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Research Article

Strong Convergence Theorems for Lipschitzian Demicontraction Semigroups in Banach Spaces

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The purpose of this paper is to introduce and study the strong convergence problem of the explicit iteration process for a Lipschitzian and demicontraction semigroups in arbitrary Banach spaces. The main results presented in this paper not only extend and improve some recent results announced by many authors, but also give an affirmative answer for the open questions raised by Suzuki (2003) and Xu (2005).

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, $\mathcal{R}^+$ is the set of nonnegative real numbers, and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|x\| = \|f\| \} \tag{1.1}$$

for all $x \in E$. Let $T : C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T. We also use " $\to$ " to stand for strong convergence and "$\rightharpoonup$" for weak convergence.

Definition 1.1. (1) The one-parameter family $\mathcal{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is called a *nonexpansive semigroup* if the following conditions are satisfied:

(a) $T(0)x = x$ for each $x \in C$;

(b) $T(t+s)x = T(t)T(s)x$ for any $t, s \in \mathcal{R}^+$ and $x \in C$;

(c) for any $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(d) for any $t \in \mathcal{R}^+$, $T(t)$ is a nonexpansive mapping on C, that is, for any $x, y \in C$,

$$\left\| T(t)x - T(t)y \right\| \leq \left\| x - y \right\| \tag{1.2}$$

for any $t > 0$.

(2) The one-parameter family $\mathcal{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is called a *pseudocontraction semigroup* if the conditions (a)–(c) and the following condition (e) are satisfied:

(e) for any $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\left\langle T(t)x - T(t)y, j(x-y) \right\rangle \leq \left\| x - y \right\|^2 \tag{1.3}$$

for any $t > 0$.

(3) A pseudocontraction semigroup $\mathcal{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is said to be *Lipschitzian* if the conditions (a)–(c), (e), and the following condition (f) are satisfied:

(f) there exists a bounded measurable function $L : [0, \infty) \rightarrow (0, \infty)$ such that, for any $x, y \in C$,

$$\left\| T(t)x - T(t)y \right\| \leq L(t) \left\| x - y \right\| \tag{1.4}$$

for any $t > 0$. In the sequel, we denote it by

$$L = \sup_{t \geq 0} L(t) < \infty. \tag{1.5}$$

From the definitions, it is easy to see that every nonexpansive semigroup is a Lipschitzian and pseudocontraction semigroup with $L(t) \equiv 1$.

(4) The one-parameter family $\mathcal{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is called a *strictly pseudocontractive semigroup* if the conditions (a)–(c) and the following condition (g) are satisfied:

(g) there exists a bounded function $\lambda : [0, \infty) \rightarrow (0, \infty)$ such that, for any given $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\left\langle T(t)x - T(t)y, j(x-y) \right\rangle \leq \left\| x - y \right\|^2 - \lambda(t) \left\| (I - T(t))x - (I - T(t))y \right\|^2 \tag{1.6}$$

for any $t > 0$.

It is easy to see that such mapping is $((1+\lambda(t))/\lambda(t))$-*Lipschitzian and pseudocontractive semigroup*.

(5) The one-parameter family $\mathcal{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is called a *demicontractive semigroup* if $F(T(t)) \neq \emptyset$ for all $t > 0$ and the conditions (a)–(c) and the following condition (h) are satisfied:

(h) there exists a bounded function $\lambda : [0,\infty) \to (0,\infty)$ such that, for any $t > 0$, $x \in C$ and $p \in F(T(t))$, there exists $j(x-p) \in J(x-p)$ such that

$$\left\langle T(t)x - p, j(x-p)\right\rangle \leq \|x-p\|^2 - \lambda(t)\|(I-T(t))x\|^2. \tag{1.7}$$

In this case, we also say that $\mathcal{T}$ is a $\lambda(t)$-*demicontractive semigroup*.

Remark 1.2. (1) It is easy to see that the condition (1.7) is equivalent to the following condition: for any $t > 0$, $x \in C$ and $p \in F(T(t))$,

$$\left\langle x - T(t)x, j(x-p)\right\rangle \geq \lambda(t)\|x - T(t)x\|^2. \tag{1.8}$$

(2) Every strictly pseudocontractive semigroup with $\mathcal{F} := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ is demicontractive and Lipschitzian.

