Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 927640, 6 pages
doi:10.1155/2010/927640

Research Article

Hyers-Ulam Stability of Nonlinear Integral Equation

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Received 8 April 2010; Revised 9 August 2010; Accepted 13 August 2010

Academic Editor: T. Domínguez Benavides

We will apply the successive approximation method for proving the Hyers-Ulam stability of a nonlinear integral equation.

1. Introduction

We say a functional equation is stable if, for every approximate solution, there exists an exact solution near it. In 1940, Ulam posed the following problem concerning the stability of functional equations [1]: we are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$\rho(f(xy), f(x)f(y)) < \delta, \tag{1.1}$$

for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$? The problem for the case of the approximately additive mappings was solved by Hyers [2] when G and G' are Banach space. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [3–5]). Recently, Y. Li and L. Hua proved the stability of Banach's fixed point theorem [6]. The interested reader can also find further details in the book of Kuczma (see [7, chapter XVII]). Examples of some recent developments, discussions, and critiques of that idea of stability can be found, for example, in [8–12].

In this paper, we study the Hyers-Ulam stability for the nonlinear Volterra integral equation of second kind. Jung was the author who investigated the Hyers-Ulam stability of Volterra integral equation on any compact interval. In 2007, he proved the following [13].

Given $a \in \mathbb{R}$ and $r > 0$, let $I(a;r)$ denote a closed interval $\{x \in \mathbb{R} \mid a - r \leq x \leq a + r\}$ and let $f : I(a;r) \times \mathbb{C} \to \mathbb{C}$ be a continuous function which satisfies a Lipschitz condition $|f(x,y) - f(x,z)| \leq L|y - z|$ for all $x \in I(a;r)$ and $y,z \in \mathbb{C}$, where L is a constant with $0 < Lr < 1$. If a continuous function $y : I(a;r) \to \mathbb{C}$ satisfies

$$\left|y(x) - b - \int_a^x f(x,t,u(t))dt\right| \leq \epsilon, \tag{1.2}$$

for all $x \in I(a;r)$ and for some $\epsilon \geq 0$, where b is a complex number, then there exists a unique continuous function $u : I(a;r) \to \mathbb{C}$ such that

$$y(x) = b + \int_a^x f(x,t,u(t))dt, \qquad |u(x) - y(x)| \leq \frac{\epsilon}{1 - Lr}, \tag{1.3}$$

for all $x \in I(a;r)$.

The purpose of this paper is to discuss the Hyers-Ulam stability of the following nonhomogeneous nonlinear Volterra integral equation:

$$u(x) = f(x) + \varphi\left(\int_a^x F(x,t,u(t))dt\right) \equiv Tu, \tag{1.4}$$

where $x \in I = [a,b], -\infty < a < b < \infty$. We will use the successive approximation method, to prove that (1.4) has the Hyers-Ulam stability under some appropriate conditions. The method of this paper is distinctive. This new technique is simpler and clearer than methods which are used in some papers, (cf. [13, 14]). On the other hand, Hyers-Ulam stability constant obtained in our paper is different to the other works, [13].

2. Basic Concepts

Consider the nonhomogeneous nonlinear Volterra integral equation (1.4). We assume that $f(x)$ is continuous on the interval $[a,b]$ and $F(x,t,u(t))$ is continuous with respect to the three variables x, t, and u on the domain $D = \{(x,t,u) : x \in [a,b], t \in [a,b], u(t) \in [c,d]\}$; and $F(x,t,u(t))$ is Lipschitz with respect to u. In this paper, we consider the complete metric space $(X := C[a,b], \|\cdot\|_\infty)$ and assume that φ is a bounded linear transformation on X.

Note that, the linear mapping $\varphi : X \to X$ is called bounded, if there exists $M > 0$ such that $\|\varphi x\| \leq M\|x\|$, for all $x \in X$. In this case, we define $\|\varphi\| = \sup\{(\|\varphi x\|/\|x\|); x \neq 0, x \in X\}$. Thus φ is bounded if and only if $\|\varphi\| < \infty$, [15].

