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Research Article

Convergence Theorems of Modified Ishikawa Iterative Scheme for Two Nonexpansive Semigroups

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We prove convergence theorems of modified Ishikawa iterative sequence for two nonexpansive semigroups in Hilbert spaces by the two hybrid methods. Our results improve and extend the corresponding results announced by Saejung (2008) and some others.

1. Introduction

Let C be a subset of real Hilbert spaces H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. $T : C \to C$ is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.1}$$

We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$.

Let $\{T(t) : t \geq 0\}$ be a family of mappings from a subset C of H into itself. We call it a nonexpansive semigroup on C if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$.

The Mann's iterative algorithm was introduced by Mann [1] in 1953. This iterative process is now known as Mann's iterative process, which is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0,1]$.

In 1967, Halpern [2] first introduced the following iterative scheme:

$$\begin{gathered} x_0 = u \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \end{gathered} \tag{1.3}$$

see also Browder [3]. He pointed out that the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$ are necessary in the sence that, if the iteration (1.3) converges to a fixed point of T, then these conditions must be satisfied.

On the other hand, in 2002, Suzuki [4] was the first to introduce the following implicit iteration process in Hilbert spaces:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)(x_n), \quad n \geq 1, \tag{1.4}$$

for the nonexpansive semigroup. In 2005, Xu [5] established a Banach space version of the sequence (1.4) of Suzuki [4].

In 2007, Chen and He [6] studied the viscosity approximation process for a nonexpansive semigroup and prove another strong convergence theorem for a nonexpansive semigroup in Banach spaces, which is defined by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \tag{1.5}$$

where $f : C \to C$ is a fixed contractive mapping.

Recently He and Chen [7] is proved a strong convergence theorem for nonexpansive semigroups in Hilbert spaces by hybrid method in the mathematical programming. Very recently, Saejung [8] proved a convergence theorem by the new iterative method introduced by Takahashi et al. [9] without Bochner integrals for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ with $F := \bigcap_{t=0}^{\infty} F(T(t)) \neq \emptyset$ in Hilbert spaces:

$$\begin{gathered} x_0 \in H \text{ taken arbitrary,} \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T(t_n)x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{gathered} \tag{1.6}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H.

In 1974, Ishikawa [10] introduced a new iterative scheme, which is defined recursively by

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)T x_n, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)T y_n, \end{aligned} \tag{1.7}$$

where the initial guess x_0 is taken in C arbitrarily and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0,1]$.

In this paper, motivated by the iterative sequences (1.6) given by Saejung in [8] and Ishikawa [10], we introduce the modified Ishikawa iterative scheme for two nonexpansive semigroups in Hilbert spaces. Further, we obtain strong convergence theorems by using the hybrid methods. This result extends and improves the result of Saejung [8] and some others.

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section.

It is known that every Hilbert space H satisfies the Opial's condition [11], that is,

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in X,\ y \neq x. \tag{2.1}$$

Recall that the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows. Given $x \in H, P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}. \tag{2.2}$$

$P_C x$ is characterized as follows.

Lemma 2.1. *Let H be a real Hilbert space, C a closed convex subset of H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C. \tag{2.3}$$

Lemma 2.2. *There holds the identity in a Hilbert space H*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.3 (see [12, Lemma 1]). *Let* $\{t_n\}$ *be a real sequence and let* τ *be a real number such that* $\liminf_n t_n \le \tau \le \limsup_n t_n$. *Suppose that either of the following holds:*

(i) $\limsup_n (t_{n+1} - t_n) \le 0$ *or*

(ii) $\liminf_n (t_{n+1} - t_n) \ge 0$,

then τ *is a cluster point of* $\{t_n\}$. *Moreover, for* $\varepsilon > 0$, $k, m \in \mathbb{N}$, *there exists* $m_0 \ge m$ *such that* $|t_j - \tau| < \varepsilon$ *for every integer* j *with* $m_0 \le j \le m_0 + k$.

3. Main Results

3.1. The Shrinking Projection Method

In this section, we prove strong convergence of an iterative sequence generated by the shrinking hybrid projection method in mathematical programming.

