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Research Article

Fixed Points of Discontinuous Multivalued Operators in Ordered Spaces with Applications

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Existence theorems of fixed points for multivalued increasing operators in partially ordered spaces are presented. Here neither the continuity nor compactness is assumed for multivalued operators. As an application, we lead to the existence principles for integral inclusions of Hammerstein type multivalued maps.

1. Introduction

The influence of fixed point theorems for contractive and nonexpansive mappings (see [1, 2]) on fixed point theory is so huge that there are many results dealing with fixed points of mappings satisfying various types of contractive and nonexpansive conditions. On the other hand, it is also huge that well-known Brouwer's and Schauder's fixed point theorems for set-contractive mappings exert an influence on this theory. However, if a mapping is not completely continuous, in general, it is difficult to verify that the mapping satisfies the set-contractive condition. In 1980, Mönch [3] has obtained the following important fixed point theorem which avoids the above mentioned difficulty.

Theorem 1.1. *Let E be a Banach space, $K \subset E$ a closed convex subset. Suppose that (single) operator $F : K \to K$ is continuous and satisfies that*

(i) *there exists $x \in K$ such that if $C \subset K \cap \overline{\mathrm{co}}(\{x\} \cup F(C))$ is countable, then C is relatively compact,*

then F has a fixed point in K.

It has been observed that continuity is an ideal and important property in the above cited works, while in some applications the mapping under consideration may not be continuous, yet at the same time it may be "not very discontinuous". this idea has motivated many authors to study corresponding problems, for instance, the stability of Brouwer's fixed point theorem [4], similar result for nonexpansive mappings [5], and existence and approximation of the synthetic approaches to fixed point theorems [6]. Recently, fixed point theory for discontinuous multivalued mappings has attracted much attention and many authors studied the existence of fixed points for such mappings. We refer to [7–11]. For example, Hong [8] has extended Mönch [3] to discontinuous multivalued operators in ordered Banach spaces by using a quite weak compactness condition; that is, assuming the following condition is satisfied.

(H) If $C = \{x_n\}$ is a countable totally ordered set and $C \subset \mathrm{wcl}(\{x_1\} \cup A(C))$, then C is weakly relatively compact. Here A is a multivalued operator and $\mathrm{wcl}(B)$ denotes the weak closure of the set B.

The purpose of this paper is to present some results on fixed point theorems of Mönch type of multivalued increasing operators for which neither the continuity nor the compactness is assumed in ordered topological spaces. However, we will use the following hypothesis.

(H1) If $C = \{x_n\} \subset K$ is a countable totally ordered set and $C \subset \mathrm{cl}(\{x_1\} \cup A(C))$, then C has a supremum.

E is a topological vector space endowed with partial ordering "$\le$", $\mathrm{cl}(B)$ stands for the closure of the set B, and $K = \{x \in E \mid x \ge u_0\}$ with $u_0 \in E$ is a given ordered set of E.

This paper is organized as follows. In Section 2, we introduce some definitions and preliminary facts from partially ordered theory and multivalued analysis which are used later. In especial, we introduce a new partial ordering of sets which forms a basis to our main results. In Section 3, we state and prove existence of fixed points, also, maximal and minimal fixed point theorem is presented for discontinuous multivalued increasing operators which are our main results. To illustrate the applicability of our theory, in Section 4, we discuss the existence of solutions to the Hammerstein integral inclusions of the form

$$u(t) \in \int_0^T k(t,s)G(s,u(s))ds \quad \text{a.e. on } [0,T]. \tag{1.1}$$

2. Preliminaries

Let $(E, \le)$ be a partially ordered topological vector space. By the notation "$x < y$" we always mean that $x \le y$ and $x \ne y$. Let 2^E stand for the collection of all nonempty subsets of E. Take $u_0 \in E$ and let $K_{u_0} = \{x \in E \mid x \ge u_0\}$ be a given ordered set of E. The ordered interval of E is written as $[u, v] = \{x \in E : u \le x \le v\}$.

