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Research Article

Iterative Approaches to Find Zeros of Maximal Monotone Operators by Hybrid Approximate Proximal Point Methods

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The purpose of this paper is to introduce and investigate two kinds of iterative algorithms for the problem of finding zeros of maximal monotone operators. Weak and strong convergence theorems are established in a real Hilbert space. As applications, we consider a problem of finding a minimizer of a convex function.

1. Introduction

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. In this paper, we always assume that $T : C \to 2^H$ is a maximal monotone operator. A classical method to solve the following set-valued equation:

$$0 \in Tx \tag{1.1}$$

is the proximal point method. To be more precise, start with any point $x_0 \in H$, and update x_{n+1} iteratively conforming to the following recursion:

$$x_n \in x_{n+1} + \lambda_n T x_{n+1}, \quad \forall n \geq 0, \tag{1.2}$$

where $\{\lambda_n\} \subset [\lambda, \infty)$ $(\lambda > 0)$ is a sequence of real numbers. However, as pointed out in [1], the ideal form of the method is often impractical since, in many cases, to solve the problem (1.2) exactly is either impossible or has the same difficulty as the original problem (1.1). Therefore, one of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximate zeros of T.

In 1976, Rockafellar [2] gave an inexact variant of the method

$$x_0 \in H, \qquad x_n + e_{n+1} \in x_{n+1} + \lambda_n T x_{n+1}, \quad \forall n \geq 0, \tag{1.3}$$

where $\{e_n\}$ is regarded as an error sequence. This is an inexact proximal point method. It was shown that, if

$$\sum_{n=0}^{\infty} \|e_n\| < \infty, \tag{1.4}$$

the sequence $\{x_n\}$ defined by (1.3) converges weakly to a zero of T provided that $T^{-1}(0) \neq \emptyset$. In [3], Güler obtained an example to show that Rockafellar's inexact proximal point method (1.3) does not converge strongly, in general.

Recently, many authors studied the problems of modifying Rockafellar's inexact proximal point method (1.3) in order to strong convergence to be guaranteed. In 2008, Ceng et al. [4] gave new accuracy criteria to modified approximate proximal point algorithms in Hilbert spaces; that is, they established strong and weak convergence theorems for modified approximate proximal point algorithms for finding zeros of maximal monotone operators in Hilbert spaces. In the meantime, Cho et al. [5] proved the following strong convergence result.

Theorem CKZ 1. *Let H be a real Hilbert space, Ω a nonempty closed convex subset of H, and $T : \Omega \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_Ω be the metric projection of H onto Ω. Suppose that, for any given $x_n \in H$, $\lambda_n > 0$, and $e_n \in H$, there exists $\overline{x}_n \in \Omega$ conforming to the following set-valued mapping equation:*

$$x_n + e_n \in \overline{x}_n + \lambda_n T \overline{x}_n, \tag{1.5}$$

where $\{\lambda_n\} \subset (0, +\infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \|e_n\|^2 < \infty. \tag{1.6}$$

Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ such that

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any fixed $u \in \Omega$, define the sequence $\{x_n\}$ iteratively as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_\Omega(\overline{x}_n - e_n), \quad \forall n \geq 0. \tag{1.7}$$

Then $\{x_n\}$ converges strongly to a zero z of T, where $z = \lim_{t \to \infty} J_t u$.

They also derived the following weak convergence theorem.

Theorem CKZ 2. *Let H be a real Hilbert space, Ω a nonempty closed convex subset of H, and $T : \Omega \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_Ω be the metric projection of H onto Ω. Suppose that, for any given $x_n \in H$, $\lambda_n > 0$, and $e_n \in H$, there exists $\overline{x}_n \in \Omega$ conforming to the following set-valued mapping equation:*

$$x_n + e_n \in \overline{x}_n + \lambda_n T\overline{x}_n, \tag{1.8}$$

where $\liminf_{n\to\infty}\lambda_n > 0$ and

$$\sum_{n=0}^{\infty}\|e_n\|^2 < \infty. \tag{1.9}$$

Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ with $\limsup_{n\to\infty}\alpha_n < 1$, and define a sequence $\{x_n\}$ iteratively as follows:

$$x_0 \in \Omega, \qquad x_{n+1} = \alpha_n x_n + \beta_n P_\Omega(\overline{x}_n - e_n), \quad \forall n \geq 0, \tag{1.10}$$

where $\alpha_n + \beta_n = 1$ for all $n \geq 0$. Then the sequence $\{x_n\}$ converges weakly to a zero x^* of T.

Very recently, Qin et al. [6] extended (1.7) and (1.10) to the iterative scheme

$$x_0 \in H, \qquad x_{n+1} = \alpha_n u + \beta_n P_C(\overline{x}_n - e_n) + \gamma_n P_C f_n, \quad \forall n \geq 0, \tag{1.11}$$

and the iterative one

$$x_0 \in C, \qquad x_{n+1} = \alpha_n x_n + \beta_n P_C(\overline{x}_n - e_n) + \gamma_n P_C f_n, \quad \forall n \geq 0, \tag{1.12}$$

respectively, where $\alpha_n+\beta_n+\gamma_n = 1$, $\sup_{n\geq 0}\|f_n\| < \infty$, and $\|e_n\| \leq \eta_n\|x_n-\overline{x}_n\|$ with $\sup_{n\geq 0}\eta_n = \eta < 1$. Under appropriate conditions, they derived one strong convergence theorem for (1.11) and another weak convergence theorem for (1.12). In addition, for other recent research works on approximate proximal point methods and their variants for finding zeros of monotone maximal operators, see, for example, [7–10] and the references therein.

