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Research Article

Relation between Fixed Point and Asymptotical Center of Nonexpansive Maps

Mohammad Reza Haddadi, Hamid Mazaheri, and Mohammad Hussein Labbaf Ghasemi

Department of Mathematics, Faculty of Mathematics, Yazd University, P. O. Box 89195-741, Yazd, Iran

Correspondence should be addressed to Mohammad Reza Haddadi, haddadi83@math.iut.ac.ir

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We introduce the concept of asymptotic center of maps and consider relation between asymptotic center and fixed point of nonexpansive maps in a Banach space.

1. Introduction

Many topics and techniques regarding asymptotic centers and asymptotic radius were studied by Edelstein [1], Bose and Laskar [2], Downing and Kirk [3], Goebel and Kirk [4], and Lan and Webb [5]. Now, We recall that definitions of asymptotic center and asymptotic radius.

Let C be a nonempty subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Consider the functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|, \quad x \in X. \tag{1.1}$$

The infimum of $r_a(\cdot, \{x_n\})$ over C is said to be the asymptotic radius of $\{x_n\}$ with respect to C and is denoted by $r_a(C, \{x_n\})$. A point $z \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to C if

$$r_a(z, \{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in C\}. \tag{1.2}$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $Z_a(C, \{x_n\})$.

We present new definitions of asymptotic center and asymptotic radius that is for a mapping and obtain new results.

Definition 1.1. Let C be a bounded closed convex subset of X. A sequence $\{x_n\} \subseteq X$ is said to be an asymptotic center for a mapping $T : C \to X$ if, for each $x \in C$,

$$\limsup_{n\to\infty} \|Tx - x_n\| \leq \limsup_{n\to\infty} \|x_n - x\|. \tag{1.3}$$

Definition 1.2. Let C be a nonempty subset of X. We say that C has the fixed-point property for continuous mappings of C with asymptotic center if every continuous mapping $T : C \to C$ admitting an asymptotic center has a fixed point.

Definition 1.3. Let C be a nonempty subset of X. We say that C has Property (Z) if for every bounded sequence $\{x_n\} \subset X \setminus C$, the set $Z_a(C, \{x_n\})$ is a nonempty and compact subset of C.

Example 1.4. Let X be a normed space and C a nonempty subset of X. It is clear that

(i) if C is a compact set, then $Z_a(C, \{x_n\})$ in nonempty compact set and so has Property (Z);

(ii) if C is a open set, since $Z_a(C, \{x_n\}) \subset \partial C$, therefore $Z_a(C, \{x_n\})$ is empty and so fail to have Property (Z).

2. Main Results

Our new results are presented in this section.

Proposition 2.1. *Let X be a Banach space and let C be a nonempty closed bounded and convex subset of X. If C satisfies Property (Z), then every continuous mapping $T : C \to C$ asymptotically admitting a center in C has a fixed point.*

Proof. Assume that $T : C \to C$ is a continuous mapping and $\{x_n\}$ is a asymptotic center. Let $\{x_n\} \subset X \setminus C$ has set of asymptotic center $Z_a(C, \{x_n\})$. Since C has Property (Z), $Z_a(C, \{x_n\})$ is nonempty and compact and it is easy to see that it is also convex. In order to obtain the result, it will be enough to show that $Z_a(C, \{x_n\})$ is T-invariant since in this case we may apply Schauder's Fixed-Point Theorem [4, Theorem 18.10]. Indeed, let $y \in Z_a(C, \{x_n\})$. Since $\{x_n\}$ is a asymptotic center for T, we have

$$r_a(C, \{x_n\}) \leq \limsup_{n\to\infty} \|Ty - x_n\| \leq \limsup_{n\to\infty} \|x_n - y\| = r_a(C, \{x_n\}). \tag{2.1}$$

Therefore $Ty \in Z_a(C, \{x_n\})$. □

Theorem 2.2. *Let X be a Banach space and let C be a nonempty closed bounded and convex subset of X. If C has the fixed-point property for continuous mappings admitting an asymptotic center, then C has Property (Z).*

Proof. Suppose that C fails to have Property (Z). There exists $\{x_n\} \subset X$ such that either $Z_a(C, \{x_n\}) = \emptyset$ or $Z_a(C, \{x_n\})$ is noncompact. In the second case, by Klee's theorem in

[6] there exists a continuous function $S : Z_a(C,\{x_n\}) \to Z_a(C,\{x_n\})$ without fixed points $(Sx = x)$. Since a closed convex subset of a normed space is a retract of the space, there exists a continuous mapping $r : C \to Z_a(C,\{x_n\})$ such that $r(x) = x$ for all $x \in Z_a(C,\{x_n\})$. Define $T : C \to Z_a(C,\{x_n\})$ by $T(x) = S(r(x))$. Clearly T is a continuous mapping. Moreover,

$$\begin{aligned}\limsup_{n\to\infty}\|T(x) - x_n\| &= \limsup_{n\to\infty}\|x_n - S(r(x))\| \\ &= \limsup_{n\to\infty}\|x_n - r(x)\| \\ &\le \limsup_{n\to\infty}\|x_n - x\|,\end{aligned} \tag{2.2}$$

that is, $\{x_n\}$ is an asymptotic center for T. Therefore, by Proposition 2.1, T has a fixed point in C, $T(x) = x \in Z_a(C,\{x_n\})$. Hence $x = S(r(x)) = S(x)$ sets a contradiction.

