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Research Article

Generalized Hyers-Ulam Stability of the Pexiderized Cauchy Functional Equation in Non-Archimedean Spaces

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We prove the generalized Hyers-Ulam stability of the Pexiderized Cauchy functional equation $f(x+y) = g(x) + h(y)$ in non-Archimedean spaces.

1. Introduction

The stability problem of functional equations was originated from a question of Ulam [1] concerning the stability of group homomorphisms.

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that, if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, that is, if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the following question.

When the solutions of an equation differing slightly from a given one must be close to the true solution of the given equation.

For Banach spaces, the Ulam problem was first solved by Hyers [2] in 1941, which states that, if $\delta > 0$ and $f : X \to Y$ is a mapping, where X, Y are Banach spaces, such that

$$\left\| f(x+y) - f(x) - f(y) \right\|_Y \leq \delta \tag{1.1}$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \leq \delta \tag{1.2}$$

for all $x \in X$. Rassias [3] succeeded in extending the result of Hyers by weakening the condition for the Cauchy difference to be unbounded. A number of mathematicians were attracted to this result of Rassias and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Rassias is called the *generalized Hyers-Ulam stability.* Forti [4] and Găvruţa [5] have generalized the result of Rassias, which permitted the Cauchy difference to become arbitrary unbounded. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. A large list of references can be found, for example, in [3, 6–30].

Definition 1.1. A field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ is called a *non-Archimedean field* if the function $|\cdot| : \mathbb{K} \to [0,\infty)$ satisfies the following conditions:

(1) $|r| = 0$ if and only if $r = 0$;

(2) $|rs| = |r||s|$;

(3) $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$.

Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Definition 1.2. Let X be a vector space over scaler field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

$(1)'$ $\|x\| = 0$ if and only if $x = 0$;

$(2)'$ $\|rx\| = |r|\|x\|$;

$(3)'$ the strong triangle inequality, namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \tag{1.3}$$

for all $x, y \in X$ and $r \in \mathbb{K}$.

The pair $(X, \|\cdot\|)$ is called a *non-Archimedean space* if $\|\cdot\|$ is non-Archimedean norm on X.

It follows from $(3)'$ that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\} \tag{1.4}$$

for all $x_n, x_m \in X$, where $m, n \in \mathbb{N}$ with $n > m$. Therefore, a sequence $\{x_n\}$ is a Cauchy sequence in non-Archimedean space $(X, \|\cdot\|)$ if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero in $(X, \|\cdot\|)$. In a complete non-Archimedean space, every Cauchy sequence is convergent.

In 1897, Hensel [31] discovered the p-adic number as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to metric $d(x,y) = |x-y|_p$, which is denoted by $\mathbb{Q}_p$, is called *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p-1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$, and it makes $\mathbb{Q}_p$ a locally compact field (see [32, 33]).

In [34], Arriola and Beyer showed that, if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous mapping for which there exists a fixed ε such that $|f(x+y) - f(x) - f(y)| \leq \varepsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive mapping $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \leq \varepsilon$ for all $x \in \mathbb{Q}_p$. The stability problem of the Cauchy functional equation and quadratic functional equation has been investigated by Moslehian and Rassias [19] in non-Archimedean spaces.

According to Theorem 6 in [16], a mapping $f : X \to Y$ satisfying $f(0) = 0$ is a solution of the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) \tag{1.5}$$

for all $x, y \in X$ if and only if it satisfies the additive Cauchy functional equation $f(x+y) = f(x) + f(y)$.

In this paper, by using the idea of Găvruţa [5], we prove the stability of the Jensen functional equation and the Pexiderized Cauchy functional equation:

$$f(x+y) = g(x) + h(y). \tag{1.6}$$

2. Generalized Hyers-Ulam Stability of the Jensen Functional Equation

Throughout this section, let X be a normed space with norm $\|\cdot\|_X$ and Y a complete non-Archimedean space with norm $\|\cdot\|_Y$.