For two subsets P, Q of E, we write $P \le Q$ (or $Q \ge P$) if

$$\forall p \in P, \quad \exists q \in Q \quad \text{such that } p \le q. \tag{2.1}$$

Given a nonempty subsets Ω of E we say that $A : \Omega \multimap 2^E$ is increasing upwards if $u, v \in \Omega$, $u \le v$, and $x \in A(u)$ imply that there exists $y \in A(v)$ such that $x \le y$. A is increasing downwards if $u, v \in \Omega$, $u \le v$, and $y \in A(v)$ imply an existence of $x \in A(u)$ such that $x \le y$. If A is increasing upwards and downwards we say that A is increasing.

Let $\Gamma \subset E$ be nonempty. The element $y \in E$ is called an upper (lower) bound of Γ if $x \leq y$ ($x \geq y$) whenever $x \in \Gamma$. Γ is called upper (lower) bounded with respect to the ordering if its upper (lower) bounds exist. The element $z \in E$ is called a supremum of Γ, written as $z = \sup \Gamma$, if z is an upper bound and $z \leq y$ as long as y is another upper bound of Γ. Similarly, we can define the infimum $\inf \Gamma$ of Γ.

Throughout this paper, unless otherwise mentioned, the partial ordering of E always introduced by a closed cone if E is a Banach space. The following lemmas will be used in after sections.

Lemma 2.1 (see [12]). *Let E be an ordered Banach space and B a totally ordered and weakly relatively compact subset of E, then there exists $x^* \in \mathrm{wcl}(B)$ such that $x \leq x^*$ foe all $x \in B$.*

An ordered topological vector space E is said to have the limit ordinal property if $x_n, y_n \in E$ with $x_n \leq y_n$ for $n = 1, 2, \ldots$, and $x_n \to x^*, y_n \to y^*$ for $n \to \infty$ imply $x^* \leq y^*$. By an analogy of the proof of Lemma 1.1.2 in [12], we have the following.

Lemma 2.2. *If E has the limit ordinal property and $\{x_n\}$ is a relatively compact monotone sequence of E, then $\{x_n\}$ is convergent. Moreover, $x_n \leq x^*$ if $\{x_n\}$ is increasing and $x^* \leq x_n$ if $\{x_n\}$ is decreasing for $n = 1, 2, \ldots$. Here $\lim_{n\to\infty} x_n = x^*$.*

Remark 2.3. Under the assumptions of Lemma 2.2, it is evident that x^* is the supremum (infimum) of increasing (decreasing) sequence $\{x_n\}$.

Lemma 2.4. *Let the increasing sequence $\{x_n\}$ have the supremum z. If $\{x_{n_i}\}$ is a infinity subsequence of $\{x_n\}$, then $\{x_{n_i}\}$ has the supremum z, too.*

Proof. Evidently, z is an upper bound of $\{x_{n_i}\}$. Let y be the other one, then $x_{n_i} \leq y$ for $i = 1, 2, \ldots$. For any given n, since $\{x_{n_i}\}$ is infinity, there exists i_0 such that $x_n \leq x_{n_{i_0}}$, which implies that $x_n \leq y$ for all $n \geq 1$. From the definition of supremums it follows that $z \leq y$, that is, z is the supremum of $\{x_{n_i}\}$. □

Lemma 2.5. *Suppose that every countable totally ordered subset of the partially ordered set Y has a supremum in Y. Let the operator $F : Y \to Y$ satisfy $F(x) \geq x$ for all $x \in Y$, then there exists $x_0 \in Y$ such that $F(x_0) = x_0$.*

