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Research Article

Strong Convergence Theorems by Shrinking Projection Methods for Class $\mathfrak{T}$ Mappings

Qiao-Li Dong,[1,2] Songnian He,[1,2] and Fang Su[3]

[1] *College of Science, Civil Aviation University of China, Tianjin 300300, China*
[2] *Tianjin Key Laboratory for Advanced Signal Processing, Civil Aviation University of China, Tianjin 300300, China*
[3] *Department of Mathematics and Systems Science, National University of Defense Technology, Changsha 410073, China*

Correspondence should be addressed to Qiao-Li Dong, dongqiaoli@ymail.com

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We prove a strong convergence theorem by a shrinking projection method for the class of $\mathfrak{T}$ mappings. Using this theorem, we get a new result. We also describe a shrinking projection method for a nonexpansive mapping on Hilbert spaces, which is the same as that of Takahashi et al. (2008).

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. Recall that a mapping $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is $\mathrm{Fix}(T) := \{x \in H : Tx = x\}$.

$T : H \to H$ is said to be quasi-nonexpansive if $\mathrm{Fix}(T)$ is nonempty and $\|Tx - p\| \le \|x - p\|$ for all $x \in H$ and $p \in \mathrm{Fix}(T)$.

Given $x, y \in H$, let

$$H(x, y) := \{z \in H : \langle z - y, x - y \rangle \le 0\} \tag{1.1}$$

be the half-space generated by (x, y). A mapping $T : H \to H$ is said to be the class $\mathfrak{T}$ (or a cutter) if $T \in \mathfrak{T} = \{T : H \to H \mid \mathrm{dom}(T) = H \text{ and } \mathrm{Fix}(T) \subset H(x, Tx), \text{ for all } x \in H\}$.

Remark 1.1. The class $\mathfrak{T}$ is fundamental because it contains several types of operators commonly found in various areas of applied mathematics and in particular in approximation and optimization theory (see [1] for details).

Combettes [2], Bauschke, and Combettes [1] studied properties of the class $\mathfrak{T}$ mappings and presented several algorithms. They introduced an abstract Haugazeau method in [1] as follows: starting $x_0 \in H$,

$$x_{n+1} = P_{H(x_0,x_n)\cap H(x_n,T_nx_n)}x_0. \tag{1.2}$$

Using Lemma 1.2 given below and the fact that a nonexpansive mapping is quasi-nonexpansive, one can easily obtain hybrid methods introduced by Nakajo and Takahashi [3] for a nonexpansive mapping.

Recently, Takahashi et al. [4] proposed a shrinking projection method for nonexpansive mappings $T_n : C \to C$. Let $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, and

$$\begin{gathered} y_n = \alpha_n + (1-\alpha_n)T_nx_n, \\ C_{n+1} = \left\{z \in C_n : \|y_n - z\| \le \|x_n - z\|\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n = 1,2,\ldots, \end{gathered} \tag{1.3}$$

where $0 \le \alpha_n \le a < 1$, P_K denotes the metric projection from H onto a closed convex subset K of H.

Inspired by Bauschke and Combettes [1] and Takahashi et al. [4], we present a shrinking projection method for the class of $\mathfrak{T}$ mappings. Furthermore, we obtain a shrinking projection method for a nonexpansive mapping on Hilbert spaces, which is the same as presented by Takahashi et al. [4].

We will use the following notations:

(1) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;

(2) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit of $\{x_n\}$.

We need some facts and tools in a real Hilbert space H which are listed below.

Lemma 1.2 (see [1]). *Let H be a Hilbert space. Let I be the identity operator of H.*

(i) *If* $\operatorname{dom} T = H$, *then* $2T - I$ *is quasi-nonexpansive if and only if* $T \in \mathfrak{T}$.

(ii) *If* $T \in \mathfrak{T}$, *then* $\lambda I + (1-\lambda)T \in \mathfrak{T}$, *for all* $\lambda \in [0,1]$.

Definition 1.3. Let $T_n \in \mathfrak{T}$ for each n. The sequence $\{T_n\}$ is called to be coherent if, for every bounded sequence $\{v_n\}$ in H, there holds

$$\left.\begin{aligned} &\sum_{n=0}^{\infty}\|v_{n+1} - v_n\|^2 < \infty, \\ &\sum_{n=0}^{\infty}\|v_n - T_nv_n\|^2 < \infty, \end{aligned}\right. \Longrightarrow \omega_w(v_n) \subset \bigcap_{n=0}^{\infty}\operatorname{Fix}(T_n). \tag{1.4}$$

Definition 1.4. T is called demiclosed at $y \in H$ if $Tx = y$ whenever $\{x_n\} \subset H$, $x_n \rightharpoonup x$ and $Tx_n \to y$.