Theorem 2.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{2.1}$$

for all $x, y \in X$ and the limit

$$\lim_{n\to\infty} \max\left\{ |2|^j \varphi\left(\frac{x}{2^j}, 0\right) : 0 \leq j < n \right\} \tag{2.2}$$

for all $x \in X$*, which is denoted by* $\widetilde{\varphi}(x)$*, exist. Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_Y \le \varphi(x,y) \tag{2.3}$$

for all $x, y \in X$*. Then the limit*

$$T(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.4}$$

exists for all $x \in X$ *and* $T : X \to Y$ *is an additive mapping satisfying*

$$\|f(x) - T(x)\|_Y \le \widetilde{\varphi}(x) \tag{2.5}$$

for all $x \in X$*. Moreover, if*

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\left\{|2|^j \varphi\left(\frac{x}{2^j}, 0\right) : k \le j < n+k\right\} = 0 \tag{2.6}$$

for all $x \in X$*, then* T *is a unique additive mapping satisfying* (2.5).

Proof. Letting $y = 0$ in (2.3), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_Y \le \varphi(x,0) \tag{2.7}$$

for all $x \in X$. If we replace x in (2.7) by $x/2^n$ and multiply both sides of (2.7) to $|2|^n$, then we have

$$\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\|_Y \le |2|^n \varphi\left(\frac{x}{2^n}, 0\right) \tag{2.8}$$

for all $x \in X$ and all nonnegative integers n. It follows from (2.1) and (2.8) that the sequence $\{2^n f(x/2^n)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{2^n f(x/2^n)\}$ converges for all $x \in X$. Hence one can define the mapping $T : X \to Y$ by (2.4).

By induction on n, one can conclude that

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\|_Y \le \max\left\{|2|^k \varphi\left(\frac{x}{2^k}, 0\right) : 0 \le k < n\right\} \tag{2.9}$$

for all $n \in \mathbb{N}$ and $x \in X$. By passing the limit $n \to \infty$ in (2.9) and using (2.2), we obtain (2.5).

Now, we show that T is additive. It follows from (2.1), (2.3), and (2.4) that

$$\begin{aligned}
&\left\|2T\left(\frac{x+y}{2}\right)-T(x)-T(y)\right\|_Y \\
&\quad=\lim_{n\to\infty}|2|^n\left\|2f\left(\frac{x+y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\|_Y \\
&\quad\le\lim_{n\to\infty}|2|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) \\
&\quad=0
\end{aligned}\tag{2.10}$$

for all $x,y\in X$. Therefore, the mapping $T:X\to Y$ is additive.

To prove the uniqueness of T, let $U:X\to Y$ be another additive mapping satisfying (2.5). Since

$$\begin{aligned}
\lim_{k\to\infty}|2|^k\tilde{\varphi}\left(\frac{x}{2^k}\right)&=\lim_{k\to\infty}\lim_{n\to\infty}|2|^k\max\left\{|2|^j\varphi\left(\frac{x}{2^{k+j}},0\right):0\le j<n\right\} \\
&=\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{|2|^j\varphi\left(\frac{x}{2^j},0\right):k\le j<k+n\right\}
\end{aligned}\tag{2.11}$$

for all $x\in X$, it follows from (2.6) that

$$\|T(x)-U(x)\|_Y=\lim_{k\to\infty}|2|^k\left\|f\left(\frac{x}{2^k}\right)-U\left(\frac{x}{2^k}\right)\right\|_Y\le\lim_{k\to\infty}|2|^k\tilde{\varphi}\left(\frac{x}{2^k}\right)=0\tag{2.12}$$

for all $x\in X$. So $T=U$. This completes the proof. □

The following theorem is an alternative result of Theorem 2.1, and its proof is similar to the proof of Theorem 2.1.

Theorem 2.2. *Let $\psi:X^2\to[0,\infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{1}{|2|^n}\psi(2^nx,2^ny)=0\tag{2.13}$$

for all $x,y\in X$ and the limit

$$\lim_{n\to\infty}\max\left\{\frac{1}{|2|^j}\psi\left(2^jx,0\right):0<j\le n\right\}\tag{2.14}$$

for all $x\in X$, denoted by $\tilde{\varphi}(x)$, exist. Suppose that a mapping $f:X\to Y$ with $f(0)=0$ satisfies the inequality

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|_Y\le\psi(x,y)\tag{2.15}$$

for all $x, y \in X$*. Then the limit*

$$T(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \tag{2.16}$$

exists for all $x \in X$*, and* $T : X \to Y$ *is an additive mapping satisfying*

$$\|f(x) - T(x)\|_Y \leq \widetilde{\varphi}(x) \tag{2.17}$$

for all $x \in X$*. Moreover, if*

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{ \frac{1}{|2|^j} \varphi\left(2^j x, 0\right) : k < j \leq n + k \right\} = 0 \tag{2.18}$$

for all $x \in X$*, then* T *is a unique additive mapping satisfying* (2.17).