Proof. Take $z_0 \in Y$ any fixed and let $z_{i+1} = F(z_i)$ for $i = 0, 1, \ldots$, then $z_{i+1} \geq z_i$ that is, $\{z_i\}$ is increasing. From our assumption it follows that $\{z_i\}$ has a supremum denoted by $z_0^1 = \sup z_i$. Let

$$\Gamma_1 = \{z_0, z_1, \ldots\} \cup \left\{z_0^1\right\}. \tag{2.2}$$

If $z_0^1 = F(z_0^1)$, then the conclusion of the lemma is proved. Otherwise, take $z_i^1 = F(z_{i-1}^1)$ for $i = 1, 2, \ldots$. Again, the set $\{z_0^1, z_1^1, \ldots\}$ has the supremum $z_0^2 = \sup z_i^1$. Denote $\Gamma_2 = \{z_0^1, z_1^1, \ldots\} \cup \{z_0^2\}$. If $z_0^2 = F(z_0^2)$, then the conclusion of the lemma is proved. Otherwise, take $z_i^2 = F(z_{i-1}^2)$ for $i = 1, 2, \ldots$, and let $\Gamma_3 = \{z_0^2, z_1^2, \ldots\} \cup \{z_0^3\}$ with $z_0^3 = \sup z_i^2$. In general, having defined $\Gamma_k = \{z_0^{k-1}, z_1^{k-1}, \ldots\} \cup \{z_0^k\}$ with $z_i^{k-1} = F(z_i^{k-1})$ and $z_0^k = \sup z_i^{k-1}$, where $z_i^0 = z_i$ and $k, i = 1, 2, \ldots$, if $z_0^k = F(z_0^k)$, which completes the proof. Otherwise, repeating this process, either the conclusion of the lemma is proved, or we can obtain a set sequence $\Gamma_1, \Gamma_2, \ldots$ satisfying

(i) $\Gamma_k = \{z_1^{k-1}, z_2^{k-1}, \dots\} \cup \{z_0^k\}$ with $z_0^k = \sup z_i^{k-1}$ and $z_i^k = F(z_{i-1}^k)$, $i, k = 1, 2, \dots$;

(ii) $z_{i-1}^k \le z_i^k$ for $i, k = 1, 2, \dots$;

(iii) $z_j^{k-1} \le z_t^k$ $j, t = 0, 1, 2, \dots$, and $z_i^0 = z_i$.

Let $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$, then Γ is a countable subset and

$$z_0 \le x \quad \forall x \in \Gamma. \tag{2.3}$$

We claim that

$$F(\Gamma) \subset \Gamma. \tag{2.4}$$

In fact, for any $y \in F(\Gamma)$, there exists $x \in \Gamma$ such that $y = F(x)$. There exists Γ_k such that $x \in \Gamma_k$. If $x = z_i^{k-1}$ for some nature number i, then $y = F(x) = z_{i+1}^{k-1} \in \Gamma_k$ which yields $y \in \Gamma$. Otherwise, we have $x = \sup z_i^{k-1} = z_0^k \in \Gamma_{k+1}$. This implies that $y = F(x) = F(\binom{k}{0}) = z_1^k \in \Gamma_{k+1}$. Consequently, $y \in \Gamma$. From the arbitrariness of y it follows that (2.4) is satisfied.

Finally, combining (ii) and (iii) we see easily that Γ is totally ordered. Our hypothesis guarantees that Γ has a supremum, written as $x^* = \sup \Gamma$. Note that (2.4) guarantees $F(x^*) \in \Gamma$, we have $F(x^*) \le x^*$. On the other hand, the definition of F ensures that $F(x^*) \ge x^*$. Hence $F(x^*) = x^*$. This proof is completed. □

Let Ω be a nonempty subset of K_{u_0}. In this section we impose the following hypotheses on the increasing upwards multivalued operator $A : \Omega \multimap 2^E$. Set

$$\mathcal{R} = \{x \in \Omega \mid \text{there exists } u \in Ax \text{ such that } x \le u\} \tag{2.5}$$

and for any $x \in \mathcal{R}$ define that

$$C(x) = \{x, u_1, u_2, \dots, u_n, \dots\}, \qquad D(x) = C(x) \cup \{w(x)\}, \tag{2.6}$$

where, $w(x) = \sup C(x)$ and u_i $(i = 1, 2, \dots)$ is given as follows: since $x \in \mathcal{R}$, there exists $u_1 \in Ax$ such that $x \le u_1$. In virtue of the fact that A is increasing upwards, there exists $u_2 \in Au_1$ such that $u_1 \le u_2$. On the analogy of this process, there exists $u_n \in Au_{n-1}$ such that $u_{n-1} \le u_n$ for $n = 2, 3, \dots$, Obviously, $C(x) \subset \mathrm{cl}(\{x\} \cup A(C))$, thus, the condition (H1) guarantees that the supremum $w(t)$ of $C(x)$ exists.

