|T_i(y_n)\|). \end{aligned} \tag{2.39}$$

On the other hand,

$$\begin{aligned}\|T_i(y_n)\| &= \|T_i(y_n) - T_i(u^*) + u^*\| \\ &\le \|y_n - u^*\| + \|u^*\| \\ &\le \|y_n\| + 2\|u^*\| \\ &\le 3\|u^*\|.\end{aligned} \tag{2.40}$$

Therefore,

$$\|y_n - \tilde{y}_n\| \le C_0 N \frac{h_n}{\alpha_n}, \tag{2.41}$$

where $C_0 = \sup\{g(t) : 0 < t \le 3\|u^*\|\}$. It means that $\lim_{n\to\infty}\tilde{y}_n = u^*$ because $\lim_{n\to\infty}(h_n/\alpha_n) = 0$.

Secondly, to prove

$$\lim_{n\to\infty}\|\tilde{x}_n - \tilde{y}_n\| = 0, \tag{2.42}$$

as in the proof of Theorem 2.8, first we need to estimate the value $\|\tilde{y}_{n+1} - \tilde{y}_n\|$. By the argument as in the proof of Theorem 2.7, we have

$$\sum_{i=1}^{N}\left\langle A_i^n(\tilde{y}_n) - A_i^{n+1}(\tilde{y}_{n+1}), \tilde{y}_n - \tilde{y}_{n+1}\right\rangle + \alpha_n\langle \tilde{y}_n, \tilde{y}_n - \tilde{y}_{n+1}\rangle - \alpha_{n+1}\langle \tilde{y}_{n+1}, \tilde{y}_n - \tilde{y}_{n+1}\rangle = 0. \tag{2.43}$$

Thus,

$$\begin{aligned}\|\tilde{y}_n - \tilde{y}_{n+1}\|^2 &= \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}\langle -\tilde{y}_{n+1}, \tilde{y}_n - \tilde{y}_{n+1}\rangle + \frac{1}{\alpha_n}\sum_{i=1}^{N}\left\langle A_i^{n+1}(\tilde{y}_{n+1}) - A_i^n(\tilde{y}_n), \tilde{y}_n - \tilde{y}_{n+1}\right\rangle \\ &\le \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}\langle -\tilde{y}_n, \tilde{y}_n - \tilde{y}_{n+1}\rangle + \frac{1}{\alpha_n}\sum_{i=1}^{N}\left\langle A_i^{n+1}(\tilde{y}_{n+1}) - A_i^n(\tilde{y}_n), \tilde{y}_n - \tilde{y}_{n+1}\right\rangle \\ &\le \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}\|\tilde{y}_n\|\|\tilde{y}_n - \tilde{y}_{n+1}\| + \frac{1}{\alpha_n}\sum_{i=1}^{N}\left\langle A_i^{n+1}(\tilde{y}_n) - A_i^n(\tilde{y}_n), \tilde{y}_n - \tilde{y}_{n+1}\right\rangle.\end{aligned} \tag{2.44}$$

Therefore,

$$\begin{aligned}\|\tilde{y}_n - \tilde{y}_{n+1}\| &\le \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}\|\tilde{y}_n\| + \frac{1}{\alpha_n}\sum_{i=1}^{N}\|A_i^{n+1}(\tilde{y}_n) - A_i(\tilde{y}_n)\| + \|A_i(\tilde{y}_n) - A_i^n(\tilde{y}_n)\| \\ &\le \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}\|\tilde{y}_n\| + N\frac{h_n + h_{n+1}}{\alpha_n}g(\|\tilde{y}_n\|).\end{aligned} \tag{2.45}$$

Using (2.14) and (2.41), we have

$$\|\widetilde{y}_n\| \leq \|u^*\| + C_0 N \frac{h_n}{\alpha_n}. \tag{2.46}$$

Consequently, there exists a positive constant C such that $\|\widetilde{y}_n\| \leq C$ for $n \geq 0$. Finally, we have

$$\|\widetilde{y}_n - \widetilde{y}_{n+1}\| \leq \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} C + N C_1 \frac{h_n + h_{n+1}}{\alpha_n}, \tag{2.47}$$

where $C_1 = \sup\{g(t) : 0 < t < C\}$. Now, set $\widetilde{\Delta}_n = \|\widetilde{x}_n - \widetilde{y}_n\|$. It is not difficult to verify that

$$\begin{aligned} \|\widetilde{x}_{n+1} - \widetilde{y}_n\| &\leq \widetilde{\Delta}_n \left[1 - 2\beta_n\alpha_n + \beta_n^2 (2N + \alpha_n)^2\right]^{1/2}, \\ \widetilde{\Delta}_{n+1} &\leq \widetilde{\Delta}_n (1 - \alpha_n\beta_n)^{1/2} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} C + N C_1 \frac{h_n + h_{n+1}}{\alpha_n}. \end{aligned} \tag{2.48}$$

Therefore, $\lim_{n\to\infty} \widehat{\Delta}_n = 0$. The proof is completed. □

Remark. The sequences $\alpha_n = (1+n)^{-p}$, $0 < p < 1/2$, and $\beta_n = \gamma_0 \alpha_n$ with

$$0 < \gamma_0 < \frac{1}{(2N + \alpha_0)^2}, \tag{2.49}$$

satisfy all the necessary conditions in Theorem 2.8.

