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Research Article

Dynamic Traffic Network Equilibrium System

Yun-Peng He,[1] Jiu-Ping Xu,[2] Nan-Jing Huang,[1,2] and Meng Wu[2,3]

[1] *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*
[2] *College of Business and Administration, Sichuan University, Chengdu, Sichuan 610064, China*
[3] *College of General Studies, Konkuk University, Seoul 143-701, South Korea*

Correspondence should be addressed to Meng Wu, shancherish@hotmail.com

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We discuss the dynamic traffic network equilibrium system problem. We introduce the equilibrium definition based on Wardrop's principles when there are some internal relationships between different kinds of goods which transported through the same traffic network. Moreover, we also prove that the equilibrium conditions of this problem can be equivalently expressed as a system of evolutionary variational inequalities. By using the fixed point theory and projected dynamic system theory, we get the existence and uniqueness of the solution for this equilibrium problem. Finally, a numerical example is given to illustrate our results.

1. Introduction

The problem of users of a congested transportation network seeking to determine their travel paths of minimal cost from origins to their respective destinations is a classical network equilibrium problem. The first author who studied the transportation networks was Pigou [1] in 1920, who considered a two-node, two-link transportation network, and it was further developed by Knight [2]. But it was only during most recent decades that traffic network equilibrium problems have attracted the attention of several researchers. In 1952, Wardrop [3] laid the foundations for the study of the traffic theory. He proposed two principles until now named after him. Wardrop's principles were stated as follows.

(i) *First Principle*. The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

(ii) *Second Principle*. The average journey time is minimal.

The rigorous mathematical formulation of Wardrop's principles was elaborated by Beckmann et al. [4] in 1956. They showed the equivalence between the traffic equilibrium stated as Wardrop's principles and the Kuhn-Tucker conditions of a particular optimization problem under some symmetry assumptions. Hence, in this case, the equilibrium flows could be obtained as the solution of a mathematical programming problem. Dafermos and Sparrow [5] coined the terms "user-optimized" and "system-optimized" transportation networks to distinguish between two distinct situations in which users act unilaterally, in their own self-interest, in selecting their routes, and in which users select routes according to what is optimal from a societal point of view, in that the total costs in the system are minimized. In the latter problem, marginal costs rather than average costs are employed.

In 1979, Smith [6] proved that the equilibrium solution could be expressed in terms of variational inequalities. This was a crucial step, because it allowed the application of the powerful tool of variational inequalities to the study of traffic equilibrium problems in the most general framework. From that starting point, many authors, such as Dafermos [7], Giannessi and Maugeri [8, 9], Nagurney [10], and Nagurney and Zhang [11], and so on, paid attention to the study of many features of the traffic equilibrium problem via variational inequality approaches.

Later in 1999, Daniele et al. [12] studied the time-dependent traffic equilibrium problems. This new concept arose from the observation that the physical structure of the networks could remain unchanged, but the phenomena which occur in these networks varied with time. They got a strict connection between equilibrium problems in dynamic networks and the evolutionary variational inequalities; in this sense that the time-dependent equilibrium conditions of this problem are equivalently expressed as evolutionary variational inequalities.

Most recently, many researches focused on the vector equilibrium problems. They examined the traffic equilibrium problem based on a vector cost consideration rather than the traditional single cost criterion. The vector equilibrium problem takes time, distance, expenses and other criterion as the component of the vector cost. Some results on vector equilibrium problem can be found in [13–17]. But the vector equilibrium model can not solve the equilibrium problem when there are many interactional kinds of goods transported through the same traffic network.

In fact, there are more than one kind of goods transported through the traffic network in reality. As we know, the transportation cost of one kind of goods can be affected by other kinds of goods under the same traffic network. In detail, the flows of different kinds of goods are not independent. For example, the transportation costs of one certain kind of goods is not only related with the flow and demand of itself, but also related with the flow and the demand of its substitution. Because the increasing of the flow and the demand of the substitution will put a whole lot of pressure on the transportation of the certain kind of goods under the same traffic network, the marginal cost will increase. Therefore, it is reasonable to consider the traffic equilibrium problem when there are many kinds of goods transported through the same traffic network. Generally, we called this problem dynamic traffic network equilibrium system. In this paper, we introduce the equilibrium definition about this problem based on Wardrop's principles and propose a mathematical model about this traffic equilibrium problem in dynamic networks. We employ marginal costs rather than average costs in our research. Moreover, we also prove that the equilibrium conditions of this problem can be equivalently expressed as a system of evolutionary variational inequalities. Furthermore, we show the existence and uniqueness of the solution for this equilibrium problem. Finally, we give a numerical example to illustrate our results.

The rest of the paper is organized as follows. In Section 2, we recall some necessary knowledge about traffic equilibrium. In Section 3, we propose the basic model about the dynamic traffic network equilibrium system. The issues regarding (i) the variational inequality approaches to express the equilibrium system and (ii) the existence and uniqueness conditions of the solution for the equilibrium system are discussed in this section too. In Section 4, we give an example to illustrate our main results. We give conclusion in Section 5.