The convergence problems of the implicit or explicit iterative sequences for nonexpansive semigroup to a common fixed has been considered by some authors in the settings of Hilbert or Banach spaces (see, e.g., [1–10]).

In 1998, Shioji and Takahashi [7] introduced the following implicit iteration:

$$x_n = \alpha_n u + (1-\alpha_n)\sigma_{t_n}(x_n) \tag{1.9}$$

for each $n \geq 1$ in a Hilbert space, where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ is a sequence of positive real numbers divergent to ∞, and, for any $t > 0$ and $x \in C$, $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x\,ds. \tag{1.10}$$

Under certain restrictions to the sequence $\{\alpha_n\}$, they proved some strong convergence theorems of $\{x_n\}$ to a point $p \in \mathcal{F} := \bigcap_{t\geq 0} F(T(t))$.

In 2003, Suzuki [8] first introduced the following implicit iteration process for the nonexpansive semigroup in a Hilbert space:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n \tag{1.11}$$

for each $n \geq 1$. Under appropriate assumptions imposed upon the sequences $\{\alpha_n\}$ and $\{t_n\}$, he proved that the sequence $\{x_n\}$ defined by (1.11) converges strongly to a common fixed point of the nonexpansive semigroup. At the same time, he also raised the following open question.

Open Question 1.3 (see [8]). Does there exist an explicit iteration concerning Suzuki's result? That is, for any given $x_0, u \in C$, if we define an explicit iterative sequence $\{x_n\}$ by

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n \tag{1.12}$$

for each $n \geq 0$, what conditions to be imposed on $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset (0,\infty)$ are sufficient to guarantee the strong convergence of $\{x_n\}$ to a common fixed point of the nonexpansive semi-group $\mathcal{T} := \{T(t) : t \geq 0\}$ of mapping from C into itself?

In 2005, Xu [9] proved that Suzuki's result holds in a uniformly convex Banach space with a weakly continuous duality mapping. At the same time, he also raised the following open question.

Open Question 1.4 (see [9]). We do not know whether or not the same result holds in a uniformly convex and uniformly smooth Banach space.

In 2005, Aleyner and Reich [1] first introduced the following explicit iteration sequence:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n \tag{1.13}$$

for each $n \geq 0$ in a reflexive Banach space with a uniformly Gâteaux differentiable norm such that each nonempty bounded closed and convex subset of E has the common fixed point property for nonexpansive mappings (note that all these assumptions are fulfilled whenever E is uniformly smooth [11]).

Also, under appropriate assumptions imposed upon the parameter sequences $\{\alpha_n\}$ and $\{t_n\}$, they proved that the sequence $\{x_n\}$ defined by (1.13) converges strongly to some point in $\mathcal{F} =: \bigcap_{t\geq 0} F(T(t))$.

Recently, in 2007, Zhang et al. [3] introduced the following composite iteration scheme in the framework of reflexive Banach with a uniformly Gâteaux differentiable norm, uniformly smooth Banach space and uniformly convex Banach space with a weakly continuous normalized duality mapping:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n \tag{1.14}$$

for each $n \geq 0$ for the nonexpansive semi-group $\mathcal{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself, where u is an arbitrary (but fixed) element in C and the sequences $\{\alpha_n\}$ in $(0,1)$, $\{\beta_n\}$ in $[0,1]$, $\{t_n\}$ in $\mathcal{R}^+$, and proved some strong convergence theorems for the iteration sequence $\{x_n\}$. In fact, the results presented in [3] not only extend and improve the corresponding results of Shioji and Takahashi [7], Suzuki [8], Xu [9], and Aleyner and Reich [1], but also give a partially affirmative answer for the open questions raised by Suzuki [8] and Xu [9].

In order to improve and develop the results mentioned above, recently, Zhang [12, 13], by using the different methods, introduce and study the weak convergence problem of the implicit iteration processes for the Lipschitzian and pseudocontraction semigroups in general Banach spaces. The results given in [12, 13] not only extend the above results, but also extend and improve the corresponding results in Li et al. [6], Osilike [14], Xu and Ori [15], and Zhou [16].

The purpose of this paper is to introduce and study the strong convergence problem of the following explicit iteration process:

$$x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(t_n)x_n \tag{1.15}$$

for each $n \geq 1$ for the Lipschitzian and demicontractive semigroup $\mathcal{T} := \{T(t) : t \geq 0\}$ in general Banach spaces. The results presented in this paper improve, extend, and replenish the corresponding results given in [1, 3–10, 12, 13].