Theorem 3.1. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $\{T(t) : t \ge 0\}$ *and* $\{S(t) : t \ge 0\}$ *be nonexpansive semigroups on* C *with a nonempty common fixed point set* F, *that is,* $F := (\bigcap_{t=0}^{\infty} F(T(t))) \cap (\bigcap_{t=0}^{\infty} F(S(t))) \ne \emptyset$. *Let* $\{\alpha_n\} \subset [0, a] \subset [0, 1)$, $\{\beta_n\} \subset [b, c] \subset (0, 1)$ *and* $\{t_n\}$ *be the sequences such that* $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, *and* $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. *Suppose that* $\{x_n\}$ *is a sequence generated by the following iterative scheme:*

$$\begin{gathered}
x_0 \in H \text{ taken arbitrary}, \\
C_1 = C, \\
x_1 = P_{C_1}(x_0), \\
z_n = \beta_n x_n + (1 - \beta_n) T(t_n) x_n, \\
y_n = \alpha_n x_n + (1 - \alpha_n) S(t_n) z_n, \\
C_{n+1} = \{u \in C_n : \|y_n - u\| \le \|x_n - u\|\}, \\
x_{n+1} = P_{C_{n+1}}(x_0),
\end{gathered} \tag{3.1}$$

then $\{x_n\}$ *converges strongly to* $P_F(x_0)$.

Proof. We first show that C_{n+1} is closed and convex for each $n \ge 0$. From the definition of C_{n+1} it is obvious that C_{n+1} is closed for each $n \ge 0$. We show that C_{n+1} is convex for any $n \ge 0$. Since

$$\|y_n - u\| \le \|x_n - u\| \Longleftrightarrow 2\langle x_n - y_n, u\rangle \le \|x_n\|^2 - \|y_n\|^2, \tag{3.2}$$

and hence C_{n+1} is convex. Next we show that $F \subset C_{n+1}$ for all $n \geq 0$. Let $p \in F$, then we have

$$\begin{aligned} \|z_n - p\| &= \|\beta_n x_n + (1-\beta_n)T(t_n)x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|T(t_n)x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n x_n + (1-\alpha_n)S(t_n)z_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|S(t_n)z_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|z_n - p\|. \end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4), we have

$$\|y_n - p\| \leq \|x_n - p\|. \tag{3.5}$$

This means that $p \in C_{n+1}$ for all $n \geq 0$. Thus, $\{x_n\}$ is well defined. Since $x_n = P_{C_n}(x_0)$ and $x_{n+1} \in C_{n+1} \subset C_n$, we get

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Consequently,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|, \end{aligned} \tag{3.7}$$

for $n \in \mathbb{N}$. This implies that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \quad \forall n \in \mathbb{N}. \tag{3.8}$$

Therefore, $\{\|x_0 - x_n\|\}$ is nondecreasing. From $x_n = P_{C_n}(x_0)$, we also have $\langle x_0 - x_n, x_n - p\rangle \geq 0$, for all $p \in C_n$.

Since $F \subseteq C_n$, we get

$$\langle x_0 - x_n, x_n - p\rangle \geq 0 \quad \forall p \in F. \tag{3.9}$$

Thus, for $p \in F$, we obtain

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - p\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - p\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\| \|x_0 - p\|. \end{aligned} \tag{3.10}$$

Thus, $\|x_n - x_0\| \leq \|x_0 - p\|$, for all $p \in F$ and $n \in \mathbb{N}$. Then $\lim_{n\to\infty}\|x_n - x_0\|$ exists and $\{x_n\}$ is bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (3.6) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned} \tag{3.11}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, then

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.12}$$

Further, as in the proof of [8, page 3], we have $\{x_n\}$ which is a Cauchy sequence. So, we have $x_n \rightharpoonup z$. By definition of y_n, we have

$$\|y_n - x_n\| = (1-\alpha_n)\|S(t_n)z_n - x_n\|. \tag{3.13}$$

Since $x_{n+1} \in C_{n+1}$ and (3.12), we obtain

$$\begin{aligned} \|S(t_n)z_n - x_n\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\leq \frac{1}{1-\alpha_n}\left(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|\right) \\ &\leq \frac{1}{1-\alpha_n}\left(\|x_n - x_{n+1}\| + \|x_{n+1} - x_n\|\right) \\ &\leq \frac{2}{1-\alpha_n}\|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.14}$$

We now show that $\|T(t_n)x_n - x_n\| \to 0$.