2. Preliminaries

Suppose that a traffic network consists of a set N of nodes, a set Ω of origin-destination (O/D) pairs, and a set $\mathcal{R}$ of routes. Each route $r \in \mathcal{R}$ links one given origin-destination pair $\omega \in \Omega$. The set of all $r \in \mathcal{R}$ which links the same origin-destination pair $\omega \in \Omega$ is denoted by $\mathcal{R}(\omega)$. Assume that n is the number of the route in $\mathcal{R}$ and m is the number of origin-destination (O/D) pairs in Ω. Let vector $H = (H_1, H_2, \dots, H_r, \dots, H_n)^T \in R^n$ denote the flow vector, where H_r, $r \in \mathcal{R}$, denotes the flow in route $r \in \mathcal{R}$. A feasible flow has to satisfy the capacity restriction principle: $\lambda_r \le H_r \le \mu_r$, for all $r \in \mathcal{R}$, and a traffic conservation law: $\sum_{r\in\mathcal{R}(\omega)} H_r = \rho_\omega$, for all $\omega \in \Omega$, where λ and μ are given in R^n, $\rho_\omega \ge 0$ is the travel demand related to the given pair $\omega \in \Omega$, and $\rho \in R^m$ denotes the travel demand vector. Thus the set of all feasible flows is given by

$$K := \{H \in R^n \mid \lambda \le H \le \mu, \Phi H = \rho\}, \tag{2.1}$$

where $\Phi = (\delta_{\omega,r})_{m\times n}$ is defined as

$$\delta_{\omega,r} := \begin{cases} 1, & \text{if } \ r \in \mathcal{R}(\omega), \\ 0, & \text{else}. \end{cases} \tag{2.2}$$

Let mapping $C : K \to R^n$ be the cost function. $C(H) \in R^n$ is the cost vector respected to feasible flow $H \in K$. $C_r(H)$ gives the marginal cost of transporting one additional unit of flow through route $r \in \mathcal{R}$.

Definition 2.1 (see [12]). $H \in R^n$ is called an equilibrium flow if and only if for all $\omega \in \Omega$ and $q, s \in \mathcal{R}(\omega)$ there holds

$$C_q(H) < C_s(H) \Longrightarrow H_q = \mu_q \quad \text{or} \quad H_s = \lambda_s. \tag{2.3}$$

Such a definition represents Wardrop's equilibrium principles in a generalized version.

Lemma 2.2 (see [12]). *Let K be given by* (2.1). *If $H \in R^n$ is an equilibrium flow, then the following conditions are equivalent:*

(1) *for all $\omega \in \Omega$ and $q, s \in \mathcal{R}(\omega)$, there holds $C_q(H) < C_s(H) \Rightarrow H_q = \mu_q$* or *$H_s = \lambda_s$,*

(2) *$H \in K$ and $\langle C(H), F - H\rangle \ge 0$, for all $F \in K$.*

Remark 2.3. Lemma 2.2 characterizes that the equilibrium flow defined by Wardrop's equilibrium principle is equivalent to a variational inequality formulation.

Lemma 2.4 (see [18]). *If K is nonempty, convex, and closed, then H^* is an equilibrium flow in the sense of Definition 2.1 if and only if there is $\alpha > 0$ such that*

$$H^* = P_K(H^* - \alpha C(H^*)), \tag{2.4}$$

where $P_K : R^n \to K$ is the projection operator from R^n to K.

Furthermore, we can get the dynamic model based on the assumption that the flow is time dependent. First of all, we need to define the flow function over time. Now the traffic network is considered at all times $t \in \mathcal{T}$, where $\mathcal{T} := [0, T]$. For each time $t \in \mathcal{T}$, we have a flow vector $H(t) \in R^n$. $H(\cdot) : \mathcal{T} \to R^n$ is the flow function over time. The feasible flows have to satisfy the time-dependent capacity constraints and traffic conservation law, that is,

$$\lambda(t) \leq H(t) \leq \mu(t), \quad \Phi H(t) = \rho(t), \quad \text{a.e. } t \in \mathcal{T}, \tag{2.5}$$

where $\lambda, \mu, \rho : \mathcal{T} \to R^n$ are given, $\lambda(\cdot) \leq \mu(\cdot)$, and Φ is defined as (2.2).

We choose the reflexive Banach space $L^p(\mathcal{T}, R^n)$ (for short $\mathcal{L}$) with $p > 1$ as the functional set of the flow functions for technical reasons. The dual space $L^q(\mathcal{T}, R^n)$, where $1/p + 1/q = 1$, will be denoted by $\mathcal{L}^*$. On $\mathcal{L}^* \times \mathcal{L}$, Daniele et al. [12] employed the definition of evolutionary variational inequalities as follows:

$$\langle\langle G, F \rangle\rangle := \int_{\mathcal{T}} \langle G(t), F(t) \rangle dt, \qquad G \in \mathcal{L}^*, F \in \mathcal{L}. \tag{2.6}$$

The set of feasible flows is defined as

$$\mathbb{K} := \left\{ H \in \mathcal{L} \mid \lambda(t) \leq H(t) \leq \mu(t), \Phi H(t) = \rho(t), \text{a.e. } t \in \mathcal{T} \right\}. \tag{2.7}$$