In this paper, motivated by the research work going on in this direction, we continue to consider the problem of finding a zero of the maximal monotone operator T. The iterative algorithms (1.7) and (1.10) are extended to develop the following new iterative ones:

$$x_0 \in H, \qquad x_{n+1} = \alpha_n u + \beta_n P_C\left[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n\right], \quad \forall n \geq 0, \tag{1.13}$$

$$x_0 \in C, \qquad x_{n+1} = \alpha_n x_n + \beta_n P_C\left[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n\right], \quad \forall n \geq 0, \tag{1.14}$$

respectively, where u is any fixed point in C, $\alpha_n + \beta_n = 1$, $\gamma_n + \delta_n \leq 1$, $\sup_{n\geq 0}\|f_n\| < \infty$, and $\|e_n\| \leq \eta_n\|x_n - \overline{x}_n\|$ with $\sup_{n\geq 0}\eta_n = \eta < 1$. Under mild conditions, we establish one strong convergence theorem for (1.13) and another weak convergence theorem for (1.14). The results presented in this paper improve the corresponding results announced by many others. It is easy to see that in the case when $\gamma_n = 1$ and $\delta_n = 0$ for all $n \geq 0$, the iterative algorithms (1.13) and (1.14) reduce to (1.7) and (1.10), respectively. Moreover, the iterative algorithms (1.13) and (1.14) are very different from (1.11) and (1.12), respectively. Indeed, it is clear that the iterative algorithm (1.13) is equivalent to the following:

$$\begin{gathered} x_0 \in H, \\ y_n = (1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n, \\ x_{n+1} = \alpha_n u + \beta_n P_C y_n, \quad \forall n \geq 0. \end{gathered} \tag{1.15}$$

Here, the first iteration step $y_n = (1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n$, is to compute the prediction value of approximate zeros of T; the second iteration step, $x_{n+1} = \alpha_n u + \beta_n P_C y_n$, is to compute the correction value of approximate zeros of T. Similarly, it is obvious that the iterative algorithm (1.14) is equivalent to the following:

$$\begin{gathered} x_0 \in C, \\ y_n = (1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C y_n, \quad \forall n \geq 0. \end{gathered} \tag{1.16}$$

Here, the first iteration step, $y_n = (1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n$, is to compute the prediction value of approximate zeros of T; the second iteration step, $x_{n+1} = \alpha_n x_n + \beta_n P_C y_n$, is to compute the correction value of approximate zeros of T. Therefore, there is no doubt that the iterative algorithms (1.13) and (1.14) are very interesting and quite reasonable.

In this paper, we consider the problem of finding zeros of maximal monotone operators by hybrid proximal point method. To be more precise, we introduce two kinds of iterative schemes, that is, (1.13) and (1.14). Weak and strong convergence theorems are established in a real Hilbert space. As applications, we also consider a problem of finding a minimizer of a convex function.

2. Preliminaries

In this section, we give some preliminaries which will be used in the rest of this paper. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let T be a set-valued mapping. The set $D(T)$ defined by

$$D(T) = \{u \in H : T(u) \neq \emptyset\} \tag{2.1}$$

is called the effective domain of T. The set $R(T)$ defined by

$$R(T) = \bigcup_{u \in H} T(u) \tag{2.2}$$

is called the range of T. The set $G(T)$ defined by

$$G(T) = \{(x,u) \in H \times H : x \in D(T),\ u \in T(x)\} \tag{2.3}$$

is called the graph of T. A mapping T is said to be monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x,u), (y,v) \in G(T). \tag{2.4}$$

T is said to be maximal monotone if its graph is not properly contained in the one of any other monotone operator.

The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. Within the past several decades, many authors have been devoted to the study of the existence and iterative algorithms of zeros for maximal monotone mappings; see [1–5, 7, 11–30]. In order to prove our main results, we need the following lemmas. The first lemma can be obtained from Eckstein [1, Lemma 2] immediately.

Lemma 2.1. *Let C be a nonempty, closed, and convex subset of a Hilbert space H. For any given $x_n \in H$, $\lambda_n > 0$, and $e_n \in H$, there exists $\overline{x}_n \in C$ conforming to the following set-valued mapping equation (SVME):*

$$x_n + e_n \in \overline{x}_n + \lambda_n T \overline{x}_n. \tag{2.5}$$

Furthermore, for any $p \in T^{-1}(0)$, we have

$$\begin{gathered} \langle x_n - \overline{x}_n, x_n - \overline{x}_n + e_n \rangle \leq \langle x_n - p, x_n - \overline{x}_n + e_n \rangle, \\ \|\overline{x}_n - e_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - \overline{x}_n\|^2 + \|e_n\|^2. \end{gathered} \tag{2.6}$$

Lemma 2.2 (see [30, Lemma 2.5, page 243]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0, \tag{2.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the conditions

(i) *$\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, or equivalently $\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$,*

(ii) *$\limsup_{n \to \infty} \beta_n \leq 0$,*

(iii) *$\{\gamma_n\} \subset [0, \infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.*

Then $\lim_{n \to \infty} s_n = 0$.