Remark 2.6. In general, the sequences of these kinds, $\{u_n\}$, may not be unique, that is, every $\{u_n\}$ corresponds to $C(x)$, moreover, corresponds to $D(x)$. For given $x \in \mathcal{R}$, we denote with $\mathcal{C}(x)$ and $\mathfrak{D}(x)$ the families of $C(x)$ and $D(x)$ as above, respectively.

In addition, if E has the limit ordinal property, $D(x)$ is a closed set for any $x \in \mathcal{R}$. In fact, let $\{u_{n_i}\}$ be any infinity subsequence of D for which

$$u_{n_i} \longrightarrow x^* \quad \text{for } i \longrightarrow \infty. \tag{2.7}$$

observing that $\{u_{n_i}\}$ is increasing, by Lemma 2.2 we get that x^* is a supremum of $\{u_{n_i}\}$ and by Lemma 2.4 we get $w(x) = x^*$.

Definition 2.7. A set Γ is said to be sup-closed if the supremum of each countable subset of Γ (provided that it exists) belongs to Γ. A multivalued operator $A : \Omega \multimap 2^E$ is said to have sup-closed values if Ax is sup-closed for each $x \in \Omega$.

Defining

$$X(x) = \{u : \text{there exists } D(x) \in \mathfrak{D}(x) \text{ such that } u \in D(x)\}. \tag{2.8}$$

Lemma 2.8. *Let E be an ordered topological space, Ω a nonempty subset of K_{u_0} with $u_0 \in E$; let $A : \Omega \multimap 2^E$ have sup-closed values and satisfy hypothesis (H1). Moreover, assume that*

(H2) *A is increasing upwards and satisfies $u_0 \leq Au_0$,*

then for any $C(x) \in \mathcal{C}(x)$, $C(x)$ has the supremum $w(x)$ which belongs to $\mathcal{R}$, that is,

$$w(x) \leq x^* \quad \text{for some } x^* \in A(w(x)). \tag{2.9}$$

Proof. It is clear that $C(x)$ has the supremum $w(x) \in E$. For any $u_i \in C(x)/\{x\}$, from $u_i \in A(u_{i-1})$ and $u_{i-1} \leq w(x)$ there exists $x_i \in A(w(x))$ such that $u_i \leq x_i$. We can assume that the sequence $\{x_i\}$ is increasing. Indeed, if $x_i \leq x_{i+1}$ for $i = 1, 2, \ldots$, our purpose is reached. Otherwise, there exists i_0 such that $x_{i_0} \nleq x_{i_0+1}$, then we take x_{i_0+1} instead of x_{i_0}. Let $M = \{w(x), x_1, x_2, \ldots, x_n, \ldots\}$, then $M \subset \text{cl}(\{w(x)\} \cup A(M))$. Condition (H1) guarantees that M has a supremum $x^* = \sup M$. Clearly, $w(x) \leq x^*$. By virtue of the fact that A has sup-closed values, we have $x^* \in A(w(x))$. This proof is complete. □