In order to guarantee that $\mathbb{K} \neq \emptyset$, the following assumption is employed (see [12])

$$\Phi\lambda(t) \leq \rho(t) \leq \Phi\mu(t), \quad \text{a.e. } t \in \mathcal{T}, \tag{2.8}$$

where $\lambda, \mu \in \mathcal{L}$ and for all $\omega \in \Omega$, $\rho_\omega \geq 0$ in $L^p(\mathcal{T}, R^m)$. It can be shown that $\mathbb{K}$ is convex, closed, and bounded, hence weakly compact. Furthermore, the mapping $C : \mathbb{K} \to \mathcal{L}^*$ assigns each flow function $H(\cdot) \in \mathbb{K}$ to the cost function $C(H(\cdot)) \in \mathcal{L}^*$.

Definition 2.5 (see [12]). $H \in \mathcal{L}$ is an equilibrium flow if and only if for all $\omega \in \Omega$ and $q, s \in \mathcal{R}(\omega)$ there holds:

$$C_q(H(t)) < C_s(H(t)) \Longrightarrow H_q(t) = \mu_q(t) \quad \text{or } H_s(t) = \lambda_s(t), \quad \text{a.e. } t \in \mathcal{T}. \tag{2.9}$$

Lemma 2.6 (see [12]). *$H \in \mathbb{K}$ is an equilibrium flow which is defined by Definition 2.5, then the following statements are equivalent:*

(1) *for all $\omega \in \Omega$ and $q, s \in \mathcal{R}(\omega)$, there holds:*

$$C_q(H(t)) < C_s(H(t)) \Longrightarrow H_q(t) = \mu_q(t) \quad \text{or} \quad H_s(t) = \lambda_s(t), \; t \in \mathcal{T}; \tag{2.10}$$

(2) *$H \in \mathbb{K}$ and $\langle\langle C(H), F - H \rangle\rangle \geq 0$, for all $F \in \mathbb{K}$.*

The statement (1) in Lemma 2.6 is called Wardrop's condition for the time-dependent traffic network equilibrium by Daniele et al. [12]. Lemma 2.6 shows that the time-dependent traffic network equilibrium can be equivalently expressed as an evolutionary variational inequality. Then we can get the following corollary from Lemmas 2.2 and 2.6 directly.

Corollary 2.7 (see [18]). *If $H \in \mathbb{K}$ is an equilibrium flow, then the following inequalities are equivalent:*

(1) $\langle\langle C(H), F - H \rangle\rangle \geq 0$, *for all $F \in \mathbb{K}$,*

(2) $\langle C(H(t)), F(t) - H(t) \rangle \geq 0$, a.e. $t \in \mathcal{T}$, *for all $F \in \mathbb{K}$.*

Corollary 2.7 is interesting because we can use it to find the solutions of the evolutionary variational inequality.

3. Dynamic Traffic Network Equilibrium System

There are more than one kind of goods transported through the traffic network in reality. As we know, the transportation cost of one kind of goods can be affected by other kinds of goods under the same traffic network. For example, the transportation costs of certain kind of goods is not only related with the flow and the demand of itself, but also related with the flow and the demand of its substitution. Therefore, it is reasonable to consider the equilibrium problem when several kinds of goods are transported through the same traffic network.

3.1. Basic Model

Without loss of generality, we consider the case that there are only two kinds of goods transported through the network. We choose space $L^2(\mathcal{T}, R^n)$ as the functional set of the flow function. Define

$$\mathbb{K}_i := \left\{ H \in L^2(\mathcal{T}, R^n) \mid \lambda_i(t) \leq H(t) \leq \mu_i(t), \; \Phi H(t) = \rho_i(t), \text{ a.e. } t \in \mathcal{T} \right\}, \quad i = 1, 2. \tag{3.1}$$

Thus the set of feasible flows is given by $\mathbb{K}_1 \times \mathbb{K}_2$. We call that $(H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ is a flow of the dynamic traffic network system.

Let mapping $C_i : \mathbb{K}_1 \times \mathbb{K}_2 \rightarrow L^2(\mathcal{T}, R^n)$ denote the marginal transportation cost function of the ith kind of goods for $i = 1, 2$. Then $C_i(H_1, H_2) \in L^2(\mathcal{T}, R^n)$ is the cost vector with respect to feasible flow $(H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ and $C_{ir}(H_1, H_2)$ is the marginal transportation cost of the ith kind of goods under the rth route.

Definition 3.1. $(H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ is an equilibrium flow if and only if for all $\omega \in \Omega$ and $q, s, p, r \in \mathcal{R}(\omega)$ there holds

$$\begin{aligned} &C_{1q}(H_1(t), H_2(t)) < C_{1s}(H_1(t), H_2(t)) \Longrightarrow H_{1q}(t) = \mu_{1q}(t) \text{ or } H_{1s}(t) = \lambda_{1s}(t), \text{ a.e. } t \in \mathcal{T}, \\ &C_{2p}(H_1(t), H_2(t)) < C_{2r}(H_1(t), H_2(t)) \Longrightarrow H_{2p}(t) = \mu_{2p}(t) \text{ or } H_{2r}(t) = \lambda_{2r}(t), \text{ a.e. } t \in \mathcal{T}. \end{aligned} \tag{3.2}$$

Remark 3.2. If the traffic network transports only one kind of good, then Definition 3.1 reduces to Definition 2.5. So, the dynamic traffic equilibrium system (3.2) generalizes the model in [12] to the case of several related goods.