Lemma 2.3 (see [28, Lemma 1, page 303]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0. \tag{2.8}$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.

Lemma 2.4 (see [11]). *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

Lemma 2.5 (see [31]). *Let E be a uniformly convex Banach space, and and $B_r(0)$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \nu z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \nu \|z\|^2 - \lambda \mu g(\|x - y\|) \tag{2.9}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \nu \in [0,1]$ with $\lambda + \mu + \nu = 1$.

It is clear that the following lemma is valid.

Lemma 2.6. *Let H be a real Hilbert space. Then there holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.10}$$

3. Main Results

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. We always assume that $T : C \to 2^H$ is a maximal monotone operator. Then, for each $t > 0$, the resolvent $J_t = (I + tT)^{-1}$ is a single-valued nonexpansive mapping whose domain is all H. Recall also that the Yosida approximation of T is defined by

$$T_t = \frac{1}{t}(I - J_t). \tag{3.1}$$

Assume that $T^{-1}(0) \neq \emptyset$, where $T^{-1}(0)$ is the set of zeros of T. Then $T^{-1}(0) = \mathrm{Fix}(J_t)$ for all $t > 0$, where $\mathrm{Fix}(J_t)$ is the set of fixed points of the resolvent J_t.

Theorem 3.1. *Let H be a real Hilbert space, C a nonempty, closed, and convex subset of H, and $T : C \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_C be a metric projection from H onto C. For any given $x_n \in H$, $\lambda_n > 0$, and $e_n \in H$, find $\overline{x}_n \in C$ conforming to SVME* (2.5)*, where $\{\lambda_n\} \subset (0,\infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\|e_n\| \leq \eta_n \|x_n - \overline{x}_n\|$ with $\sup_{n\geq 0} \eta_n = \eta < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real sequences in $[0,1]$ satisfying the following control conditions:*

(i) *$\alpha_n + \beta_n = 1$ and $\gamma_n + \delta_n \leq 1$,*

(ii) *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,*

(iii) *$\lim_{n\to\infty} \gamma_n = 1$ and $\sum_{n=0}^{\infty} \delta_n < \infty$.*

Let $\{x_n\}$ be a sequence generated by the following manner:

$$x_0 \in H, \qquad x_{n+1} = \alpha_n u + \beta_n P_C\left[(1 - \gamma_n - \delta_n) x_n + \gamma_n (\overline{x}_n - e_n) + \delta_n f_n\right], \quad \forall n \geq 0, \tag{3.2}$$

where $u \in C$ is a fixed point and $\{f_n\}$ is a bounded sequence in H. Then the sequence $\{x_n\}$ generated by (3.2) *converges strongly to a zero z of T, where $z = \lim_{t\to\infty} J_t u$, if and only if $e_n \to 0$ as $n \to \infty$.*

Proof. First, let us show the necessity. Assume that $x_n \to z$ as $n \to \infty$, where $z \in T^{-1}(0)$. It follows from (2.5) that

$$\begin{aligned} \|\overline{x}_n - z\| &= \|J_{\lambda_n}(x_n + e_n) - J_{\lambda_n}(z)\| \\ &\le \|x_n - z\| + \|e_n\| \\ &\le \|x_n - z\| + \eta_n\|x_n - \overline{x}_n\| \\ &\le (1 + \eta_n)\|x_n - z\| + \eta_n\|\overline{x}_n - z\|, \end{aligned} \tag{3.3}$$

and hence

$$\|\overline{x}_n - z\| \le \frac{1 + \eta_n}{1 - \eta_n}\|x_n - z\| \le \frac{1 + \eta}{1 - \eta}\|x_n - z\|. \tag{3.4}$$

This implies that $\overline{x}_n \to z$ as $n \to \infty$. Note that

$$\|e_n\| \le \eta_n\|x_n - \overline{x}_n\| \le \eta_n(\|x_n - z\| + \|z - \overline{x}_n\|). \tag{3.5}$$

This shows that $e_n \to 0$ as $n \to \infty$.

Next, let us show the sufficiency. The proof is divided into several steps.

Step 1 ($\{x_n\}$ is bounded). Indeed, from the assumptions $\|e_n\| \le \eta_n\|x_n - \overline{x}_n\|$ and $\sup_{n\ge 0}\eta_n = \eta < 1$, it follows that

$$\|e_n\| \le \|x_n - \overline{x}_n\|. \tag{3.6}$$

Take an arbitrary $p \in T^{-1}(0)$. Then it follows from Lemma 2.1 that

$$\|\overline{x}_n - e_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - \overline{x}_n\|^2 + \|e_n\|^2 \le \|x_n - p\|^2, \tag{3.7}$$