For $p \in F$, we have $\|x_n - p\| \leq \|x_n - S(t_n)z_n\| \leq \|S(t_n)z_n - p\|$. This implies that $0 \leq \|x_n - p\| - \|z_n - p\| \leq \|x_n - S(t_n)z_n\| \to 0$ and hence $\|x_n - p\|^2 - \|z_n - p\|^2 \to 0$. Moreover, since

$$\|z_n - p\|^2 = \beta_n \|x_n - p\|^2 + (1-\beta_n)\|T(t_n)x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T(t_n)x_n\|^2, \tag{3.15}$$

we have

$$\begin{aligned} bc\|x_n - T(t_n)x_n\|^2 &\leq \beta_n(1-\beta_n)\|x_n - T(t_n)x_n\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|T(t_n)x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \longrightarrow 0. \end{aligned} \tag{3.16}$$

And since $S(t_n)$ is a nonexpansive mapping, we obtain

$$\begin{aligned} \|x_n - S(t_n)x_n\| &\leq \|x_n - S(t_n)z_n\| + \|S(t_n)z_n - S(t_n)x_n\|, \\ &\leq \|x_n - S(t_n)z_n\| + \|z_n - x_n\|. \end{aligned} \tag{3.17}$$

Since $\|z_n - x_n\| = (1-\beta_n)\|T(t_n)x_n - x_n\| \to 0$ and $\|x_n - S(t_n)z_n\| \to 0$, we obtain

$$\lim_{n\to\infty} \|x_n - S(t_n)x_n\| = 0. \tag{3.18}$$

As in the proof of [12, Theorem 4], by Lemma 2.3, we can choose a sequence $\{t_{n_k}\}$ of positive real numbers such that

$$t_{n_k} \longrightarrow 0, \quad \frac{1}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.19}$$

In similar way, we also have

$$t_{n_k} \longrightarrow 0, \quad \frac{1}{t_{n_k}}\|x_{n_k} - S(t_{n_k})x_{n_k}\| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.20}$$

Next, we show that $z \in F$. To see this, we fix $t > 0$,

$$\begin{aligned}
\|x_{n_k} - T(t)z\| &\le \sum_{j=0}^{[t/t_{n_k}]-1} \left\|T(jt_{n_k})x_{n_k} - T((j+1)t_{n_k})x_{n_k}\right\| \\
&\quad + \left\|T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)x_{n_k} - T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)z\right\| + \left\|T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)z - T(t)z\right\| \\
&\le \left[\frac{t}{t_{n_k}}\right]\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - z\| + \left\|T\left(t - \left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)z - z\right\| \\
&\le \frac{t}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - z\| + \sup\{\|T(s)z - z\| : 0 \le s \le t_{n_k}\}.
\end{aligned} \tag{3.21}$$

As $x_{n_k} \to z$ and (3.19), we obtain $x_{n_k} \to T(t)z$ and so $T(t)z = z$. Similarly, we have $S(t)z = z$. Thus $z \in F$.

Finally, we show that $z = P_F(x_0)$. Since $F \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}(x_0)$,

$$\|x_{n+1} - x_0\| \le \|q - x_0\| \quad \forall n \in \mathbb{N},\ q \in F. \tag{3.22}$$

But $x_n \to z$ as $n \to \infty$, we have

$$\|z - x_0\| \le \|q - x_0\| \quad \forall q \in F. \tag{3.23}$$

Hence $z = P_F(x_0)$ as required. This completes the proof. □

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \ge 0\}$ be nonexpansive semigroups on C with a nonempty common fixed point set F, that is, $F := \bigcap_{t=0}^{\infty} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,a] \subset [0,1)$, $\{\beta_n\} \subset [b,c] \subset (0,1)$ and $\{t_n\}$ be the sequences such that $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Suppose that $\{x_n\}$ is a sequence iteratively generated by the following iterative scheme:*

$$\begin{gathered}
x_0 \in H \text{ taken arbitrary}, \\
C_1 = C, \\
x_1 = P_{C_1}(x_0), \\
y_n = \alpha_n x_n + (1 - \alpha_n)T(t_n)z_n, \\
z_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n, \\
C_{n+1} = \{u \in C_n : \|y_n - u\| \le \|x_n - u\|\}, \\
x_{n+1} = P_{C_{n+1}}(x_0),
\end{gathered} \tag{3.24}$$

then $\{x_n\}$ converges strongly to $P_F(x_0)$.