The following result establishes relationship between the system of dynamic traffic equilibrium problem and a system of evolutionary variational inequalities.

Theorem 3.3. $(H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ *is an equilibrium flow if and only if*

$$\begin{aligned} &\langle\langle C_1(H_1, H_2), F_1 - H_1 \rangle\rangle \geq 0, \quad \forall F_1 \in \mathbb{K}_1, \\ &\langle\langle C_2(H_1, H_2), F_2 - H_2 \rangle\rangle \geq 0, \quad \forall F_2 \in \mathbb{K}_2. \end{aligned} \tag{3.3}$$

Proof. First assume that (3.3) holds and (3.2) does not hold. Then there exist $\omega \in \Omega$ and $q, s \in \mathcal{R}(\omega)$ together with a set $E \subseteq \mathcal{T}$ having positive measure such that

$$C_{iq}(H_1(t), H_2(t)) < C_{is}(H_1(t), H_2(t)), \quad H_{iq}(t) < \mu_{iq}(t), \quad H_{is}(t) > \lambda_{is}(t), \quad \text{a.e. } t \in E, i = 1, 2. \tag{3.4}$$

For $t \in E$, let $\delta_i(t) = \min\{\mu_{iq}(t) - H_{iq}(t), H_{\mathrm{is}}(t) - \lambda_{\mathrm{is}}(t)\}$. Then $\delta_i(t) > 0$, a.e. $t \in E$. We define a vector $F_i \in \mathbb{K}_i$ whose components are

$$F_{iq}(t) = H_{iq}(t) + \delta_i(t), \quad F_{is}(t) = H_{is}(t) - \delta_i(t), \quad F_{ir}(t) = H_{ir}(t), \quad \text{a.e. } t \in E \tag{3.5}$$

when $r \neq q, s$, and we can construct $F_i \in \mathbb{K}_i$ such that $F_i = H_i$ outside E. Thus,

$$\begin{aligned} \langle\langle C_i(H_1, H_2), F_i - H_i \rangle\rangle &= \int_{\mathcal{T}} \langle C_i(H_1(t), H_2(t)), F_i(t) - H_i(t) \rangle dt \\ &= \int_E \delta_i(t) \left(C_{iq}(H_1(t), H_2(t)) - C_{\mathrm{is}}(H_1(t), H_2(t)) \right) dt \\ &< 0, \end{aligned} \tag{3.6}$$

and so (3.3) is not satisfied. Therefore, it is proved that (3.3) implies (3.2).

Next, assume that (3.2) holds. That is

$$\begin{aligned} C_{iq}(H_1(t), H_2(t)) < C_{\text{is}}(H_1(t), H_2(t)) \\ \Longrightarrow H_{iq}(t) = \mu_{iq}(t), \text{or} \\ H_{\text{is}}(t) = \lambda_{\text{is}}(t), \quad \text{a.e. } t \in \mathcal{T}, \; i = 1, 2. \end{aligned} \tag{3.7}$$

Let $F_i \in \mathbb{K}_i$ for $i = 1, 2$. Then (3.3) holds from Lemma 2.6. □

Furthermore, we can get the following corollary directly from Corollary 2.7 and Theorem 3.3.

Corollary 3.4. $(H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ *is an equilibrium flow if and only if, for all* $F_i \in \mathbb{K}_i$ *with* $i = 1, 2$,

$$\begin{aligned} \langle C_1(H_1(t), H_2(t)), F_1(t) - H_1(t) \rangle \geq 0, \quad \textit{a.e. } t \in \mathcal{T}, \\ \langle C_2(H_1(t), H_2(t)), F_2(t) - H_2(t) \rangle \geq 0, \quad \textit{a.e. } t \in \mathcal{T}. \end{aligned} \tag{3.8}$$

3.2. Existence and Uniqueness Theorem

In this subsection, we discuss the existence and uniqueness of the solution for the dynamic traffic equilibrium system (3.3). In order to get our main results, the following definitions will be employed.

Definition 3.5. $C_i(x, y)$ $(i = 1, 2)$ is said to be θ-strictly monotone with respect to x on $\mathbb{K}_1 \times \mathbb{K}_2$ if there exists $\theta > 0$ such that

$$\langle \langle C_i(x_1, y) - C_i(x_2, y), x_1 - x_2 \rangle \rangle \geq \theta \|x_1 - x_2\|_{L^2}^2, \quad \forall x_1, x_2 \in \mathbb{K}_1, \; y \in \mathbb{K}_2, \tag{3.9}$$

where

$$\|x\|_{L^2}^2 = \int_{\mathcal{T}} \|x(t)\|^2 dt \tag{3.10}$$

and $\| \cdot \|$ is Euclidean norm.