For the sake of convenience, in this paper, by $w(x)$ we always stand for the supremum of $C(x)$. For given $x \in \mathcal{R}$, let $\mathcal{W}(x)$ be a set consisting of all $w(x)$ given as in Lemma 2.8, then $\mathcal{W}$ is an increasing map. Now for any $u_n \in C(x)$ Lemma 2.4 shows $D(u_n) \subset D(x)$, thus, $\mathcal{W}(u_n) \subset \mathcal{W}(x)$. Define

$$Z = \{D(x) : x \in \mathcal{R}\}. \tag{2.10}$$

It is obvious that $D(u_0) \in Z$. Hence, Z is nonempty. A relation "$\leq_1''$ on Z is defined as follows (it is easy to see that $(Z, \leq_1)$ is a partially ordered set):

$$D(x) = D(y) \Leftrightarrow x = y,\ w(x) = w(y);$$
$$D(x) <_1 D(y) \Leftrightarrow x < y \text{ and } w(x) \leq w(y).$$

Remark 2.9. It is clear we may assume that, for any $u \in D(x)$, there exists $v \in D(y)$ such that $u \leq v$ if $D(x) <_1 D(y)$.

Let us assume that there exists some $u_0 \in \mathcal{R}$ such that

(H3) $\mathcal{W}(u_0) \subset \text{cl}(A(X(u_0)))$.

Define

$$S = \{\mathfrak{D}(x) : x \in \mathcal{R},\ \mathcal{W}(x) \subset \text{cl}(A(X(x)))\}. \tag{2.11}$$

Obviously, $\mathfrak{D}(u_0) \in S$, that is, S is nonempty if A is increasing upwards. Now we denote $\leq_2$ as a relation on S defined by, for any $\mathfrak{D}(x), \mathfrak{D}(y) \in S$,

(I) $\mathfrak{D}(x) = \mathfrak{D}(y) \Leftrightarrow x = y$;

(II) $\mathfrak{D}(x) <_2 \mathfrak{D}(y) \Leftrightarrow (a)$ for all $D(x) \in \mathfrak{D}(x)$, there exists $D(y) \in \mathfrak{D}(y)$ such that $D(x) <_1 D(y)$ and

(b) there exists a countable at most and totally ordered subset $Q \subset \mathcal{R}$ such that

(b_1) $x < q < y$ for any $q \in Q$.

(b_2) There exists $D(x) \in \mathfrak{D}(x)$ such that

$$y \in \mathrm{cl}\left(\{w(x)\} \cup \bigcup_{q \in Q} \mathcal{W}(q) \right), \quad \{y\} \geq \mathcal{W}(q), \quad w(x) \leq q (\forall q \in Q); \tag{2.12}$$

(b_3) $\bigcup_{q\in Q} X(q)$ is a totally ordered set and satisfies $\bigcup_{q\in Q} X(q) \subset \mathrm{cl}(\mathcal{W}(x) \cup A(\bigcup_{q\in Q} X(q)))$. Q is called a link of linking $\mathfrak{D}(x)$ with $\mathfrak{D}(y)$.

Remark 2.10. (b_2) may be satisfied. In fact, we can take empty set as a link of linking $\mathfrak{D}(x)$ and $\mathfrak{D}(y)$. Thus, $\mathfrak{D}(x) <_2 \mathfrak{D}(y)$ implies that for any $D(x) \in \mathfrak{D}(x)$ we can find $D(y) \in \mathfrak{D}(y)$ such that $D(x) <_1 D(y)$. In this case, we take $w(x) = y$. Besides, Q can be a finite set, for example, $Q = \{q_1, q_2, \ldots, q_m\}$ with $q_1 < q_2 < \cdots < q_m$, then $q_1 = \inf \mathcal{W}(x),\ y = \sup \mathcal{W}(q_m)$. ($b_3$) and the condition (H1) ensure $\bigcup_{q\in Q} X(q)$ to exist the supremum, so, from Lemma 2.2 the element y satisfying (b_2) exists.

Lemma 2.11. *The relation "$\leq_2$" satisfies that*

(i) $\mathfrak{D}(x) \leq_2 \mathfrak{D}(x)$;

(ii) $\mathfrak{D}(x) \leq_2 \mathfrak{D}(y)$ *and* $\mathfrak{D}(y) \leq_2 \mathfrak{D}(x)$ *implies* $\mathfrak{D}(x) = \mathfrak{D}(y)$;

(iii) $\mathfrak{D}(x) \leq_2 \mathfrak{D}(y),\ D(y) \leq_2 \mathfrak{D}(z)$ *implies* $\mathfrak{D}(x) \leq_2 \mathfrak{D}(z)$.