3. Generalized Hyers-Ulam Stability of the Pexiderized Cauchy Functional Equation

Throughout this section, let X be a normed space with norm $\|\cdot\|_X$ and Y a complete non-Archimedean space with norm $\|\cdot\|_Y$.

Theorem 3.1. *Let* $\Phi : X^2 \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} |2|^n \Phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.1}$$

for all $x, y \in X$ *and the limits*

$$\widetilde{\Phi_1}(x) := \lim_{n\to\infty} \max_{0\leq j<n} \left\{ |2|^j \Phi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), |2|^j \Phi\left(\frac{x}{2^{j+1}}, 0\right), |2|^j \Phi\left(0, \frac{x}{2^{j+1}}\right), |2|^j \Phi(0,0) \right\}, \tag{3.2}$$

$$\widetilde{\Phi_2}(x) := \lim_{n\to\infty} \max_{0\leq j<n} \left\{ |2|^j \Phi\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), |2|^j \Phi\left(\frac{x}{2^{j+1}}, 0\right), |2|^j \Phi\left(\frac{x}{2^{j}}, \frac{-x}{2^{j+1}}\right), |2|^j \Phi(0,0) \right\}, \tag{3.3}$$

$$\widetilde{\Phi_3}(x) := \lim_{n\to\infty} \max_{0\leq j<n} \left\{ |2|^j \Phi\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), |2|^j \Phi\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j}}\right), |2|^j \Phi\left(0, \frac{x}{2^{j+1}}\right), |2|^j \Phi(0,0) \right\} \tag{3.4}$$

exist for all $x \in X$*. Suppose that mappings* $f, g, h : X \to Y$ *with* $f(0) = g(0) = h(0) = 0$ *satisfy the inequality*

$$\|f(x+y) - g(x) - h(y)\|_Y \leq \Phi(x, y) \tag{3.5}$$

for all $x, y \in X$*. Then the limits*

$$T(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right) \tag{3.6}$$

exist for all $x \in X$ *and* $T : X \to Y$ *is an additive mapping satisfying*

$$\|f(x) - T(x)\|_Y \le \widetilde{\Phi_1}(x), \tag{3.7}$$

$$\|g(x) - T(x)\|_Y \le \widetilde{\Phi_2}(x), \tag{3.8}$$

$$\|h(x) - T(x)\|_Y \le \widetilde{\Phi_3}(x) \tag{3.9}$$

for all $x \in X$. *Moreover, if*

$$\lim_{k\to\infty} |2|^k \widetilde{\Phi_1}\left(\frac{x}{2^k}\right) = \lim_{k\to\infty} |2|^k \widetilde{\Phi_2}\left(\frac{x}{2^k}\right) = \lim_{k\to\infty} |2|^k \widetilde{\Phi_3}\left(\frac{x}{2^k}\right) = 0 \tag{3.10}$$

for all $x \in X$, *then* T *is a unique additive mapping satisfying* (3.7), (3.8), *and* (3.9).

Proof. It follows from (3.5) that

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_Y \\
&\quad\le \max\left\{\left\|f\left(\frac{x+y}{2}\right) - g\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right)\right\|_Y, \left\|f\left(\frac{x+y}{2}\right) - g\left(\frac{y}{2}\right) - h\left(\frac{x}{2}\right)\right\|_Y,\right. \\
&\qquad\qquad \left.\left\|f(x) - g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\|_Y, \left\|f(y) - g\left(\frac{y}{2}\right) - h\left(\frac{y}{2}\right)\right\|_Y\right\} \\
&\quad\le \max\left\{\Phi\left(\frac{x}{2},\frac{y}{2}\right), \Phi\left(\frac{y}{2},\frac{x}{2}\right), \Phi\left(\frac{x}{2},\frac{x}{2}\right), \Phi\left(\frac{y}{2},\frac{y}{2}\right)\right\}
\end{aligned} \tag{3.11}$$

for all $x, y \in X$. Let

$$\Psi_f(x,y) := \max\left\{\Phi\left(\frac{x}{2},\frac{y}{2}\right), \Phi\left(\frac{y}{2},\frac{x}{2}\right), \Phi\left(\frac{x}{2},\frac{x}{2}\right), \Phi\left(\frac{y}{2},\frac{y}{2}\right)\right\} \tag{3.12}$$