Proof. Putting $S(t_n) = T(t_n)$, in Theorem 3.1, we obtain the conclusion immediately. □

Corollary 3.3 (see [8, Theorem 2.1]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroups on C with a nonempty common fixed point set F, that is, $F := \bigcap_{t=0}^{\infty} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,a] \subset [0,1)$ and $\{t_n\}$ be the sequences such that* $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, *and* $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. *Suppose that $\{x_n\}$ is a sequence iteratively generated by the following iterative scheme:*

$$\begin{gathered}
x_0 \in H \text{ taken arbitrary}, \\
C_1 = C, \\
x_1 = P_{C_1}(x_0), \\
z_n = \alpha_n x_n + (1-\alpha_n) T(t_n) x_n, \\
C_{n+1} = \{ u \in C_n : \| y_n - u \| \leq \| x_n - u \| \}, \\
x_{n+1} = P_{C_{n+1}}(x_0),
\end{gathered} \tag{3.25}$$

then $x_n \to P_F(x_0)$.

Proof. If $S(t_n) = T(t_n)$ for all $n \in \mathbb{N}$ and $T(t) = I$ for every $t > 0$ in Theorem 3.1 then (3.1) reduced to (3.25). By using Theorem 3.1, we get the following conclusion. □

3.2. The CQ Hybrid Method

In this section, we consider the modified Ishikawa iterative scheme computing by the CQ hybrid method [13–15]. We use the same idea as Saejung's Theorem 2.2 in [8] and our Theorem 3.1 to obtain the following result and the proof is omitted.

Theorem 3.4. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \geq 0\}$ and $\{S(t) : t \geq 0\}$ be nonexpansive semigroups on C with a nonempty common fixed point set F, that is, $F := (\bigcap_{t=0}^{\infty} F(T(t))) \cap (\bigcap_{t=0}^{\infty} F(S(t))) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,a] \subset [0,1)$, $\{\beta_n\} \subset [b,c] \subset (0,1)$ and $\{t_n\}$ be the sequences such that* $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, *and* $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. *Suppose that $\{x_n\}$ is a sequence generated by the following iterative scheme:*

$$\begin{gathered}
x_0 \in H \text{ taken arbitrary}, \\
y_n = \alpha_n x_n + (1-\alpha_n) S(t_n) z_n, \\
z_n = \beta_n x_n + (1-\beta_n) T(t_n) x_n, \\
C_n = \{ u \in C : \| y_n - u \| \leq \| x_n - u \| \}, \\
Q_n = \{ u \in C : \langle x_n - x_0, u - x_n \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{3.26}$$

then $\{x_n\}$ converges strongly to $P_F(x_0)$.

Proof. First, we show that both C_n and Q_n are closed and convex, and $C_n \cap Q_n \neq \emptyset$ for all $n \in \mathbb{N} \cup \{0\}$. It follows easily from the definition that C_n and Q_n are just intersection of C and the half-spaces see also [9]. As in the proof of the preceding theorem, we have $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly, $F \subset C = Q_0$. Suppose that $F \subset Q_k$ for some $k \in \mathbb{N} \cup \{0\}$, we have $p \in C_k \cap Q_k$. In particular, $\langle x_{k+1} - x_0, p - x_{k+1} \rangle \geq 0$, that is, $p \in Q_{k+1}$. It follows from the induction that $F \subset Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. This proves the claim.

Next, we show that $\|x_n - T(t_n)x_n\| \to 0$, and $\|x_n - S(t_n)x_n\| \to 0$.