Therefore, $(S, \leq_2)$ is a partially ordered set.

Proof. (i) and (ii) are satisfied. Trivial by (I) and (II)(a). To prove (iii), for any given $D(x) \in \mathfrak{D}(x)$ we take $D(y) \in \mathfrak{D}(y)$ such that $D(x) \leq_1 D(y)$ and we can find $D(z) \in \mathfrak{D}(z)$ such that $D(y) \leq_1 D(z)$. It is sufficient to assume that at least one of the above equalities does not hold. The definition of $<_1$ guarantees that

$$x \leq y \leq z, \tag{2.13}$$

$$w(x) \leq w(y) \leq w(z), \tag{2.14}$$

and at least one strictly inequality in (2.13) holds. The links linking $\mathfrak{D}(x)$ with $\mathfrak{D}(y)$ and linking $\mathfrak{D}(y)$ with $\mathfrak{D}(z)$ are written, respectively, as Q' and Q''. Let $Q = Q' \cup Q'' \cup \{y\}$, for any $q' \in Q',\ q'' \in Q''$, if none of equalities in (2.13) holds, then by (b_1) we have

$$x < q' < y, \qquad y < q'' < z. \tag{2.15}$$

If at least one equality in (2.13) holds, for instance, $x = y$, then (b_1) and (b_2) show that

$$w(x) \le q' \le w(y), \quad w(y) \le q'' < z. \tag{2.16}$$

Hence, $Q \subset \mathcal{R}$ is a countable totally ordered subset and satisfies (b_1).

Next we will prove that Q satisfies (b_2). It is clear that the following consequences are true, that is, $z \ge w(y) \ge w(x)$ and $z \in \mathrm{cl}(\{w(y)\} \cup \bigcup_{q'' \in Q''} \mathcal{W}(q'')) \subset \mathrm{cl}(\{w(x)\} \cup \bigcup_{q \in Q} \mathcal{W}(q))$. It is easy to see that $\{z\} \ge \mathcal{W}(q)$ for all $q \in Q$ by $\{z\} \ge \mathcal{W}(q'')$ and $\mathcal{W}(q'') \ge \mathcal{W}(q')$ for all $q' \in Q'$. Also, $w(x) \le q$ for all $q \in Q$.

Finally, we prove that Q satisfies (b_3). For all $x_1,\ x_2 \in \bigcup_{q \in Q} X(q)$, there exist $q', q'' \in Q$ with $q' \le q''$ (because Q is totally ordered) and $D(q') \in \mathfrak{D}(q')$, $D(q'') \in \mathfrak{D}(q'')$ such that $x_1 \in D(q')$, $x_2 \in D(q'')$. If $q', q'' \in Q'$ (or $q', q'' \in Q''$), then x_1 and x_2 are ordered by (b_3). If $q' \in Q'$, $q'' \in Q''$, from (b_1) and (b_2) it follows that $D(q') <_1 D(q'')$, which shows that $x_1 \le w(q') \le q'' \le x_2$. To conclude, $\bigcup_{q \in Q} X(q)$ is totally ordered. Noting that both Q' and Q'' have supremums, by the definition of S, we have

$$\mathcal{W}(y) \subset \mathrm{cl}(A(X(y))). \tag{2.17}$$

Therefore,

$$\begin{aligned}
\bigcup_{q\in Q} X(q) &= \left(\bigcup_{q\in Q'} X(q)\right) \bigcup \left(\bigcup_{q\in Q''} X(q)\right) \bigcup X(y) \\
&\subset \mathrm{cl}\left(\mathcal{W}(x) \cup A\left(\bigcup_{q\in Q'} X(q)\right)\right) \bigcup \mathrm{cl}\left(w(y) \cup A\left(\bigcup_{q\in Q''} X(q)\right)\right) \bigcup X(y) \\
&\subset \mathrm{cl}\left(\mathcal{W}(x) \cup A\left(\bigcup_{q\in Q'} X(q)\right)\right) \bigcup \left(\mathcal{W}(y) \cup A\left(\bigcup_{q\in Q''} X(q)\right)\right) \bigcup \mathrm{cl}(A(X(y))) \\
&\subset \left(\mathcal{W}(x) \cup A\left(\bigcup_{q\in Q} X(q)\right)\right).
\end{aligned} \tag{2.18}$$