In the sequel, we make use of the following lemmas for our main results.

Lemma 1.5. *Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{1.16}$$

for all $j(x+y) \in J(x+y)$.

Lemma 1.6 (see [17]). *Let $\{a_n\}$ and $\{\sigma_n\}$ be the sequences of nonnegative real numbers satisfying the following condition:*

$$a_{n+1} \leq (1+\sigma_n)a_n \tag{1.17}$$

for all $n \geq n_0$, where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists. In addition, if there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

Now, we are ready to give our main results in this paper.

Theorem 2.1. *Let E be a real Banach space; let C be a nonempty closed convex subset of E, and let $\mathcal{T} := \{T(t) : t \geq 0\} : C \to C$ be a Lipschitzian and demicontractive semigroup with a bounded measurable function $L : [0,\infty) \to (0,\infty)$ and a bounded function $\lambda : [0,\infty) \to (0,\infty)$, respectively, such that*

$$L := \sup_{t\geq 0} L(t) < \infty, \qquad \lambda := \inf_{t\geq 0} \lambda(t) > 0, \qquad \mathcal{F} := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset. \tag{2.1}$$

Let $\{x_n\}$ *be the sequence defined by* (1.15), *where* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *and* $\{t_n\}$ *is an increasing sequence in* $[0,\infty)$. *If the following conditions are satisfied:*

(a) $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\sum_{n=1}^{\infty}\alpha_n^2<\infty$;

(b) *for any bounded subset* $D\subset C$,

$$\lim_{n\to\infty}\sup_{x\in D, s\in\mathcal{R}^+}\|T(s+t_n)x-T(t_n)x\|=0, \tag{2.2}$$

then we have the following:

(1) $\lim_{n\to\infty}\|x_n-p\|$ *exists for all* $p\in\mathcal{F}$.

(2) $\liminf_{n\to\infty}\|x_n-T(t_n)x_n\|=0$.

Proof. (1) For any $p\in\mathcal{F}$, we have

$$\|T(t_n)x_n-p\|\le L(t_n)\|x_n-p\|\le L\|x_n-p\|. \tag{2.3}$$

It follows from (2.3) that

$$\begin{aligned}\|x_{n+1}-p\|&\le(1-\alpha_n)\|x_n-p\|+\alpha_n\|T(t_n)x_n-p\|\\&\le(1-\alpha_n+\alpha_nL)\|x_n-p\|\\&\le(1+L)\|x_n-p\|.\end{aligned} \tag{2.4}$$

Consequently, it follows from (2.3) and (2.4) that

$$\|x_n-T(t_n)x_n\|\le\|x_n-p\|+\|p-T(t_n)x_n\|\le(1+L)\|x_n-p\|, \tag{2.5}$$

$$\|x_{n+1}-T(t_n)x_{n+1}\|\le\|x_{n+1}-p\|+\|p-T(t_n)x_{n+1}\|\le(1+L)^2\|x_n-p\|. \tag{2.6}$$

From (2.5), we have

$$\|x_{n+1}-x_n\|=\alpha_n\|T(t_n)x_n-x_n\|\le\alpha_n(1+L)\|x_n-p\|. \tag{2.7}$$

Since $\mathcal{T}:=\{T(t):t\ge 0\}$ is an demicontractive semigroup with $\lambda=\inf_{t\ge\ge 0}\lambda(t)>0$, for the points x_{n+1} and p, there exists $j(x_{n+1}-p)\in J(x_{n+1}-p)$ such that

$$\langle x_{n+1}-T(t_n)x_{n+1}, j(x_{n+1}-p)\rangle\ge\lambda\|x_{n+1}-T(t_n)x_{n+1}\|^2. \tag{2.8}$$