Definition 3.6. $C_i(x, y)$ $(i = 1, 2)$ is said to be L-*Lipschitz* continuous with respect to x on $\mathbb{K}_1 \times \mathbb{K}_2$ if there exists $L > 0$ such that

$$\left\| C_i(x_1, y) - C_i(x_2, y) \right\|_{L^2} \leq L \|x_1 - x_2\|_{L^2}, \quad \forall x_1, x_2 \in \mathbb{K}_1, y \in \mathbb{K}_2. \tag{3.11}$$

Remark 3.7. Based on Definitions 3.5 and 3.6, we can similarly define the θ-strict monotonicity and L-Lipschitz continuity of $C_i(x, y)$ with respect to y on $\mathbb{K}_1 \times \mathbb{K}_2$ for $i = 1, 2$.

Theorem 3.8. $(H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2$ *is an equilibrium flow if and only if there exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\begin{aligned} H_1 &= P_{\mathbb{K}_1}\left(H_1 - \alpha C_1(H_1, H_2)\right), \\ H_2 &= P_{\mathbb{K}_2}\left(H_2 - \beta C_2(H_1, H_2)\right), \end{aligned} \tag{3.12}$$

where $P_{\mathbb{K}_i} : L^2(\mathcal{T}; R^n) \to \mathbb{K}_i$ *is a projection operator for* $i = 1, 2$.

Proof. The proof is analogous to that of Theorem 5.2.4 of [18]. □

Let $\|(x, y)\|_1$ be the norm on space $\mathbb{K}_1 \times \mathbb{K}_2$ defined as follows:

$$\left\|(x, y)\right\|_1 = \|x\|_{L^2} + \left\|y\right\|_{L^2}, \quad \forall x \in \mathbb{K}_1, y \in \mathbb{K}_2. \tag{3.13}$$

It is easy to see that $(\mathbb{K}_1 \times \mathbb{K}_2\ , \|\cdot\|_1)$ is a Banach space.

Theorem 3.9. *Suppose that* $C_1(H_1, H_2)$ *is* θ_1*-strictly monotone and* L_{11}*-Lipschitz continuous with respect to* H_1*, and* L_{12}*-Lipschitz continuous with respect to* H_2 *on* $\mathbb{K}_1 \times \mathbb{K}_2$*. Suppose that* $C_2(H_1, H_2)$ *is* L_{21}*-Lipschitz continuous with respect to* H_1*,* θ_2*-strictly monotone, and* L_{22}*-Lipschitz continuous with respect to* H_2 *on* $\mathbb{K}_1 \times \mathbb{K}_2$*. If there exist* $\gamma > 0$ *and* $\eta > 0$ *such that*

$$\begin{aligned} &\sqrt{1 - 2\gamma\theta_1 + \gamma^2 L_{11}^2} + \eta L_{21} < 1, \\ &\sqrt{1 - 2\eta\theta_2 + \eta^2 L_{22}^2} + \gamma L_{12} < 1, \end{aligned} \tag{3.14}$$

then problem (3.3) *admits unique solution.*

Proof. For any $(H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2$, let

$$\begin{aligned} F_1(H_1, H_2) &= P_{\mathbb{K}_1}\left(H_1 - \gamma C_1(H_1, H_2)\right), \\ F_2(H_1, H_2) &= P_{\mathbb{K}_2}\left(H_2 - \eta C_2(H_1, H_2)\right), \end{aligned} \tag{3.15}$$

where $P_{\mathbb{K}_i} : L^2(\mathcal{T}, R^n) \to \mathbb{K}_i$ is a projection operator for $i = 1, 2$. Define $F : \mathbb{K}_1 \times \mathbb{K}_2 \to \mathbb{K}_1 \times \mathbb{K}_2$ as follows:

$$F(H_1, H_2) = (F_1(H_1, H_2), F_2(H_1, H_2)), \quad \forall (H_1, H_2) \in \mathbb{K}_1 \times \mathbb{K}_2. \tag{3.16}$$

Since $P_{\mathbb{K}_i}$ is nonexpansive, it follows that, for any $(H_1, H_2), (\widetilde{H}_1, \widetilde{H}_2) \in \mathbb{K}_1 \times \mathbb{K}_2$,