Next lemma shows that nonexpansive mappings are demeiclosed at 0.

Lemma 1.5 (Goebel and Kirk [5]). *Let C be a closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

Lemma 1.6 (see [6]). *Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If x_n is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n, \tag{1.5}$$

then $x_n \to q$.

Lemma 1.7 (Goebel and Kirk [5]). *Let K be a closed convex subset of real Hilbert space H, and let P_K be the (metric or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$, then $z = P_K x$ if and only if there holds the relation*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in K. \tag{1.6}$$

2. Main Results

In this section, we will introduce a shrinking projection method for the class of $\mathfrak{T}$ mappings and prove strong convergence theorem.

Theorem 2.1. *Let $T_n \in \mathfrak{T}$ for each n such that $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Suppose that the sequence $\{T_n\}$ is coherent. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:*

$$\begin{gathered} x_{n+1} = P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots, \\ C_{n+1} = \{z \in C_n : \langle z - T_n x_n, x_n - T_n x_n \rangle \leq 0\}. \end{gathered} \tag{2.1}$$

Then, $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof. We first show by induction that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. $\mathcal{F} \subset C_1$ is obvious. Suppose $\mathcal{F} \subset C_k$ for some $k \in \mathbb{N}$. Note that, by the definition of $T_k \in \mathfrak{T}$, we always have $\mathcal{F} \subset \mathrm{Fix}(T_k) \subset H(x_k, T_k x_k)$, that is,

$$\langle z - T_k x_k, x_k - T_k x_k \rangle \leq 0, \quad \forall z \in \mathcal{F}. \tag{2.2}$$

From the definition of C_{k+1} and $\mathcal{F} \subset C_k$, we obtain $\mathcal{F} \subset C_{k+1}$. This implies that

$$\mathcal{F} \subset C_n, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

It is obvious that $C_1 = H$ is closed and convex. So, from the definition, C_n is closed and convex for all $n \in \mathbb{N}$. So we get that $\{x_n\}$ is well defined.

Since x_n is the projection of x_0 onto C_n which contains $\mathcal{F}$, we have

$$\|x_0 - x_n\| \leq \|x_0 - y\|, \quad \forall y \in C_n. \tag{2.4}$$

Taking $y = P_{\mathcal{F}} x_0 \in \mathcal{F}$, we get

$$\|x_0 - x_n\| \le \|x_0 - P_{\mathcal{F}} x_0\|. \tag{2.5}$$

The last inequality ensures that $\{\|x_0 - x_n\|\}$ is bounded. From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, using Lemma 1.7, we get

$$\langle x_{n+1} - x_n, x_0 - x_n \rangle \le 0. \tag{2.6}$$

It follows that

$$\begin{aligned}
\|x_0 - x_{n+1}\|^2 &= \|(x_0 - x_n) - (x_{n+1} - x_n)\|^2 \\
&= \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_{n+1} - x_n \rangle + \|x_{n+1} - x_n\|^2 \\
&\ge \|x_0 - x_n\|^2 + \|x_{n+1} - x_n\|^2 \\
&\ge \|x_0 - x_n\|^2.
\end{aligned} \tag{2.7}$$

Thus $\{\|x_n - x_0\|\}$ is increasing. Since $\{\|x_n - x_0\|\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From (2.7), it follows that

$$\|x_{n+1} - x_n\|^2 \le \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2, \tag{2.8}$$

and $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < \infty$. On the other hand, by $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\langle x_{n+1} - T_n x_n, x_n - T_n x_n \rangle \le 0. \tag{2.9}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - T_n x_n) - (x_n - T_n x_n)\|^2 \\
&= \|x_{n+1} - T_n x_n\|^2 - 2\langle x_{n+1} - T_n x_n, x_n - T_n x_n \rangle + \|x_n - T_n x_n\|^2 \\
&\ge \|x_{n+1} - T_n x_n\|^2 + \|x_n - T_n x_n\|^2.
\end{aligned} \tag{2.10}$$

We therefore get $\sum_{n=1}^{\infty} \|x_n - T_n x_n\|^2 < \infty$. Since the sequence $\{T_n\}$ is coherent, we have $\omega_w(x_n) \subset \mathcal{F}$. From Lemma 1.6 and (2.5), the result holds. □

Remark 2.2. We take $C_1 = H$ so that $\mathcal{F} \subset C_1$ is satisfied.