Thus, by Lemma 1.5, (2.4), (2.7), and (2.8), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(x_n-p)+\alpha_n(T(t_n)x_n-x_n)\|^2\\ &\le \|x_n-p\|^2+2\alpha_n\langle T(t_n)x_n-x_n, j(x_{n+1}-p)\rangle\\ &= \|x_n-p\|^2+2\alpha_n\langle T(t_n)x_n-T(t_n)x_{n+1}, j(x_{n+1}-p)\rangle\\ &\quad -2\alpha_n\langle x_{n+1}-T(t_n)x_{n+1}, j(x_{n+1}-p)\rangle+2\alpha_n\langle x_{n+1}-x_n, j(x_{n+1}-p)\rangle\\ &\le \|x_n-p\|^2+2\alpha_n L\|x_n-x_{n+1}\|\|x_{n+1}-p\|\\ &\quad -2\alpha_n\lambda\|T(t_n)x_{n+1}-x_{n+1}\|^2+2\alpha_n^2L(1+L)^2\|x_n-p\|^2\\ &\le \left(1+2\alpha_n^2(1+L)^3\right)\|x_n-p\|^2-2\alpha_n\lambda\|T(t_n)x_{n+1}-x_{n+1}\|^2.\end{aligned} \tag{2.9}$$

This implies that

$$\|x_{n+1}-p\|^2 \le \left(1+2\alpha_n^2(1+L)^3\right)\|x_n-p\|^2. \tag{2.10}$$

By the assumption $\sum_{n=1}^{\infty}\alpha_n^2<\infty$, it follows from Lemma 1.6 that the limit $\lim_{n\to\infty}\|x_n-p\|$ exists and so the sequence $\{x_n\}$ is bounded in C.

(2) We first prove that

$$\liminf_{n\to\infty}\|x_{n+1}-T(t_n)x_{n+1}\|=0. \tag{2.11}$$

If it is not the case, suppose $\liminf_{n\to\infty}\|x_{n+1}-T(t_n)x_{n+1}\|=\delta>0$. There exists a positive integer n_0 such that

$$\|x_{n+1}-T(t_n)x_{n+1}\| \ge \frac{\delta}{2} \tag{2.12}$$

for each $n\ge n_0$. Since $\{x_n\}$ is bounded, denote by

$$M=\sup_{n\ge 1}\|x_n-p\|. \tag{2.13}$$

Thus it follows from (2.9) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|x_n-p\|^2-2\alpha_n\lambda\|T(t_n)x_{n+1}-x_{n+1}\|^2+2\alpha_n^2(1+L)^3\|x_n-p\|^2\\ &\le \|x_n-p\|^2-\alpha_n\lambda\frac{\delta^2}{2}+2\alpha_n^2(1+L)^3M^2\end{aligned} \tag{2.14}$$

for each $n \geq n_0$. This implies that

$$\alpha_n \lambda \frac{\delta^2}{2} \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n^2 (1+L)^3 M^2 \tag{2.15}$$

for each $n \geq n_0$. Hence, for each $m > n_0$, we have

$$\begin{aligned} \lambda \frac{\delta^2}{2} \sum_{n=n_0}^{m} \alpha_n &\leq \sum_{n=n_0}^{m} \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \right) + 2(1+L)^3 M^2 \sum_{n=n_0}^{m} \alpha_n^2 \\ &\leq \|x_{n_0} - p\|^2 + 2(1+L)^3 M^2 \sum_{n=n_0}^{m} \alpha_n^2. \end{aligned} \tag{2.16}$$

Letting $m \to \infty$ in (2.16), we have

$$\lambda \frac{\delta^2}{2} \sum_{n=n_0}^{\infty} \alpha_n \leq \|x_{n_0} - p\|^2 + 2(1+L)^3 M^2 \sum_{n=n_0}^{\infty} \alpha_n^2, \tag{2.17}$$

which is a contradiction since, by the condition (a), $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Therefore, the conclusion (2.11) is proved.

On the other hand, since $\{x_n\}$ is bounded and $\{t_n\}$ is increasing, it follows from (2.11) and the condition (b) that

$$\begin{aligned} &\liminf_{n\to\infty} \|x_{n+1} - T(t_{n+1})x_{n+1}\| \\ &\quad \leq \liminf_{n\to\infty} \{\|x_{n+1} - T(t_n)x_{n+1}\| + \|T(t_{n+1})x_{n+1} - T(t_n)x_{n+1}\|\} \\ &\quad = \liminf_{n\to\infty} \{\|x_{n+1} - T(t_n)x_{n+1}\| + \|T((t_{n+1} - t_n) + t_n)x_{n+1} - T(t_n)x_{n+1}\|\} \\ &\quad \leq \liminf_{n\to\infty} \left\{ \|x_{n+1} - T(t_n)x_{n+1}\| + \sup_{z\in\{x_n\},\, s\in\mathcal{R}^+} \|T(s+t_n)z - T(t_n)z\| \right\} \\ &\quad = 0. \end{aligned} \tag{2.18}$$

This completes the proof. □

By using Theorem 2.1, we have the following.