$$\begin{aligned}
&\left\| F(H_1, H_2) - F(\widetilde{H}_1, \widetilde{H}_2) \right\|_1 \\
&\quad = \left\| F_1(H_1, H_2) - F_1(\widetilde{H}_1, \widetilde{H}_2) \right\|_{L^2} + \left\| F_2(H_1, H_2) - F_2(\widetilde{H}_1, \widetilde{H}_2) \right\|_{L^2} \\
&\quad = \left\| P_{\mathbb{K}_1}(H_1 - \gamma C_1(H_1, H_2)) - P_{\mathbb{K}_1}(\widetilde{H}_1 - \gamma C_1(\widetilde{H}_1, \widetilde{H}_2)) \right\|_{L^2} \\
&\qquad + \left\| P_{\mathbb{K}_2}(H_2 - \eta C_2(H_1, H_2)) - P_{\mathbb{K}_2}(\widetilde{H}_2 - \eta C_2(\widetilde{H}_1, \widetilde{H}_2)) \right\|_{L^2} \\
&\quad \leq \left\| H_1 - \widetilde{H}_1 - \gamma [C_1(H_1, H_2) - C_1(\widetilde{H}_1, \widetilde{H}_2)] \right\|_{L^2} \\
&\qquad + \left\| H_2 - \widetilde{H}_2 - \eta [C_2(H_1, H_2) - C_2(\widetilde{H}_1, \widetilde{H}_2)] \right\|_{L^2} \\
&\quad \leq \left\| H_1 - \widetilde{H}_1 - \gamma [C_1(H_1, H_2) - C_1(\widetilde{H}_1, H_2)] \right\|_{L^2} + \gamma \left\| C_1(\widetilde{H}_1, H_2) - C_1(\widetilde{H}_1, \widetilde{H}_2) \right\|_{L^2} \\
&\qquad + \left\| H_2 - \widetilde{H}_2 - \eta [C_2(H_1, H_2) - C_2(H_1, \widetilde{H}_2)] \right\|_{L^2} + \eta \left\| C_2(H_1, \widetilde{H}_2) - C_2(\widetilde{H}_1, \widetilde{H}_2) \right\|_{L^2}.
\end{aligned} \tag{3.17}$$

Since $C_1(H_1, H_2)$ is θ_1-strictly monotone and L_{11}-Lipschitz continuous with respect to H_1, we have

$$\begin{aligned}
&\left\| H_1 - \widetilde{H}_1 - \gamma [C_1(H_1, H_2) - C_1(\widetilde{H}_1, H_2)] \right\|_{L^2}^2 \\
&\quad = \left\| H_1 - \widetilde{H}_1 \right\|_{L^2}^2 - 2\gamma \left\langle \left\langle C_1(H_1, H_2) - C_1\left(\widetilde{H}_1, H_2\right), H_1 - \widetilde{H}_1 \right\rangle \right\rangle \\
&\qquad + \gamma^2 \left\| C_1(H_1, H_2) - C_1(\widetilde{H}_1, H_2)) \right\|_{L^2}^2 \\
&\quad \leq \left\| H_1 - \widetilde{H}_1 \right\|_{L^2}^2 - 2\gamma\theta_1 \left\| H_1 - \widetilde{H}_1 \right\|_{L^2}^2 + \gamma^2 L_{11}^2 \left\| H_1 - \widetilde{H}_1 \right\|_{L^2}^2 \\
&\quad = \left[1 - 2\gamma\theta_1 + \gamma^2 L_{11}^2 \right] \left\| H_1 - \widetilde{H}_1 \right\|_{L^2}^2.
\end{aligned} \tag{3.18}$$

Thus,

$$\begin{aligned}
&\left\| H_1 - \widetilde{H}_1 - \gamma [C_1(H_1, H_2) - C_1(\widetilde{H}_1, H_2)] \right\|_{L^2} \\
&\quad \leq \sqrt{1 - 2\gamma\theta_1 + \gamma^2 L_{11}^2} \; \left\| H_1 - \widetilde{H}_1 \right\|_{L^2}.
\end{aligned} \tag{3.19}$$

Furthermore, $C_1(H_1, H_2)$ is L_{12}-*Lipschitz* continuous with respect to H_2, we get

$$\begin{aligned}
&\left\| H_1 - \widetilde{H}_1 - \gamma [C_1(H_1, H_2) - C_1(\widetilde{H}_1, H_2)] \right\|_{L^2} + \gamma \left\| (C_1(\widetilde{H}_1, H_2) - C_1(\widetilde{H}_1, \widetilde{H}_2)) \right\|_{L^2} \\
&\quad \leq \sqrt{1 - 2\gamma\theta_1 + \gamma^2 L_{11}^2} \left\| H_1 - \widetilde{H}_1 \right\|_{L^2} + \gamma L_{12} \left\| H_2 - \widetilde{H}_2 \right\|_{L^2}.
\end{aligned} \tag{3.20}$$

Similarly, we can prove that

$$\begin{aligned}
&\left\| H_2 - \widetilde{H}_2 - \eta[C_2(H_1, H_2) - C_2(H_1, \widetilde{H}_2)] \right\|_{L^2} + \eta \left\| C_2(H_1, \widetilde{H}_2) - C_2(\widetilde{H}_1, \widetilde{H}_2) \right\|_{L^2} \\
&\quad \leq \sqrt{1 - 2\eta\theta_2 + \eta^2 L_{22}^2} \left\| H_2 - \widetilde{H}_2 \right\|_{L^2} + \eta L_{21} \left\| H_1 - \widetilde{H}_1 \right\|_{L^2}.
\end{aligned} \tag{3.21}$$

Let

$$M := \max\left\{ \sqrt{1 - 2\gamma\theta_1 + \gamma^2 L_{11}^2} + \eta L_{21}, \sqrt{1 - 2\eta\theta_2 + \eta^2 L_{22}^2} + \gamma L_{12} \right\}. \tag{3.22}$$