Concerning the first case we proceed as follows.

Let $d := r_a(C,\{x_n\}) > \circ$. We take $a > 0$ such that $a + d < \sup\{\|x - x_n\| : x \in C\}$. For each positive integer n, we consider the following nonempty sets:

$$B_m := B\left[\{x_n\}, d + \frac{a}{m}\right] \cap C, \tag{2.3}$$

where $B[\{x_n\}, r] := \{x \in X : \limsup_{n\to\infty}\|x_n - x\| < r\}$

$$\begin{aligned}A_m &:= B_m \setminus B_{m+1}, \\ S_m &:= \left\{x \in C : \limsup_{n\to\infty}\|x - x_n\| = d + \frac{a}{m}\right\}.\end{aligned} \tag{2.4}$$

Since $Z_a(C,\{x_n\}) = \emptyset$, we have that

$$B_1 = \bigcup_{m=1}^{\infty} A_m. \tag{2.5}$$

Fix an arbitrary $x_1 \in S_1$ and define, by induction, a sequence $\{y_m\}$ such that $\{y_m\} \in S_m$ and the segment $(y_{m+1}, y_m]$ does not meet B_{m+1}. Given $x \in B_1$, there exists a unique positive integer n such that $x \in A_n$. In this case we define

$$\begin{aligned}S(x) = &\frac{\limsup_{n\to\infty}\|x - x_n\| - (d + a/(m+1))}{a/m(m+1)} y_{m+1} \\ &+ \left(1 - \frac{\limsup_{n\to\infty}\|x - x_n\| - (d + a/(m+1))}{a/m(m+1)}\right) y_{m+2}.\end{aligned} \tag{2.6}$$

It is a routine to check that S is a continuous mapping from B_1 to B_1. Furthermore, $S(A_m) \subset (y_{m+2}, y_{m+1}] \subset A_{m+1}$ for every $m \ge 1$.

Let r be a continuous retraction from C into the closed convex subset B_1. We can define $T : C \to C$ by $T(x) = S(r(x))$. It is clear that $\{x_n\}$ is a asymptotic center for T and that T is fixed-point free. □

Proposition 2.1 (Theorem 2.2) is a generalizations of Theorem 3.1 (Theorem 3.3) in [1]. It can be verified that definition of $L(\tau)$ space is not necessary here.

As an easy consequence of both Proposition 2.1 and Theorem 2.2, we deduce the following result.

Corollary 2.3. *Let C be a nonempty closed bounded and convex subset of a Banach space X. The following conditions are equivalent.*

(1) *C has the fixed-point property for continuous mappings admitting asymptotic center in C.*

(2) *C has Property (Z).*

Let C be a nonempty closed convex bounded subset of a Banach space X. By $KC(C)$ we denote the family of all nonempty compact convex subsets of C. On $KC(C)$ we consider the well-known Hausdorff metric H. Recall that a mapping $T : C \to KC(C)$ is said to be nonexpansive whenever

$$H(Tx, Ty) \le d(x, y), \quad x, y \in C. \tag{2.7}$$

Theorem 2.4. *Let X be a Banach space and let C be a nonempty closed convex and bounded subset of X satisfying Property (Z). If $T : C \to KC(C)$ is a nonexpansive mapping, then T has a fixed point.*

Proof. Let $T : C \to KC(C)$ be a nonexpansive mapping. The multivalued analog of Banach's Contraction Principle allows us to find a sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$.

For each $n \ge 1$, the compactness of Tx_n guarantees that there exists $y_n \in Tx_n$ satisfying $\|x_n - y_n\| = d(x_n, Tx_n)$.

Now we are going to show that for every $z \in Z_a(C, \{x_n\})$,

$$Z_a(C, \{x_n\}) \cap Tz \ne \emptyset. \tag{2.8}$$

Taking any $z \in Z_a(C, \{x_n\})$, from the compactness of Tz we can find $z_n \in Tz$ such that

$$\|y_n - z_n\| = d(y_n, Tz) \le H(Tx_n, Tz) \le \|x_n - z\|. \tag{2.9}$$

By compactness again we can assume that $\{z_n\}$ converges to a point $w_0 \in Tz$. From above it follows that

$$\limsup_{n\to\infty}\|x_n - w_0\| \le \limsup_{n\to\infty}\|y_n - w_0\| \le \limsup_{n\to\infty}\|y_n - z_n\| \le \limsup_{n\to\infty}\|x_n - z\|. \tag{2.10}$$

Therefore $w_0 \in Z_a(C, \{x_n\})$.