and hence

$$\begin{aligned} &\|P_C[(1 - \gamma_n - \delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] - p\|^2 \\ &\quad \le \|(1 - \gamma_n - \delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n - p\|^2 \\ &\quad = \|(1 - \gamma_n - \delta_n)(x_n - p) + \gamma_n(\overline{x}_n - e_n - p) + \delta_n(f_n - p)\|^2 \\ &\quad \le (1 - \gamma_n - \delta_n)\|x_n - p\|^2 + \gamma_n\|\overline{x}_n - e_n - p\|^2 + \delta_n\|f_n - p\|^2 \\ &\quad \le (1 - \gamma_n - \delta_n)\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|f_n - p\|^2 \\ &\quad = (1 - \delta_n)\|x_n - p\|^2 + \delta_n\|f_n - p\|^2. \end{aligned} \tag{3.8}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n u + \beta_n P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] - p\|^2 \\
&\le \alpha_n\|u-p\|^2 + \beta_n\|P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] - p\|^2 \\
&\le \alpha_n\|u-p\|^2 + \beta_n\Big[(1-\delta_n)\|x_n-p\|^2 + \delta_n\|f_n-p\|^2\Big] \\
&= \alpha_n\|u-p\|^2 + \beta_n(1-\delta_n)\|x_n-p\|^2 + \beta_n\delta_n\|f_n-p\|^2 \\
&\le \alpha_n\|u-p\|^2 + \beta_n(1-\delta_n)\|x_n-p\|^2 + \beta_n\delta_n\sup_{n\ge0}\|f_n-p\|^2.
\end{aligned} \tag{3.9}$$

Putting

$$M = \max\left\{\|x_0-p\|^2, \|u-p\|^2, \sup_{n\ge0}\|f_n-p\|^2\right\}, \tag{3.10}$$

we show that $\|x_n-p\|^2 \le M$ for all $n \ge 0$. It is easy to see that the result holds for $n = 0$. Assume that the result holds for some $n \ge 0$. Next, we prove that $\|x_{n+1}-p\|^2 \le M$. As a matter of fact, from (3.9), we see that

$$\|x_{n+1}-p\|^2 \le M. \tag{3.11}$$

This shows that the sequence $\{x_n\}$ is bounded.

Step 2 ($\limsup_{n\to\infty}\langle u-z, x_{n+1}-z\rangle \le 0$, where $z = \lim_{t\to\infty} J_t u$). The existence of $\lim_{t\to\infty} J_t u$ is guaranteed by Lemma 1 of Bruck [12].

Since T is maximal monotone, $T_t u \in TJ_t u$ and $T_{\lambda_n}x_n \in TJ_{\lambda_n}x_n$, we deduce that

$$\begin{aligned}
\langle u - J_t u, J_{\lambda_n}x_n - J_t u\rangle &= -t\langle T_t u, J_t u - J_{\lambda_n}x_n\rangle \\
&= -t\langle T_t u - T_{\lambda_n}x_n, J_t u - J_{\lambda_n}x_n\rangle - t\langle T_{\lambda_n}x_n, J_t u - J_{\lambda_n}x_n\rangle \\
&\le -\frac{t}{\lambda_n}\langle x_n - J_{\lambda_n}x_n, J_t u - J_{\lambda_n}x_n\rangle.
\end{aligned} \tag{3.12}$$

Since $\lambda_n \to \infty$ as $n \to \infty$, for each $t > 0$, we have

$$\limsup_{n\to\infty}\langle u - J_t u, J_{\lambda_n}x_n - J_t u\rangle \le 0. \tag{3.13}$$

On the other hand, by the nonexpansivity of J_{λ_n}, we obtain that

$$\|J_{\lambda_n}(x_n+e_n) - J_{\lambda_n}x_n\| \le \|(x_n+e_n) - x_n\| = \|e_n\|. \tag{3.14}$$

From the assumption $e_n \to 0$ as $n \to \infty$ and (3.13), we get

$$\limsup_{n\to\infty}\langle u - J_t u, J_{\lambda_n}(x_n + e_n) - J_t u\rangle \le 0. \tag{3.15}$$

From (2.5), we see that

$$\begin{aligned}
&\|P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] - J_{\lambda_n}(x_n+e_n)\| \\
&\quad\le \|(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n - J_{\lambda_n}(x_n+e_n)\| \\
&\quad\le (1-\gamma_n-\delta_n)\|x_n - J_{\lambda_n}(x_n+e_n)\| + \gamma_n\|(\overline{x}_n - e_n) - J_{\lambda_n}(x_n+e_n)\| + \delta_n\|f_n - J_{\lambda_n}(x_n+e_n)\| \\
&\quad= (1-\gamma_n-\delta_n)\|x_n - J_{\lambda_n}(x_n+e_n)\| + \gamma_n\|e_n\| + \delta_n\|f_n - J_{\lambda_n}(x_n+e_n)\|.
\end{aligned} \tag{3.16}$$

Since $\lim_{n\to\infty}\gamma_n = 1$ and $\sum_{n=0}^{\infty}\delta_n < \infty$, we conclude from $\|e_n\| \to 0$ and the boundedness of $\{f_n\}$ that

$$\lim_{n\to\infty}\|P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] - J_{\lambda_n}(x_n+e_n)\| = 0. \tag{3.17}$$

Combining (3.15) with (3.17), we have

$$\limsup_{n\to\infty}\langle u - J_t u, P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] - J_t u\rangle \le 0. \tag{3.18}$$

In the meantime, from algorithm (3.2) and assumption $\alpha_n + \beta_n = 1$, it follows that

$$\begin{aligned}
&x_{n+1} - P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] \\
&\quad= \alpha_n\{u - P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n]\}.
\end{aligned} \tag{3.19}$$

Thus, from the condition $\lim_{n\to\infty}\alpha_n = 0$, we have

$$x_{n+1} - P_C[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

This together with (3.18) implies that

$$\limsup_{n\to\infty}\langle u - J_t u, x_{n+1} - J_t u\rangle \le 0, \quad \forall t > 0. \tag{3.21}$$