Then, applying previous bounds to the final terms appearing in (3.17), we get

$$\begin{aligned}
&\left\| F(H_1, \mathrm{H}_2) - F(\widetilde{H}_1, \widetilde{H}_2) \right\|_1 \\
&\quad = \left\| F_1(H_1, H_2) - F_1(\widetilde{H}_1, \widetilde{H}_2) \right\|_{L^2} + \left\| F_2(H_1, H_2) - F_2(\widetilde{H}_1, \widetilde{H}_2) \right\|_{L^2} \\
&\quad \leq \sqrt{1 - 2\gamma\theta_1 + \gamma^2 L_{11}^2} \left\| H_1 - \widetilde{H}_1 \right\| + \gamma L_{12} \left\| H_2 - \widetilde{H}_2 \right\|_{L^2} \\
&\qquad + \sqrt{1 - 2\eta\theta_2 + \eta^2 L_{22}^2} \left\| H_2 - \widetilde{H}_2 \right\| + \eta L_{21}^2 \left\| H_1 - \widetilde{H}_1 \right\|_{L^2} \\
&\quad = \left(\sqrt{1 - 2\gamma\theta_1 + \gamma^2 L_{11}^2} + \eta L_{21} \right) \left\| H_1 - \widetilde{H}_1 \right\|_{L^2} \\
&\qquad + \left(\sqrt{1 - 2\eta\theta_2 + \eta^2 L_{22}^2} + \gamma L_{12} \right) \left\| H_2 - \widetilde{H}_2 \right\|_{L^2} \\
&\quad \leq M \left(\left\| H_1 - \widetilde{H}_1 \right\|_{L^2} + \left\| H_2 - \widetilde{H}_2 \right\|_{L^2} \right) \\
&\quad = M \left\| (H_1 - \widetilde{H}_1, H_2 - \widetilde{H}_2) \right\|_1 \\
&\quad = M \left\| (H_1, H_2) - (\widetilde{H}_1, \widetilde{H}_2) \right\|_1.
\end{aligned} \tag{3.23}$$

It follows from (3.14) that $M < 1$. Therefore, $F(\cdot)$ is a contraction mapping. By Banach fixed point theorem, $F(\cdot)$ has a unique fixed point $(\overline{H}_1, \overline{H}_2)$ on $\mathbb{K}_1 \times \mathbb{K}_2$. That is,

$$\left(\overline{H}_1, \overline{H}_2\right) = F\left(\overline{H}_1, \overline{H}_2\right) = \left(F_1\left(\overline{H}_1, \overline{H}_2\right), F_2\left(\overline{H}_1, \overline{H}_2\right) \right), \tag{3.24}$$

and so

$$\begin{aligned}\overline{H}_1 = F_1\left(\overline{H}_1, \overline{H}_2\right) &= P_{\mathbb{K}_1}\left(\overline{H}_1 - \gamma C_1\left(\overline{H}_1, \overline{H}_2\right)\right),\\ \overline{H}_2 = F_2\left(\overline{H}_1, \overline{H}_2\right) &= P_{\mathbb{K}_2}\left(\overline{H}_2 - \eta C_2\left(\overline{H}_1, \overline{H}_2\right)\right).\end{aligned} \tag{3.25}$$

By Theorem 3.8, we know that $(\overline{H}_1, \overline{H}_2)$ is an equilibrium flow. This completes the proof. □

4. An Example

In order to illustrate our results, we consider a simple traffic network consisting of a single O/D pair of nodes and two paths connecting these two nodes. The feasible sets are given by

$$\mathbb{K}_1 = \mathbb{K}_2 = \left\{F \in L^2\left([0,2]; R^2\right) \mid 0 \le F_1(t) \le t, 0 \le F_2(t) \le 3,\ F_1(t) + F_2(t) = t, \text{a.e. } t \in [0,2]\right\}. \tag{4.1}$$

Let us assume that the cost functions on the paths are defined by

$$\begin{aligned}C_{11}(H_1(t), H_2(t)) &= H_{11}(t) + 0.01H_{21}(t) + 0.01H_{22}(t),\\ C_{12}(H_1(t), H_2(t)) &= H_{12}(t) + 0.01H_{21}(t) + 0.01H_{22}(t),\\ C_{21}(H_1(t), H_2(t)) &= 0.01H_{11}(t) + 0.01H_{12}(t) + H_{21}(t),\\ C_{22}(H_1(t), H_2(t)) &= 0.01H_{11}(t) + 0.01H_{12}(t) + H_{22}(t),\end{aligned} \tag{4.2}$$

where the following vector notation is introduced:

$$\begin{aligned}C_1(H_1(t), H_2(t)) &= (C_{11}(H_1(t), H_2(t)), C_{12}(H_1(t), H_2(t)))^T,\\ C_2(H_1(t), H_2(t)) &= (C_{21}(H_1(t), H_2(t)), C_{22}(H_1(t), H_2(t)))^T,\\ H_1(t) &= (H_{11}(t), H_{12}(t))^T \in \mathbb{K}_1,\\ H_2(t) &= (H_{21}(t), H_{22}(t))^T \in \mathbb{K}_2.\end{aligned} \tag{4.3}$$