Definition 2.1 (cf. [5, 13]). One says that (1.4) has the Hyers-Ulam stability if there exists a constant $K \geq 0$ with the following property: for every $\epsilon > 0$, $y \in X$, if

$$\left|y(x) - f(x) - \varphi\left(\int_a^x F(x,t,y(t))dt\right)\right| \leq \epsilon, \tag{2.1}$$

then there exists some $u \in X$ satisfying $u(x) = f(x) + \varphi(\int_a^x F(x,t,u(t))dt)$ such that

$$\left|u(x) - y(x)\right| \leq K\epsilon. \tag{2.2}$$

We call such K a Hyers-Ulam stability constant for (1.4).

3. Existence of the Solution of Nonlinear Integral Equations

Consider the iterative scheme

$$u_{n+1}(x) = f(x) + \varphi\left(\int_a^x F(x,t,u_n(t))dt\right) \equiv Tu_n, \quad n = 1,2,\ldots. \tag{3.1}$$

Since $F(x,t,u(t))$ is assumed Lipschitz, we can write

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &= \left|\varphi\left(\int_a^x F(x,t,u_n(t))dt\right) - \varphi\left(\int_a^x F(x,t,u_{n-1}(t))dt\right)\right| \\ &= \left|\varphi\left(\int_a^x F(x,t,u_n(t))dt - \int_a^x F(x,t,u_{n-1}(t))dt\right)\right| \\ &\leq \|\varphi\| \int_a^x |F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))|dt \\ &\leq \|\varphi\| L \int_a^x |u_n(t) - u_{n-1}(t)|dt. \end{aligned} \tag{3.2}$$

Hence,

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &\leq \|\varphi\| L \int_a^x |u_n(t_1) - u_{n-1}(t_1)|dt_1 \\ &\leq \left(\|\varphi\| L\right)^2 \int_a^x \int_a^{t_1} |u_{n-1}(t_2) - u_{n-2}(t_2)|dt_2\, dt_1 \\ &\vdots \\ &\leq \left(\|\varphi\| L\right)^{n-1} \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-2}} |u_2(t_{n-1}) - u_1(t_{n-1})|dt_{n-1} \cdots dt_2\, dt_1 \\ &\leq \left(\|\varphi\| L\right)^{n-1} d(Tu_1, u_1) \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-2}} dt_{n-1} \cdots dt_2\, dt_1, \end{aligned} \tag{3.3}$$

in which $d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$, for all $f, g \in C[a,b]$. So, we can write

$$|u_{n+1}(x) - u_n(x)| \leq \left(\|\varphi\| L\right)^{n-1} \frac{(x-a)^{n-1}}{(n-1)!} d(Tu_1, u_1). \tag{3.4}$$

Therefore, since x is complete metric space, if $u_1 \in X$, then

$$\sum_{n=1}^{\infty}[u_{n+1}(x) - u_n(x)] \tag{3.5}$$

is absolutely and uniformly convergent by Weirstrass's M-test theorem. On the other hand, $u_n(x)$ can be written as follows:

$$u_n(x) = u_1(x) + \sum_{k=1}^{n-1}[u_{k+1}(x) - u_k(x)]. \tag{3.6}$$

So there exists a unique solution $u \in X$ such that $\lim_{n\to\infty} u_n(x) = u$. Now by taking the limit of both sides of (3.1), we have

$$\begin{aligned} u = \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty}\left(f(x) + \varphi\left(\int_a^x F(x,t,u_n(t))dt\right)\right) \\ &= f(x) + \varphi\left(\int_a^x F\left(x,t,\lim_{n\to\infty} u_n(t)\right)dt\right) \\ &= f(x) + \varphi\left(\int_a^x F(x,t,u(t))dt\right). \end{aligned} \tag{3.7}$$

So, there exists a unique solution $u \in X$ such that $Tu = u$.

4. Main Results

In this section, we prove that the nonlinear integral equation in (1.4) has the Hyers-Ulam stability.

Theorem 4.1. *The equation $Tx = x$, where T is defined by* (1.4)*, has the Hyers-Ulam stability; that is, for every $\xi \in X$ and $\epsilon > 0$ with*

$$d(T\xi, \xi) \le \epsilon, \tag{4.1}$$

there exists a unique $u \in X$ such that

$$\begin{gathered} Tu = u, \\ d(\xi, u) \le K\epsilon, \end{gathered} \tag{4.2}$$

for some $K \ge 0$.