This shows that Q satisfies (b_3). Consequently, $\mathfrak{D}(x) <_2 \mathfrak{D}(z)$, which completes this proof. □

3. Main Results

Now we can state and prove our main results.

Definition 3.1. $u \in E$ is said to be a fixed point of the multivalued operator A if $u \in A(u)$. The fixed point x^* of A is said to be a maximal fixed point of A if $u = x^*$ whenever $u \in A(u)$ and $x^* \le u$. If x_* is a fixed point and if $x_* = u$ whenever $u \in A(u)$ and $u \le x_*$, we say that x_* is a minimal fixed point of A.

Theorem 3.2. *Assume that E is an ordered topological space. Let $u_0 \in E$, $\Omega \subset K_{u_0}$ be nonempty and the multivalued operator $A : \Omega \multimap 2^E$ have sup-closed values such that hypotheses (H1)–(H3) hold. Then A admits at least one fixed point in K_{u_0}.*

Proof. If S has a maximal element $\mathfrak{D}(x^*)$, then x^* is a fixed point of A. In fact, since $x^* \in \mathcal{R}$, we can find $u \in A(x^*)$ such that $x^* \le u$. From the definition of $C(x^*)$ we can let $u \in D(x^*) \in \mathfrak{D}(x^*)$. This implies $u \le w(x^*)$. We claim that $x^* = u$. Suppose that $x^* < u$, then $x^* < w(x^*)$ and $D(x^*) <_1 D(w(x^*))$. Take empty set as a link of linking $\mathfrak{D}(x^*)$ with $\mathfrak{D}(w(x^*))$, we have $\mathfrak{D}(x^*) <_2 \mathfrak{D}(w(x^*))$, which contradicts the definition of maximal element.

To prove the existence of maximal element of S, by Zorn's lemma, is thus sufficient to show that every totally ordered subset of S has an upper bound. Let M be any such a subset of S. To this purpose, we consider the set $N = \bigcup_{\mathfrak{D}(x) \in M} X(x)$. Obviously, $N \subset K_{u_0}$. We claim that N is totally ordered. Indeed, for any $y_1, y_2 \in N$, there exist $\mathfrak{D}(x_1), \mathfrak{D}(x_2) \in M$ and $D(x_1) \in \mathfrak{D}(x_1)$, $D(x_2) \in \mathfrak{D}(x_2)$ such that $y_1 \in D(x_1)$, $y_2 \in D(x_2)$. If $D(x_1) = D(x_2)$, then y_1, y_2 is ordered. Otherwise, we can assume that $D(x_1) <_1 D(x_2)$, thus, from Lemma 2.8 and (b_2) it follows that $y_1 \le w(x_1) \le x_2 \le y_2$. Conclusively, N is a totally ordered subset.

We will prove that any countable totally ordered subset of N has a supremum. It is enough to prove that any given strictly monotone sequence $\{y_n\}$ of N there is a supremum. From the definition of N, there exist $\mathfrak{D}(x_n) \in M$ and $D(x_n) \in \mathfrak{D}(x_n)$ such that $y_n \in D(x_n)$ for $n = 1, 2, \ldots$. For any $x \in \mathcal{R}$, from the definition of $D(x)$, it follows $D(x)$ has a supremum. Moreover, Lemma 2.4 guarantees that $\{y_n\}$ has a supremum if $y_n \in D(x_m)$ with $n \ge m$ for some given m. It is suffices to consider the fact that there exists a subsequence of $\{y_n\}$ (without loss of generality, we may assume that it is $\{y_n\}$ itself) such that $y_n \notin D(x_m)$ $(n \ne m)$.