From $z = \lim_{t\to\infty} J_t u$ and (3.21), we can obtain that

$$\limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle \le 0, \quad \forall t > 0. \tag{3.22}$$

Step 3 ($x_n \to z$ as $n \to \infty$). Indeed, utilizing (3.8), we deduce from algorithm (3.2) that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \left\|(1-\alpha_n)\{P_C[(1-\gamma_n-\delta_n)x_n+\gamma_n(\overline{x}_n-e_n)+\delta_n f_n]-z\}+\alpha_n(u-z)\right\|^2\\ &\le (1-\alpha_n)^2\|P_C[(1-\gamma_n-\delta_n)x_n+\gamma_n(\overline{x}_n-e_n)+\delta_n f_n]-z\|^2\\ &\quad+2\alpha_n\langle u-z,x_{n+1}-z\rangle\\ &\le (1-\alpha_n)\left[(1-\delta_n)\|x_n-z\|^2+\delta_n\|f_n-z\|^2\right]+2\alpha_n\langle u-z,x_{n+1}-z\rangle\\ &\le (1-\alpha_n)\|x_n-z\|^2+\alpha_n\cdot 2\langle u-z,x_{n+1}-z\rangle+\delta_n\|f_n-z\|^2.\end{aligned} \tag{3.23}$$

Note that $\sum_{n=0}^{\infty}\delta_n<\infty$ and $\{f_n\}$ is bounded. Hence it is known that $\sum_{n=0}^{\infty}\delta_n\|f_n-z\|^2<\infty$. Since $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\limsup_{n\to\infty}2\langle u-z,x_{n+1}-z\rangle\le 0$, and $\sum_{n=0}^{\infty}\delta_n\|f_n-z\|^2<\infty$, in terms of Lemma 2.2, we conclude that

$$\|x_n-z\|\longrightarrow 0\quad \text{as } n\longrightarrow\infty. \tag{3.24}$$

This completes the proof. □

Remark 3.2. The maximal monotonicity of T is only used to guarantee the existence of solutions of SVME (2.4), for any given $x_n\in H$, $\lambda_n>0$, and $e_n\in H$. If we assume that $T:C\to 2^H$ is monotone (not necessarily maximal) and satisfies the range condition

$$\overline{D(T)}=C\subset\bigcap_{r>0}R(I+rT), \tag{3.25}$$

we can see that Theorem 3.1 still holds.

Corollary 3.3. *Let H be a real Hilbert space, C a nonempty, closed, and convex subset of H, and $S:C\to C$ a demicontinuous pseudocontraction with a fixed point in C. Let P_C be a metric projection from H onto C. For any $x_n\in C$, $\lambda_n>0$, and $e_n\in H$, find $\overline{x}_n\in C$ such that*

$$x_n+e_n=(1+\lambda_n)\overline{x}_n-\lambda_n S\overline{x}_n, \tag{3.26}$$

where $\{\lambda_n\}\subset(0,\infty)$ with $\lambda_n\to\infty$ as $n\to\infty$ and $\|e_n\|\le\eta_n\|x_n-\overline{x}_n\|$ with $\sup_{n\ge0}\eta_n=\eta<1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real sequences in $[0,1]$ satisfying the following control conditions:

(i) $\alpha_n+\beta_n=1$ *and* $\gamma_n+\delta_n\le 1$,

(ii) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$,

(iii) $\lim_{n\to\infty}\gamma_n=1$ *and* $\sum_{n=0}^{\infty}\delta_n<\infty$.

Let $\{x_n\}$ be a sequence generated by the following manner:

$$x_0\in C,\qquad x_{n+1}=\alpha_n u+\beta_n P_C[(1-\gamma_n-\delta_n)x_n+\gamma_n(\overline{x}_n-e_n)+\delta_n f_n],\quad \forall n\ge 0, \tag{3.27}$$

where $u \in C$ is a fixed point and $\{f_n\}$ is a bounded sequence in H. If the sequence $\{e_n\}$ satisfies the condition $e_n \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ converges strongly to a fixed point z of S, where $z = \lim_{t\to\infty}[I + t(I - S)]^{-1}u$.

Proof. Let $T = I - S$. Then $T : C \to H$ is demicontinuous, monotone, and satisfies the range condition:

$$\overline{D(T)} = C \subset \bigcap_{r>0} R(I + rT). \tag{3.28}$$

For any $y \in C$, define an operator $G : C \to C$ by

$$Gx = \frac{t}{1+t}Sx + \frac{1}{1+t}y. \tag{3.29}$$

Then G is demicontinuous and strongly pseudocontractive. By the study of Lan and Wu [21, Theorem 2.2], we see that G has a unique fixed point $x \in C$; that is,

$$y = x + t(I - S)x. \tag{3.30}$$

This implies that $y \in R(I + tT)$ for all $t > 0$. In particular, for any given $x_n \in C$, $\lambda_n > 0$, and $e_n \in H$, there exists $\overline{x}_n \in C$ such that

$$x_n + e_n = \overline{x}_n + \lambda_n T\overline{x}_n, \quad \forall n \geq 0, \tag{3.31}$$

that is,

$$x_n + e_n = (1 + \lambda_n)\overline{x}_n - \lambda_n S\overline{x}_n. \tag{3.32}$$

Finally, from the proof of Theorem 3.1, we can derive the desired conclusion immediately. □

From Theorem 3.1, we also have the following result immediately.