for all $x, y \in X$. It follows from (3.1) and (3.2) that

$$\begin{gathered}
\lim_{n\to\infty} |2|^n \Psi_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \\
\widetilde{\Phi_1}(x) = \lim_{n\to\infty} \max\left\{|2|^j \Psi_f\left(\frac{x}{2^j}, 0\right) : 0 \le j < n\right\}
\end{gathered} \tag{3.13}$$

for all $x, y \in X$. By Theorem 2.1, there exists an additive mapping $T_1 : X \to Y$ satisfying (3.7) and

$$T_1(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{3.14}$$

for all $x \in X$. From (3.5), we get

$$\begin{aligned}
&\left\|2g\left(\frac{x+y}{2}\right)-g(x)-g(y)\right\|_Y \\
&\quad\le \max\left\{\left\|f\left(\frac{y}{2}\right)-g\left(\frac{x+y}{2}\right)-h\left(\frac{-x}{2}\right)\right\|_Y, \left\|f\left(\frac{x}{2}\right)-g\left(\frac{x+y}{2}\right)-h\left(\frac{-y}{2}\right)\right\|_Y,\right. \\
&\qquad\quad \left.\left\|-f\left(\frac{x}{2}\right)+g(x)+h\left(\frac{-x}{2}\right)\right\|_Y, \left\|-f\left(\frac{y}{2}\right)+g(y)+h\left(\frac{-y}{2}\right)\right\|_Y\right\} \\
&\quad\le \max\left\{\Phi\left(\frac{x+y}{2},-\frac{x}{2}\right),\Phi\left(\frac{x+y}{2},-\frac{y}{2}\right),\Phi\left(x,-\frac{x}{2}\right),\Phi\left(y,-\frac{y}{2}\right)\right\}
\end{aligned} \tag{3.15}$$

for all $x, y \in X$. Let

$$\Psi_g(x,y) := \max\left\{\Phi\left(\frac{x+y}{2},-\frac{x}{2}\right),\Phi\left(\frac{x+y}{2},-\frac{y}{2}\right),\Phi\left(x,-\frac{x}{2}\right),\Phi\left(y,-\frac{y}{2}\right)\right\} \tag{3.16}$$

for all $x, y \in X$. By (3.1) and (3.3), we have

$$\begin{gathered}
\lim_{n\to\infty}|2|^n\Psi_g\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0, \\
\widetilde{\Phi_2}(x)=\lim_{n\to\infty}\max\left\{|2|^j\Psi_g\left(\frac{x}{2^j},0\right):0\le j<n\right\}
\end{gathered} \tag{3.17}$$

for all $x, y \in X$. By Theorem 2.1, there exists an additive mapping $T_2 : X \to Y$ satisfying (3.8) and

$$T_2(x)=\lim_{n\to\infty}2^n g\left(\frac{x}{2^n}\right) \tag{3.18}$$

for all $x \in X$. Similarly, (3.5) implies that

$$\begin{aligned}
&\left\|2h\left(\frac{x+y}{2}\right)-h(x)-h(y)\right\|_Y \\
&\quad\le \max\left\{\left\|f\left(\frac{y}{2}\right)-g\left(\frac{-x}{2}\right)-h\left(\frac{x+y}{2}\right)\right\|_Y, \left\|f\left(\frac{x}{2}\right)-g\left(\frac{-y}{2}\right)-h\left(\frac{x+y}{2}\right)\right\|_Y,\right. \\
&\qquad\quad \left.\left\|-f\left(\frac{x}{2}\right)+g\left(\frac{-x}{2}\right)+h(x)\right\|_Y, \left\|-f\left(\frac{y}{2}\right)+g\left(-\frac{y}{2}\right)+h(y)\right\|_Y\right\} \\
&\quad\le \max\left\{\Phi\left(-\frac{x}{2},\frac{x+y}{2}\right),\Phi\left(-\frac{y}{2},\frac{x+y}{2}\right),\Phi\left(-\frac{x}{2},x\right),\Phi\left(-\frac{y}{2},y\right)\right\}
\end{aligned} \tag{3.19}$$

for all $x, y \in X$. Let

$$\Psi_h(x,y) := \max\left\{\Phi\left(-\frac{x}{2},\frac{x+y}{2}\right),\Phi\left(-\frac{y}{2},\frac{x+y}{2}\right),\Phi\left(-\frac{x}{2},x\right),\Phi\left(-\frac{y}{2},y\right)\right\} \tag{3.20}$$