Theorem 2.2. *Let E be a real Banach space; let C be a nonempty closed convex subset of E, and let $\mathcal{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself be a Lipschitzian and demicontractive semigroup with a bounded measurable function $L : [0,\infty) \to (0,\infty)$ and a bounded function $\lambda : [0,\infty) \to (0,\infty)$, respectively, such that*

$$L := \sup_{t\geq 0} L(t) < \infty, \qquad \lambda := \inf_{t\geq 0} \lambda(t) > 0, \qquad \mathcal{F} := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset. \tag{2.19}$$

Let $\{x_n\}$ be the sequence defined by (1.15)*, where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{t_n\}$ is an increasing sequence in $[0,\infty)$. If there exists a compact subset K of E such that $\bigcup_{t\geq 0} T(t)(C) \subset K$ and the following conditions are satisfied:*

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(b) *for any bounded subset $D \subset C$,*

$$\lim_{n\to\infty} \sup_{x\in D, s\in\mathcal{R}^+} \|T(s+t_n)x - T(t_n)x\| = 0, \tag{2.20}$$

then $\{x_n\}$ converges strongly to a common fixed point of the semigroup $\mathcal{T} := \{T(t) : t \geq 0\}$.

Proof. By Theorem 2.1, we have $\liminf_{n\to\infty}\|x_n - T(t_n)x_n\| = 0$. Again, by the assumption, it follows that there exists a compact subset $K \subset E$ such that $\bigcup_{t\geq 0} T(t)(C) \subset K$ and so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{n_k\to\infty} \|x_{n_k} - T(t_{n_k})x_{n_k}\| = 0, \qquad \lim_{n_k\to\infty} T(t_{n_k})x_{n_k} = q \tag{2.21}$$

for some point $q \in C$. Hence it follows from (2.21) that $x_{n_k} \to q$ as $n_k \to \infty$.

Next, we prove that

$$\lim_{n_k\to\infty} \|T(t)x_{n_k} - x_{n_k}\| = 0 \tag{2.22}$$

for all $t \geq 0$. In fact, it follows from the condition (b) and (2.21) that, for any $t > 0$,

$$\begin{aligned}
&\|T(t)x_{n_k} - x_{n_k}\| \\
&\quad \leq \|T(t)x_{n_k} - T(t+t_{n_k})x_{n_k}\| + \|T(t+t_{n_k})x_{n_k} - T(t_{n_k})x_{n_k}\| + \|T(t_{n_k})x_{n_k} - x_{n_k}\| \\
&\quad \leq (1+L)\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|T(t+t_{n_k})x_{n_k} - T(t_{n_k})x_{n_k}\| \\
&\quad \leq (1+L)\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \sup_{z\in\{x_n\}, s\in\mathcal{R}^+} \|T(s+t_{n_k})z - T(t_{n_k})z\| \\
&\quad \longrightarrow 0
\end{aligned} \tag{2.23}$$

as $n_k \to \infty$. Since $x_{n_k} \to q$ as $n_k \to \infty$ and the semigroup $\mathcal{T} := \{T(t) : t \geq 0\}$ is Lipschitzian, it follows from (2.23) that $q = T(t)(q)$ for all $t \geq 0$, that is,

$$q \in \mathcal{F} := \bigcap_{t\geq 0} F(T(t)). \tag{2.24}$$

Since $x_{n_k} \to$ q as $n_k \to \infty$ and the limit $\lim_{n\to\infty}\|x_n - q\|$ exists by Theorem 2.1 (1), which implies that $x_n \to q \in \mathcal{F}$ as $n \to \infty$. This completes the proof. □

Remark 2.3. Theorem 2.2 not only extends and improves the corresponding results of Shioji and Takahashi [7], Suzuki [8], Xu [9], and Aleyner and Reich [1], but also gives an affirmative answer to the open questions raised by Suzuki [8] and Xu [9].

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