Corollary 3.4. *Let H be a real Hilbert space, C a nonempty, closed, and convex subset of H, and $T : C \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_C be a metric projection from H onto C. For any $x_n \in H$, $\lambda_n > 0$ and $e_n \in H$, find $\overline{x}_n \in C$ conforming to SVME (2.5), where $\{\lambda_n\} \subset (0, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\|e_n\| \leq \eta_n\|x_n - \overline{x}_n\|$ with $\sup_{n\geq 0}\eta_n = \eta < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following control conditions:*

(i) $\alpha_n + \beta_n = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty}\gamma_n = 1$.

Let $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in H, \qquad x_{n+1} = \alpha_n u + \beta_n P_C\left[(1-\gamma_n)x_n + \gamma_n(\overline{x}_n - e_n)\right], \quad \forall n \geq 0, \tag{3.33}$$

where $u \in C$ *is a fixed point. Then the sequence* $\{x_n\}$ *converges strongly to a zero* z *of* T*, where* $z = \lim_{t\to\infty} J_t u$*, if and only if* $e_n \to 0$ *as* $n \to \infty$.

Proof. In Theorem 3.1, put $\delta_n = 0$ for all $n \geq 0$. Then, from Theorem 3.1, we obtain the desired result immediately. □

Next, we give a hybrid Mann-type iterative algorithm and study the weak convergence of the algorithm.

Theorem 3.5. *Let* H *be a real Hilbert space,* C *a nonempty, closed, and convex subset of* H*, and* $T : C \to 2^H$ *a maximal monotone operator with* $T^{-1}(0) \neq \emptyset$*. Let* P_C *be a metric projection from* H *onto* C*. For any given* $x_n \in C$*,* $\lambda_n > 0$*, and* $e_n \in H$*, find* $\overline{x}_n \in C$ *conforming to SVME* (2.5)*, where* $\liminf_{n\to\infty}\lambda_n > 0$ *and* $\|e_n\| \leq \eta_n\|x_n - \overline{x}_n\|$ *with* $\sup_{n\geq 0}\eta_n = \eta < 1$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*, and* $\{\delta_n\}$ *be real sequences in* $[0,1]$ *satisfying the following control conditions:*

(i) $\alpha_n + \beta_n = 1$ *and* $\gamma_n + \delta_n \leq 1$,

(ii) $\liminf_{n\to\infty}\beta_n > 0$,

(iii) $\liminf_{n\to\infty}\gamma_n > 0$ *and* $\sum_{n=0}^{\infty}\delta_n < \infty$.

Let $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in C, \qquad x_{n+1} = \alpha_n x_n + \beta_n P_C\left[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n\right], \quad \forall n \geq 0, \tag{3.34}$$

where $\{f_n\}$ *is a bounded sequence in* H*. Then the sequence* $\{x_n\}$ *generated by* (3.34) *converges weakly to a zero* x^* *of* T.

Proof. Take an arbitrary $p \in T^{-1}(0)$. Utilizing Lemma 2.1, from the assumption $\|e_n\| \leq \eta_n\|x_n - \overline{x}_n\|$ with $\sup_{n\geq 0}\eta_n = \eta < 1$, we conclude that

$$\begin{aligned}
\|(\overline{x}_n - e_n) - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - \overline{x}_n\|^2 + \|e_n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_n - \overline{x}_n\|^2 + \eta_n^2\|x_n - \overline{x}_n\|^2 \\
&\leq \|x_n - p\|^2 - \left(1-\eta^2\right)\|x_n - \overline{x}_n\|^2.
\end{aligned} \tag{3.35}$$

It follows from Lemma 2.5 that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n x_n+\beta_n P_C[(1-\gamma_n-\delta_n)x_n+\gamma_n(\overline{x}_n-e_n)+\delta_n f_n]-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+\beta_n\|P_C[(1-\gamma_n-\delta_n)x_n+\gamma_n(\overline{x}_n-e_n)+\delta_n f_n]-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+\beta_n\|(1-\gamma_n-\delta_n)x_n+\gamma_n(\overline{x}_n-e_n)+\delta_n f_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+\beta_n\Big[(1-\gamma_n-\delta_n)\|x_n-p\|^2+\gamma_n\|(\overline{x}_n-e_n)-p\|^2+\delta_n\|f_n-p\|^2\Big]\\
&\le \alpha_n\|x_n-p\|^2+\beta_n\Big\{(1-\gamma_n-\delta_n)\|x_n-p\|^2+\gamma_n\Big[\|x_n-p\|^2-\Big(1-\eta^2\Big)\|x_n-\overline{x}_n\|^2\Big]\\
&\qquad\qquad +\delta_n\|f_n-p\|^2\Big\}\\
&= \alpha_n\|x_n-p\|^2+\beta_n(1-\delta_n)\|x_n-p\|^2-\beta_n\gamma_n\Big(1-\eta^2\Big)\|x_n-\overline{x}_n\|^2+\beta_n\delta_n\|f_n-p\|^2\\
&\le \|x_n-p\|^2-\beta_n\gamma_n\Big(1-\eta^2\Big)\|x_n-\overline{x}_n\|^2+\delta_n\|f_n-p\|^2\\
&\le \|x_n-p\|^2+\delta_n\|f_n-p\|^2.
\end{aligned} \tag{3.36}$$