Now we define the mapping $S : Z_a(C, \{x_n\}) \to KC(Z_a(C, \{x_n\}))$ by $S(z) = Z_a(C, \{x_n\}) \cap T(z)$. Since the mapping S is upper semicontinuous and $S(z)$ for every $z \in Z_a(C, \{x_n\})$ is a compact convex set we can apply the Kakutani-Bohnenblust-Karlin Theorem in [5] to obtain a fixed point for $S(z)$ and hence for T. □

Let X be a metric space and $T : X \to X$ a mapping. Then a sequence $\{x_n\}$ in X is said to be an approximating fixed-point sequence of T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let C be a bounded closed and convex subset of a Banach space X, $T : C \to C$ a nonexpansive mapping and $\alpha \in (0,1)$. Then a mappings $T_\alpha : C \to C$ define by $T_\alpha(x) = \alpha x + (1-\alpha)Tx$ is always asymptotically regular, that is, for every $x \in C$, $\lim_{n\to\infty}\|T_\alpha^{n+1}x - T_\alpha^n x\| = 0$.

Proposition 2.5. *Let X be a Banach space and C a closed bounded convex subset of X, $x_0 \in C$ and $\alpha \in (0,1)$. If $T : C \to C$ is a nonexpansive mapping, then the sequence $\{T_\alpha^n x_0\}$ is an asymptotic center for T.*

Proof. The above comments guarantee that $\{T_\alpha^n x_0\}$ is an approximated fixed-point sequence for T_α^n. Let us see that the sequence $\{T_\alpha^n x_0\}$ an asymptotic center for T. Given $x \in C$ we have

$$\begin{aligned}\limsup_{n\to\infty}\|Tx - T_\alpha^n x_0\| &\le \limsup_{n\to\infty}\|Tx - T(T_\alpha^n x_0)\| + \limsup_{n\to\infty}\|T(T_\alpha^n x_0) - T_\alpha^n x_0\| \\ &= \limsup_{n\to\infty}\|Tx - T(T_\alpha^n x_0)\| \\ &\le \limsup_{n\to\infty}\|x - T_\alpha^n x_0\|.\end{aligned} \tag{2.11}$$

Therefore $\{T_\alpha^n x_0\}$ is asymptotic center for T. □

Theorem 2.6. *Let X be a normed space, $T : X \to X$ a nonexpansive mapping with an approximating fixed point sequence $\{x_n\} \subseteq X$ and C be a nonempty subset of X such that $Z_a(C, \{x_n\})$ is a nonempty star-shaped subset of X. Then T has an approximating fixed-point sequence in $Z_a(C, \{x_n\})$.*

Proof. Suppose $y \in Z_a(C, \{x_n\})$. Therefore

$$\begin{aligned}\limsup_{n\to\infty}\|Ty - x_n\| &\le \limsup_{n\to\infty}\|Ty - Tx_n\| + \limsup_{n\to\infty}\|Tx_n - x_n\| \\ &= \limsup_{n\to\infty}\|Ty - Tx_n\| \\ &\le \limsup_{n\to\infty}\|y - x_n\| = r_a(C, \{x_n\}),\end{aligned} \tag{2.12}$$

and so $Ty \in Z_a(C, \{x_n\})$.

Now, let p be the star center of $Z_a(C, \{x_n\})$. For every $n \in \mathbb{N}$ define $T_n : Z_a(C, \{x_n\}) \to Z_a(C, \{x_n\})$ by

$$T_n(x) = \left(1 - \frac{1}{n}\right)Tx + \frac{1}{n}p. \tag{2.13}$$

For every $n \in \mathbb{N}$, T_n is a contraction, so there exists exactly one fixed point y_n of T_n. Now

$$\|y_n - Ty_n\| = \left(1 - \frac{1}{n}\right)\|Ty_n - p\| = \left(1 - \frac{1}{n}\right)k \longrightarrow 0. \tag{2.14}$$

Therefore $\{y_n\}$ is the approximating fixed-point sequence in $Z_a(C, \{x_n\})$ of T. □

Corollary 2.7. *Let X be a normed space, $T : X \to X$ a nonexpansive mapping with an approximating fixed-point sequence $\{x_n\} \subseteq X$ and C be a nonempty subset of X such that $Z_a(C, \{x_n\}) \neq \emptyset$. Suppose $Z_a(C, \{x_n\})$ is a nonempty weakly compact star-shaped subset of K. If $I - T$ is demiclosed, then T has a fixed point in $Z_a(C, \{x_n\})$.*

Proof. By the last theorem, T has an approximating fixed-point sequence $\{y_n\} \in Z_a(C, \{x_n\})$. Because $Z_a(C, \{x_n\})$ is weakly compact, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \to z \in Z_a(C, \{x_n\})$. Since $I - T$ is demiclosed on $Z_a(C, \{x_n\})$ and $y_{n_i} - Ty_{n_i} \to 0$, it follows that $z \in F(T)$. Therefore, $Z_a(C, \{x_n\}) \cap F(T) \neq \emptyset$. □

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