Proof. Let $\xi \in X$, $\epsilon > 0$, and $d(T\xi, \xi) \le \epsilon$. In the previous section we have proved that

$$u(t) \equiv \lim_{n\to\infty} T^n \xi(t) \tag{4.3}$$

is an exact solution of the equation $Tx = x$. Clearly there is n with $d(T^n\xi, u) \leq \epsilon$, because $T^n\xi$ is uniformly convergent to u as $n \to \infty$. Thus

$$\begin{aligned}
d(\xi, u) &\leq d(\xi, T^n\xi) + d(T^n\xi, u) \\
&\leq d(\xi, T\xi) + d\left(T\xi, T^2\xi\right) + d\left(T^2\xi, T^3\xi\right) + \cdots + d\left(T^{n-1}\xi, T^n\xi\right) + d(T^n\xi, u) \\
&\leq d(\xi, T\xi) + \frac{k}{1!}d(\xi, T\xi) + \frac{k^2}{2!}d(\xi, T\xi) + \cdots + \frac{k^{n-1}}{(n-1)!}d(\xi, T\xi) + d(T^n\xi, u) \\
&\leq d(\xi, T\xi)\left(1 + \frac{k}{1!} + \frac{k^2}{2!} + \cdots + \frac{k^{n-1}}{(n-1)!}\right) + \epsilon \\
&\leq \epsilon\left(e^k\right) + \epsilon = \left(1 + e^k\right)\epsilon,
\end{aligned} \tag{4.4}$$

where $k = \|\varphi\| L(b-a)$. This completes the proof. □

Corollary 4.2. *For infinite interval, Theorem 4.1 is not true necessarily. For example, the exact solution of the integral equation $u(x) = 1 + \int_a^x u(t)dt \equiv T(u)$, $x \in [0, \infty)$, is $u(x) = e^x$. By choosing $\epsilon = 1$ and $\xi(x) = 0$, $T(\xi) = 1$ is obtained, so $d(T\xi, \xi) \leq \epsilon = 1$, $d(\xi, u) = \infty$. Hence, there exists no Hyers-Ulam stability constant $K \geq 0$ such that the relation $d(\xi, u) \leq K\epsilon$ is true.*

Corollary 4.3. *Theorem 4.1 holds for every finite interval $[a, b]$, $[a, b)$, $(a, b]$, and (a, b), when $-\infty < a < b < \infty$.*

Corollary 4.4. *If one applies the successive approximation method for solving* (1.4) *and $u_i(x) = u_{i+1}(x)$ for some $i = 1, 2, \ldots$, then $u(x) = u_i(x)$, such that $u(x)$ is the exact solution of* (1.4).

Example 4.5. If we put $F(x, t, u(t)) = K(x, t)u(t)$ and $\varphi(x) = \lambda x$ (λ is constant), (1.4) will be a linear Volterra integral equation of second kind in the following form:

$$u(x) = f(x) + \lambda \int_a^x k(x, t)u(t)dt. \tag{4.5}$$

In this example, if $|k(x, t)| < M$ on square $R = \{(x, y) : x \in [a, b], y \in [a, b]\}$, then $F(x, t, u(t)) = K(x, t)u(t)$ satisfies in the Lipschitz condition, where M is the Lipschitz constant. Also $\|\varphi\| = |\lambda|$; therefore, if $|\lambda| < \infty$, the Volterra equation (4.5) has the Hyers-Ulam stability.

5. Conclusions

Let $I = [a, b]$ be a finite interval, and let $X = C[a, b]$ and $y = Ty$ be integral equations in which $T : X \to X$ is a nonlinear integral map. In this paper, we showed that T has the Hyers-Ulam stability; that is, if y° is an approximate solution of the integral equation and $d(y^\circ, Ty^\circ) \leq \varepsilon$ for all $t \in I$ and $\varepsilon \geq 0$, then $d(y^*, y^\circ) \leq K\varepsilon$, in which y^* is the exact solution and K is positive constant.

6. Ideas

In this paper, we proved that (1.4) has the Hyers-Ulam stability. In (1.4), φ is a linear transformation. It is an open problem that raises the following question: "What can we say about the Hyers-Ulam stability of the general nonlinear Volterra integral equation (1.4) when φ is not necessarily linear?"

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