3. Applications

Consider the following problem: find an element $u^* \in C$ such that

$$\langle A_i(u^*), v - u^* \rangle \geq 0, \quad \forall v \in C, \ i = 1, \ldots, N, \tag{3.1}$$

where A_i are N monotone hemicontinuous mappings from a closed convex subset C of a Hilbert space H into H.

Theorem 3.1. *Let $x_0 = x$ be an arbitrary element in H. If $\{\alpha_n\}$, $\{\beta_n\}$ are chosen as in Theorem 2.8, and the iteration sequence $\{x_n\}$ is defined as follows:*

$$\begin{gathered} u_n^i \in C, \\ \langle A_i(u_n^i), v - u_n^i \rangle + \left\langle u_n^i - x_n, v - u_n^i \right\rangle \geq 0, \quad \forall v \in C, \ i = 1, \ldots, N, \\ x_{n+1} = x_n - \beta_n \left[\sum_{i=1}^{N} \left(x_n - u_n^i\right) + \alpha_n x_n\right], \end{gathered} \tag{3.2}$$

then the sequence $\{x_n\}$ converges strongly to a common solution for (3.1).

If $C \equiv H$, then we have a problem of finding a common zero for a system of monotone hemicontinous mappings A_i, $i = 1, \ldots, N$. In this case, variational inequality in (3.2) has the form $A_i(u_n^i) + u_n^i = x_n$. Therefore, we have the following result.

Theorem 3.2. *Let $A_i, i = 1, \ldots, N$ be N hemicontinuous monotone mappings defined on H. Let $x_0 = x$ be an arbitrary element in H, let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences that are chosen as in Theorem 2.8, and, the iteration sequence $\{x_n\}$ be defined as follows:*

$$u_n^i : A_i\left(u_n^i\right) + u_n^i = x_n,$$
$$x_{n+1} = x_n - \beta_n\left[\sum_{i=1}^{N}\left(x_n - u_n^i\right) + \alpha_n x_n\right]. \tag{3.3}$$

Then the sequence $\{x_n\}$ converges strongly to an element u^* such that

$$A_i(u^*) = 0, \quad i = 1, \ldots, N. \tag{3.4}$$

Without the strongly or uniformly monotone property for A_i, each problem of (3.1), in general, is ill-posed. Some methods for finding a solution of each variational inequality in (3.1) are presented in [39].

Here we show an iterative regularization method for finding a common solution of these problems. Suppose that instead of A_i, we have their monotone approximations A_i^n such that $D(A_i^n) = C$ and

$$\|A_i^n(x) - A_i(x)\| \le h_n g(\|x\|), \quad i = 1, \ldots, N, \tag{3.5}$$

where the positive parameter $h_n \to 0$ as $n \to \infty$, and $g(t)$ is a real positive and bounded function. Obviously, the bifunctions

$$F^n(u, v) := \left\langle A_i^n(u), v - u\right\rangle, \quad i = 1, \ldots, N, \tag{3.6}$$

satisfy the Condition 1 and (2.4). Therefore, we have the following theorem.

Theorem 3.3. *Let $\tilde{x}_0 = x$ be an arbitrary element in H. If $\{\alpha_n\}$, $\{\beta_n\}$ are chosen as in Theorem 2.9, and the iteration sequence $\{\tilde{x}_n\}$ is defined as follows:*

$$\tilde{u}_n^i \in C : \left\langle A_i\left(\tilde{u}_n^i\right), v - \tilde{u}_n^i\right\rangle + \left\langle \tilde{u}_n^i - x_n, v - \tilde{u}_n^i\right\rangle \ge 0 \quad \forall v \in C,\ i = 1, \ldots, N,$$
$$\tilde{x}_{n+1} = \tilde{x}_n - \beta_n\left[\sum_{i=1}^{N}\left(\tilde{x}_n - \tilde{u}_n^i\right) + \alpha_n \tilde{x}_n\right], \tag{3.7}$$

then the sequence $\{\tilde{x}_n\}$ converges strongly to a common solution for (3.1).

If $C \equiv H$, then a common zero for a system of monotone hemicontinuous mappings $A_i, i = 1, \ldots, N$, could be found by the following.

Theorem 3.4. *Let A_i, $i = 1, \ldots, N$ be N hemicontinuous monotone mappings defined on H. Let $\tilde{x}_0 = x$ be an arbitrary element in H, let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences that are chosen as in Theorem 2.9, and the iteration sequence $\{\tilde{x}_n\}$ be defined as follows:*

$$\tilde{u}_n^i : A_i\left(\tilde{u}_n^i\right) + \tilde{u}_n^i = \tilde{x}_n,$$
$$\tilde{x}_{n+1} = \tilde{x}_n - \beta_n\left[\sum_{i=1}^{N}\left(\tilde{x}_n - \tilde{u}_n^i\right) + \alpha_n\tilde{x}_n\right]. \tag{3.8}$$

Then the sequence $\{\tilde{x}_n\}$ converges strongly to an element u^* such that

$$A_i(u^*) = 0, \quad i = 1, \ldots, N. \tag{3.9}$$

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