Utilizing Lemma 2.3, we know that $\lim_{n\to\infty}\|x_n-p\|$ exists. We, therefore, obtain that the sequence $\{x_n\}$ is bounded. It follows from (3.36) that

$$\beta_n\gamma_n(1-\eta^2)\|x_n-\overline{x}_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\delta_n\|f_n-p\|^2. \tag{3.37}$$

From the conditions $\liminf_{n\to\infty}\beta_n>0$, $\liminf_{n\to\infty}\gamma_n>0$, and $\sum_{n=0}^{\infty}\delta_n<\infty$, we conclude that

$$\lim_{n\to\infty}\|x_n-\overline{x}_n\|=0. \tag{3.38}$$

Note that

$$\begin{aligned}
\|x_n-J_{\lambda_n}x_n\| &= \|x_n-\overline{x}_n+\overline{x}_n-J_{\lambda_n}x_n\|\\
&\le \|x_n-\overline{x}_n\|+\|\overline{x}_n-J_{\lambda_n}x_n\|\\
&\le (1+\eta_n)\|x_n-\overline{x}_n\|.
\end{aligned} \tag{3.39}$$

In view of (3.38), we obtain that

$$\lim_{n\to\infty}\|x_n-J_{\lambda_n}x_n\|=0. \tag{3.40}$$

Also, note that

$$\begin{aligned}\|J_{\lambda_n}x_n - J_1 J_{\lambda_n}x_n\| &= \|T_1 J_{\lambda_n}x_n\| \\ &\le \inf\{\|w\| : w \in T J_{\lambda_n}x_n\} \\ &\le \|T_{\lambda_n}x_n\| \\ &= \frac{\|x_n - J_{\lambda_n}x_n\|}{\lambda_n}.\end{aligned} \tag{3.41}$$

In view of the assumption $\liminf_{n\to\infty}\lambda_n > 0$ and (3.40), we see that

$$\lim_{n\to\infty}\|J_{\lambda_n}x_n - J_1 J_{\lambda_n}x_n\| = 0. \tag{3.42}$$

Let $x^* \in C$ be a weakly subsequential limit of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to x^* as $i \to \infty$. From (3.40), we see that $J_{\lambda_{n_i}}x_{n_i}$ also converges weakly to x^*. Since J_1 is nonexpansive, we can obtain that $x^* \in \mathrm{Fix}(J_1) = T^{-1}(0)$ by Lemma 2.4. Opial's condition (see [23]) guarantees that the sequence $\{x_n\}$ converges weakly to x^*. This completes the proof. □

By the careful analysis of the proof of Corollary 3.3 and Theorem 3.5, it is not hard to derive the following result.

Corollary 3.6. *Let H be a real Hilbert space, C a nonempty, closed, and convex subset of H, and $S : C \to C$ a demicontinuous pseudocontraction with a fixed point in C. Let P_C be a metric projection from H onto C. For any $x_n \in C$, $\lambda_n > 0$, and $e_n \in H$, find $\overline{x}_n \in C$ such that*

$$x_n + e_n = (1+\lambda_n)\overline{x}_n - \lambda_n S\overline{x}_n, \quad \forall n \ge 0, \tag{3.43}$$

where $\liminf_{n\to\infty}\lambda_n > 0$ and $\|e_n\| \le \eta_n\|x_n - \overline{x}_n\|$ with $\sup_{n\ge 0}\eta_n = \eta < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real sequences in $[0,1]$ satisfying the following control conditions:

(i) $\alpha_n + \beta_n = 1$ *and* $\gamma_n + \delta_n \le 1$,

(ii) $\liminf_{n\to\infty}\beta_n > 0$,

(iii) $\liminf_{n\to\infty}\gamma_n > 0$ *and* $\sum_{n=0}^{\infty}\delta_n < \infty$.

Let $\{x_n\}$ be a sequence generated by the following manner:

$$x_0 \in C, \qquad x_{n+1} = \alpha_n x_n + \beta_n P_C\left[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n\right], \quad \forall n \ge 0, \tag{3.44}$$

where $\{f_n\}$ is a bounded sequence in H. Then the sequence $\{x_n\}$ converges weakly to a fixed point x^* of S.

Utilizing Theorem 3.5, we also obtain the following result immediately.

Corollary 3.7. *Let H be a real Hilbert space, C a nonempty, closed, and convex subset of H, and $T : C \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_C be a metric projection from H*

onto C. For any $x_n \in C$, $\lambda_n > 0$, and $e_n \in H$, find $\overline{x}_n \in C$ conforming to SVME (2.5)*, where* $\liminf_{n\to\infty}\lambda_n > 0$ *and* $\|e_n\| \le \eta_n\|x_n - \overline{x}_n\|$ *with* $\sup_{n\ge 0}\eta_n = \eta < 1$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$*, and* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *satisfying the following control conditions:*

(i) $\alpha_n + \beta_n = 1$,

(ii) $\limsup_{n\to\infty}\alpha_n < 1$,

(iii) $\liminf_{n\to\infty}\gamma_n > 0$.