We first claim that $\|x_{n+1} - x_n\| \to 0$. Indeed, as $x_{n+1} \in Q_n$ and $x_n = P_{Q_n}(x_0)$,

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n \in \mathbb{N}. \tag{3.27}$$

For fixed $z \in F$. It follows from $F \subset Q_n$ for all $n \in \mathbb{N}$ that

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall n \in \mathbb{N}. \tag{3.28}$$

This implies that sequence $\{x_n\}$ is bounded and

$$\lim_{n \to \infty} \|x_n - x_0\| \text{ exists.} \tag{3.29}$$

Notice that

$$\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0. \tag{3.30}$$

This implies that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle - \|x_0 - x_n\|^2 \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0. \end{aligned} \tag{3.31}$$

By using the same argument of Saejung [8, Theorem 2.2, page 6] and in the proof of Theorem 3.1, we have $\|T(t_n)x_n - x_n\| \to 0$ and $\|S(t_n)x_n - x_n\| \to 0$. And we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup z \in C$, $t_{n_k} \to 0$, $(1/t_{n_k})\|x_{n_k} - T(t_{n_k})x_{n_k}\| \to 0$ and $(1/t_{n_k})\|x_{n_k} - S(t_{n_k})x_{n_k}\| \to 0$ as $k \to \infty$.

From (3.21), we obtain

$$\begin{aligned} \limsup_{k \to \infty} \|x_{n_k} - T(t)z\| &\leq \limsup_{k \to \infty} \|x_{n_k} - z\|, \\ \limsup_{k \to \infty} \|x_{n_k} - S(t)z\| &\leq \limsup_{k \to \infty} \|x_{n_k} - z\|. \end{aligned} \tag{3.32}$$

By the Opial's condition of H, we have $z = T(t)z$ and $z = S(t)z$ for all $t > 0$, that is, $z \in F$.

We note that

$$\|x_0 - P_F(x_0)\| \leq \|x_0 - z\| \leq \liminf_{k \to \infty} \|x_0 - x_{n_k}\| \leq \limsup_{k \to \infty} \|x_0 - x_{n_k}\| \leq \|x_0 - P_F(x_0)\|. \tag{3.33}$$

This implies that

$$\lim_{k \to \infty} \|x_0 - x_{n_k}\| = \|x_0 - P_F(x_0)\| = \|x_0 - z\|. \tag{3.34}$$

Therefore,

$$x_{n_k} \longrightarrow P_F(x_0) = z, \quad \text{as } k \longrightarrow \infty. \tag{3.35}$$

Hence the whole sequence must converge to $P_F(x_0) = z$, as required. This completes the proof. □

Corollary 3.5. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \geq 0\}$ be nonexpansive semigroups on C with a nonempty common fixed point set F, that is, $F := \bigcap_{t=0}^{\infty} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,a] \subset [0,1)$, $\{\beta_n\} \subset [b,c] \subset (0,1)$ and $\{t_n\}$ be the sequences such that $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Suppose that $\{x_n\}$ is a sequence iteratively generated by the following iterative scheme:*

$$\begin{aligned}
&x_0 \in H \text{ taken arbitrary},\\
&y_n = \alpha_n x_n + (1-\alpha_n) T(t_n) z_n,\\
&z_n = \beta_n x_n + (1-\beta_n) T(t_n) x_n,\\
&C_n = \{u \in C : \|y_n - u\| \leq \|x_n - u\|\},\\
&Q_n = \{u \in C : \langle x_n - x_0, u - x_n\rangle \geq 0\},\\
&x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{aligned} \tag{3.36}$$

then $\{x_n\}$ converges strongly to $P_F(x_0)$.

Proof. If $S(t_n) = T(t_n)$ for all $n \in \mathbb{N} \cup \{0\}$, in Theorem 3.4 then (3.26) reduced to (3.36). So, we obtain the result immediately. □

We also deduce the following corollary.

Corollary 3.6 (see [8, Theorem 2.2]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroups on C with a nonempty common fixed point set F, that is, $F := \bigcap_{t=0}^{\infty} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,a] \subset [0,1)$ and $\{t_n\}$ be the sequences such that $\liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Suppose that $\{x_n\}$ is a sequence iteratively generated by the following iterative scheme:*

$$\begin{aligned}
&x_0 \in H \text{ taken arbitrary},\\
&z_n = \alpha_n x_n + (1-\alpha_n) T(t_n) x_n,\\
&C_n = \{u \in C : \|y_n - u\| \leq \|x_n - u\|\},\\
&Q_n = \{u \in C : \langle x_n - x_0, u - x_n\rangle \geq 0\},\\
&x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{aligned} \tag{3.37}$$

then $x_n \to P_F(x_0)$.

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