By Corollary 3.4, for any $F_1 \in \mathbb{K}_1$ and $F_2 \in \mathbb{K}_2$,

$$\begin{aligned}&C_{11}(H_1(t), H_2(t))(F_{11}(t) - H_{11}(t)) + C_{12}(H_1(t), H_2(t))(F_{12}(t) - H_{12}(t)) \ge 0, \quad \text{a.e. } t \in [0,2],\\ &C_{21}(H_1(t), H_2(t))(F_{21}(t) - H_{21}(t)) + C_{22}(H_1(t), H_2(t))(F_{22}(t) - H_{22}(t)) \ge 0, \quad \text{a.e. } t \in [0,2].\end{aligned} \tag{4.4}$$

From the traffic conservation law, we get

$$F_{i2}(t) = t - F_{i1}(t), \quad G_{i2}(t) = t - G_{i1}(t), \quad \text{a.e. } t \in [0,2]. \tag{4.5}$$

Thus, for any $F_1 \in \mathbb{K}_1$ and $F_2 \in \mathbb{K}_2$, we have

$$\begin{aligned}
&(C_{11}(H_1(t), H_2(t)) - C_{12}(H_1(t), H_2(t)))(F_{11}(t) - H_{11}(t)) \geq 0, \quad \text{a.e. } t \in [0,2],\\
&(C_{21}(H_1(t), H_2(t)) - C_{22}(H_1(t), H_2(t)))(F_{21}(t) - H_{21}(t)) \geq 0, \quad \text{a.e. } t \in [0,2].
\end{aligned} \tag{4.6}$$

It follows that, for any $F_1 \in \mathbb{K}_1$ and $F_2 \in \mathbb{K}_2$,

$$\begin{aligned}
&(2H_{11}(t) - t)(F_{11}(t) - H_{11}(t)) \geq 0, \quad \text{a.e. } t \in [0,2],\\
&(2H_{21}(t) - t)(F_{21}(t) - H_{21}(t)) \geq 0, \quad \text{a.e. } t \in [0,2].
\end{aligned} \tag{4.7}$$

Now we can prove that problem (4.7) has unique solution by Theorem 3.9. In fact, let

$$\theta_1 = \theta_2 = 1, \qquad L_{11} = L_{22} = 1, \qquad L_{12} = L_{21} = 0.01, \qquad \gamma = \eta = 1. \tag{4.8}$$

Then it is easy to check that $C_1(H_1, H_2)$ and $C_2(H_1, H_2)$ satisfy all the conditions of Theorem 3.9.

Furthermore, we can obtain the unique exact solution of problem (4.7). Clearly, (4.7) is equivalent to

$$\begin{aligned}
&F_{11}(t)(2H_{11}(t) - t) \geq H_{11}(t)(2H_{11}(t) - t), \quad \text{a.e. } t \in [0,2],\\
&F_{21}(t)(2H_{21}(t) - t) \geq H_{21}(t)(2H_{21}(t) - t), \quad \text{a.e. } t \in [0,2],
\end{aligned} \tag{4.9}$$

for any $F_1 \in \mathbb{K}_1$ and $F_2 \in \mathbb{K}_2$. If $H_{11}(t) > (1/2)t$, then $F_{11}(t) \geq H_{11}(t)$, for any $0 \leq F_{11}(t) \leq t$. However, the inequality holds if and only if $H_{11}(t) = 0$. It is in contradiction with $H_{11}(t) > (1/2)t$. If $H_{11}(t) < (1/2)t$, then $F_{11}(t) \leq H_{11}(t)$, for any $0 \leq F_{11}(t) \leq t$. However, this is in contradiction with $H_{11}(t) < (1/2)t$. Therefore, $H_{11}(t) = (1/2)t$. Similarly, we can prove that $H_{21}(t) = (1/2)t$. Thus,

$$\begin{aligned}
H_1(t) &= \left(\frac{1}{2}t, \frac{1}{2}t\right)^T,\\
H_2(t) &= \left(\frac{1}{2}t, \frac{1}{2}t\right)^T,
\end{aligned} \tag{4.10}$$

is the unique solution of problem (4.7).

5. Conclusions

Since the transportation costs of certain kind of goods is not only related with the flow of itself, but also related with the flow of other kinds of goods, the equilibrium problem when some kinds of goods are transported through the same traffic network should be considered. In this paper, we study the dynamic traffic equilibrium system based on Wardrop's principles and propose a basic model for the new equilibrium problem. In detail, the dynamic traffic equilibrium system can be equivalently expressed as a system of evolutionary variational inequalities. Thus some classical results of system of variational inequalities could be applied to the study of dynamic traffic equilibrium system. By using the fixed point theory and projected dynamic system theory, we get the existence and uniqueness of the solution for this equilibrium problem. A numerical example is also given to illustrate our results about the dynamic traffic equilibrium system. Our results improve and generalize the classic dynamic traffic network equilibrium problem and the results of [12].

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