Let $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in C, \qquad x_{n+1} = \alpha_n x_n + \beta_n P_C\left[(1-\gamma_n)x_n + \gamma_n(\overline{x}_n - e_n)\right], \quad \forall n \ge 0. \tag{3.45}$$

Then the sequence $\{x_n\}$ *converges weakly to a zero* x^* *of* T.

4. Applications

In this section, as applications of the main Theorems 3.1 and 3.5, we consider the problem of finding a minimizer of a convex function f.

Let H be a real Hilbert space, and let $f : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \left\{y \in H : f(z) \ge f(x) + \langle z - x, y\rangle,\ z \in H\right\}, \quad \forall x \in H. \tag{4.1}$$

Theorem 4.1. *Let H be a real Hilbert space and $f : H \to (-\infty,+\infty]$ a proper convex lower semi-continuous function such that* $(\partial f)^{-1}(0) \neq \emptyset$*. Let* $\{\lambda_n\}$ *be a sequence in* $(0,+\infty)$ *with* $\lambda_n \to \infty$ *as* $n \to \infty$ *and* $\{e_n\}$ *a sequence in H such that* $\|e_n\| \le \eta_n\|x_n - \overline{x}_n\|$ *with* $\sup_{n\ge 0}\eta_n = \eta < 1$*. Let* $\overline{x}_n$ *be the solution of SVME* (2.5) *with T replaced by ∂f; that is, for any given $x_n \in H$,*

$$x_n + e_n \in \overline{x}_n + \lambda_n \partial f(\overline{x}_n), \quad \forall n \ge 0. \tag{4.2}$$

Let $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*, and* $\{\delta_n\}$ *be real sequences in* $[0,1]$ *satisfying the following control conditions:*

(i) $\alpha_n + \beta_n = 1$ *and* $\gamma_n + \delta_n \le 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(iii) $\lim_{n\to\infty}\gamma_n = 1$ *and* $\sum_{n=0}^{\infty}\delta_n < \infty$.

Let $\{x_n\}$ *be a sequence generated by the following manner:*

$$\begin{gathered} x_0 \in H, \\ \overline{x}_n = \arg\min_{x\in H}\left\{f(x) + \frac{1}{2\lambda_n}\|x - x_n - e_n\|^2\right\}, \\ x_{n+1} = \alpha_n u + \beta_n P_C\left[(1-\gamma_n-\delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n\right], \quad \forall n \ge 0, \end{gathered} \tag{4.3}$$

where $u \in H$ is a fixed point and $\{f_n\}$ *is a bounded sequence in H. If the sequence* $\{e_n\}$ *satisfies the condition* $e_n \to 0$ *as* $n \to \infty$*, then the sequence* $\{x_n\}$ *converges strongly to a minimizer of f nearest to u.*

Proof. Since $f : H \to (-\infty, +\infty]$ is a proper convex lower semi-continuous function, we have that the subdifferential ∂f of f is maximal monotone by the study of Rockafellar [2]. Notice that

$$\overline{x}_n = \arg\min_{x\in H}\left\{ f(x) + \frac{1}{2\lambda_n}\|x - x_n - e_n\|^2 \right\} \tag{4.4}$$

is equivalent to the following:

$$0 \in \partial f(\overline{x}_n) + \frac{1}{\lambda_n}(\overline{x}_n - x_n - e_n). \tag{4.5}$$

It follows that

$$x_n + e_n \in \overline{x}_n + \lambda_n \partial f(\overline{x}_n), \quad \forall n \geq 0. \tag{4.6}$$

By using Theorem 3.1, we can obtain the desired result immediately. □

Theorem 4.2. *Let H be a real Hilbert space and $f : H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function such that $(\partial f)^{-1}(0) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0, +\infty)$ with $\liminf_{n\to\infty}\lambda_n > 0$ and $\{e_n\}$ a sequence in H such that $\|e_n\| \leq \eta_n\|x_n - \overline{x}_n\|$ with $\sup_{n\geq 0}\eta_n = \eta < 1$. Let $\overline{x}_n$ be the solution of SVME (2.5) with T replaced by ∂f; that is, for any given $x_n \in H$,*

$$x_n + e_n \in \overline{x}_n + \lambda_n \partial f(\overline{x}_n), \quad \forall n \geq 0. \tag{4.7}$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real sequences in $[0, 1]$ satisfying the following control conditions:

(i) $\alpha_n + \beta_n = 1$ *and* $\gamma_n + \delta_n \leq 1$,

(ii) $\liminf_{n\to\infty}\beta_n > 0$,

(iii) $\liminf_{n\to\infty}\gamma_n > 0$ *and* $\sum_{n=0}^{\infty}\delta_n < \infty$.

Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{gathered} x_0 \in H, \\ \overline{x}_n = \arg\min_{x\in H}\left\{ f(x) + \frac{1}{2\lambda_n}\|x - x_n - e_n\|^2 \right\}, \\ x_{n+1} = \alpha_n x_n + \beta_n P_C\left[(1 - \gamma_n - \delta_n)x_n + \gamma_n(\overline{x}_n - e_n) + \delta_n f_n\right], \quad \forall n \geq 0, \end{gathered} \tag{4.8}$$

where $\{f_n\}$ is a bounded sequence in H. Then the sequence $\{x_n\}$ converges weakly to a minimizer of f.

Proof. We can obtain the desired result readily from the proof of Theorems 3.5 and 4.1. □

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