for all $x, y \in X$. By (3.1) and (3.4), we have

$$\lim_{n\to\infty} |2|^n \Psi_h\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0,$$
$$\widetilde{\Phi_3}(x) = \lim_{n\to\infty} \max\left\{ |2|^j \Psi_h\left(\frac{x}{2^j}, 0\right) : 0 \le j < n \right\} \tag{3.21}$$

for all $x, y \in X$. By Theorem 2.1, there exists an additive mapping $T_3 : X \to Y$ satisfying (3.9) and

$$T_3(x) = \lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right) \tag{3.22}$$

for all $x \in X$. The uniqueness of T_1, T_2, and T_3 follows from (3.10).

Now, we show that $T_1 = T_2 = T_3$. Replacing x and y by $2^n x$ and 0 in (3.5), respectively, and dividing both sides of (3.5) by $|2|^n$, we get

$$\left\| 2^n f\left(\frac{x}{2^n}\right) - 2^n g\left(\frac{x}{2^n}\right) \right\|_Y \le |2|^n \Phi\left(\frac{x}{2^n}, 0\right) \tag{3.23}$$

for all $x \in X$. By passing the limit $n \to \infty$ in (3.23), we conclude that

$$T_1(x) = T_2(x) \tag{3.24}$$

for all $x \in X$. Similarly, we get $T_1(x) = T_3(x)$ for all $x \in X$. Therefore, (3.6) follows from (3.14), (3.18), and (3.22). This completes the proof. □

The next theorem is an alternative result of Theorem 3.1.

Theorem 3.2. *Let $\Psi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{|2|^n} \Psi(2^n x, 2^n y) = 0 \tag{3.25}$$

for all $x, y \in X$ and the limits

$$\widetilde{\Psi_1}(x) := \lim_{n\to\infty} \max_{0<j\le n} \left\{ \frac{1}{|2|^j} \Psi\left(2^{j-1}x, 2^{j-1}x\right), \frac{1}{|2|^j} \Psi\left(2^{j-1}x, 0\right), \frac{1}{|2|^j} \Psi\left(0, 2^{j-1}x\right) \right\},$$
$$\widetilde{\Psi_2}(x) := \lim_{n\to\infty} \max_{0<j\le n} \left\{ \frac{1}{|2|^j} \Psi\left(2^{j-1}x, -2^{j-1}x\right), \frac{1}{|2|^j} \Psi\left(2^{j-1}x, 0\right), \frac{1}{|2|^j} \Psi\left(2^j x, -2^{j-1}x\right) \right\}, \tag{3.26}$$
$$\widetilde{\Psi_3}(x) := \lim_{n\to\infty} \max_{0<j\le n} \left\{ \frac{1}{|2|^j} \Psi\left(-2^{j-1}x, 2^{j-1}x\right), \frac{1}{|2|^j} \Psi\left(-2^{j-1}x, 2^j x\right), \frac{1}{|2|^j} \Psi\left(0, 2^{j-1}x\right) \right\}$$

exist for all $x \in X$. *Suppose that mappings* $f, g, h : X \to Y$ *with* $f(0) = g(0) = h(0) = 0$ *satisfy the inequality*

$$\|f(x+y) - g(x) - h(y)\|_Y \leq \Psi(x,y) \tag{3.27}$$

for all $x, y \in X$. *Then the limits*

$$T(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{3.28}$$

exist for all $x \in X$ *and* $T : X \to Y$ *is an additive mapping satisfying*

$$\begin{aligned} \|f(x) - T(x)\|_Y &\leq \widetilde{\Psi_1}(x), \\ \|g(x) - T(x)\|_Y &\leq \widetilde{\Psi_2}(x), \\ \|h(x) - T(x)\|_Y &\leq \widetilde{\Psi_3}(x) \end{aligned} \tag{3.29}$$

for all $x \in X$. *Moreover, if*

$$\lim_{k\to\infty} \frac{1}{|2|^k} \widetilde{\Psi_1}\left(2^k x\right) = \lim_{k\to\infty} \frac{1}{|2|^k} \widetilde{\Psi_2}\left(2^k x\right) = \lim_{k\to\infty} \frac{1}{|2|^k} \widetilde{\Psi_3}\left(2^k x\right) = 0 \tag{3.30}$$

for all $x \in X$, *then* T *is a unique additive mapping satisfying the above inequalities.*

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