Theorem 2.3. *Let $T_n \in \mathfrak{T}$ for each n such that $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Suppose that the sequence $\{T_n\}$ is coherent. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ C_{n+1} &= \{ z \in C_n : \langle z - y_n, x_n - y_n \rangle \leq 0 \}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots, \end{aligned} \tag{2.11}$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof. Set $S_n = \alpha_n I + (1-\alpha_n) T_n$. By Lemma 1.2(ii), we have that $S_n \in \mathfrak{T}$. From $\|x_n - S_n x_n\| = (1-\alpha_n)\|x_n - T_n x_n\|$, it follows that $(1-a)\|x_n - T_n x_n\| \leq \|x_n - S_n x_n\| \leq \|x_n - T_n x_n\|$ which implies that the sequence $\{S_n\}$ is coherent. It is obvious that $\mathrm{Fix}(S_n) = \mathrm{Fix}(T_n)$, for all $n \in \mathbb{N}$. Hence $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Using Theorem 2.1, we get the desired result. □

3. Deduced Results

In this section, using Theorem 2.3, we obtain some new strong convergence results for the class of $\mathfrak{T}$ mappings, a quasi-nonexpansive mapping and a nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let $T \in \mathfrak{T}$ such that $\mathrm{Fix}(T) \neq \emptyset$ and satisfying that $I - T$ is demiclosed at 0. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_{n+1} &= \{ z \in C_n : \langle z - y_n, x_n - y_n \rangle \leq 0 \}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots, \end{aligned} \tag{3.1}$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)} x_0$.

Proof. Let $T_n = T$ in (2.11) for all $n \in \mathbb{N}$. Following the proof of Theorem 2.1, we can easily get (2.5) and $\sum_{n=1}^{\infty} \|x_n - T x_n\|^2 < \infty$. By (2.5), we obtain that $\{x_n\}$ is bounded and $\omega_w(x_n)$ is nonempty. For any $\widehat{x} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $x_{n_j} \rightharpoonup \widehat{x}$. From $\sum_{n=1}^{\infty} \|x_n - T x_n\|^2 < \infty$, it follows that $\|x_n - T x_n\| \to 0$. Since $I - T$ is demiclosed at 0, we get $\widehat{x} \in \mathrm{Fix}(T)$. Thus $\omega_w(x_n) \subset \mathrm{Fix}(T)$ which together with Lemma 1.6 and (2.5) implies that $x_n \to P_{\mathrm{Fix}(T)} x_0$. □

Theorem 3.2. *Let H be a Hilbert space. Let S be a quasi-nonexpansive mapping on H such that $\mathrm{Fix}(S) \neq \emptyset$ and satisfying that $I - S$ is demiclosed at 0. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} u_n &= \alpha_n x_n + (1-\alpha_n) S x_n, \\ C_{n+1} &= \{ z \in C_n : \|z - u_n\| \leq \|x_n - z\| \}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots, \end{aligned} \tag{3.2}$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(S)} x_0$.

Proof. By Lemma 1.2(i), $(S+I)/2 \in \mathfrak{T}$. Substitute T in (3.1) by $(S+I)/2$. Then $y_n = ((1+\alpha_n)/2)x_n + ((1-\alpha_n)/2)Sx_n$. Set $u_n = 2y_n - x_n = \alpha_n x_n + (1-\alpha_n)Sx_n$, then $y_n = (u_n + x_n)/2$. So, we have

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \langle z - y_n, x_n - y_n \rangle \le 0\} \\ &= \{z \in C_n : \langle 2z - (x_n + u_n), x_n - u_n \rangle \le 0\} \\ &= \{z \in C_n : \|z - u_n\| \le \|x_n - z\|\}. \end{aligned} \tag{3.3}$$

Since $I - S$ is demiclosed at 0, $I - (S+I)/2 = (I-S)/2$ is demiclosed at 0. So we can obtain the result by using Theorem 3.1. □

Since a nonexpansive mapping is quasi-nonexpansive, using Lemma 1.5 and Theorem 3.2, we have following corollary.

Corollary 3.3. *Let H be a Hilbert space. Let S be a nonexpansive mapping H such that* $\mathrm{Fix}(S) \neq \emptyset$*. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} u_n &= \alpha_n x_n + (1-\alpha_n)Sx_n, \\ C_{n+1} &= \{z \in C_n : \|z - u_n\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots, \end{aligned} \tag{3.4}$$

where $0 \le \alpha_n \le a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(S)} x_0$.

Remark 3.4. Corollary 3.3 is a special case of Theorem 4.1 in [4] when $C_1 = H$.

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