Case 1. If $\{y_n\}$ is strictly increasing, then $y_i < y_{i+1}$ $(i = 1, 2, \ldots)$. We claim that

$$\mathfrak{D}(x_1) <_2 \mathfrak{D}(x_2) <_2 \cdots <_2 \mathfrak{D}(x_n) <_2 \cdots . \tag{3.1}$$

If it is contrary, there exists some i such that $\mathfrak{D}(x_{i+1}) \le_2 \mathfrak{D}(x_i)$. It is easy to know that $\mathfrak{D}(x_{i+1}) \ne \mathfrak{D}(x_i)$. ($b_2$) implies that $w(x_{i+1}) \le x_i$, therefore, $y_{i+1} \le w(x_{i+1}) \le x_i \le y_i$. This contradicts $\{y_n\}$ increasing. The claim follows.

Taking Q_i as the link of linking $\mathfrak{D}(x_{i+1})$ with $\mathfrak{D}(x_i)$ for $i = 1, 2, \ldots$. Let

$$C = \bigcup_{i=1}^{\infty} \left(X(x_i) \cup \left(\bigcup_{q \in Q_i} X(q) \right) \right), \tag{3.2}$$

(b_3) shows that C is countable totally ordered. For any $z \in C \setminus \{x_1\}$, there exists j such that $z \in X(x_j) \cup (\bigcup_{q \in Q_j} X(q))$. If $z = x_j$, by means of (b_2) and

$$\mathcal{W}(x_{j-1}) \in \mathrm{cl}(A(X(x_{j-1}))), \tag{3.3}$$

we have

$$z \in \mathrm{cl}\left(\mathcal{W}(x_{j-1}) \cup A\left(\bigcup_{q\in Q_{j-1}} X(q)\right)\right) \subset \mathrm{cl}\left(A(X(x_{j-1})) \cup A\left(\bigcup_{q\in Q_{j-1}} X(q)\right)\right)$$
$$= \mathrm{cl}\left(A\left(X(x_{j-1}) \cup \left(\bigcup_{q\in Q_{j-1}} X(q)\right)\right)\right) \subset \mathrm{cl}(A(C)). \tag{3.4}$$

If $z \in X_j$ with $z \neq x_j$, then, by (3.3), $z \in \mathrm{cl}(A(X(x_j))) \subset \mathrm{cl}(A(C))$. If $z \in \bigcup_{q\in Q_j} X(q)$ with $z \notin X(x_j)$, then from condition (b$_3$) it follows that $z \in \mathrm{cl}(A(\bigcup_{q\in Q_j} X(q))) \subset \mathrm{cl}(A(C))$. To sum up, $C \subset \{x_1\} \cup \mathrm{cl}(A(C)) \subset \mathrm{cl}(\{x_1\} \cup A(C))$, which, combining the condition (H1), yields that C has a supremum. Hence, by Lemma 2.4, $\{y_n\}$ has a supremum.

Case 2. It is clear that $\{y_n\}$ has a supremum when $\{y_n\}$ is decreasing.

Now, we prove that N has a maximal element. Suppose, on the contrary, for any $y \in N$, that there exists $y_1 \in N$ such that $y \leq y_1$ and $y_1 \neq y$. Let $F(y) = y_1$, then F is an operator mapping N into N and satisfies $F(y) \geq y$ and $F(y) \neq y$ for every $y \in N$. In virtue of Lemma 2.5, there exists $y^* \in N$ such that $F(y^*) = y^*$. On the other hand, by the definition of F, we have $F(y^*) \geq y^*$ and $F(y^*) \neq y^*$, a contradiction. Therefore, N has a maximal element, that is, there exists $x^* \in N$ such that $x \leq x^*$ for all $x \in N$.

Finally, we shall prove that $\mathfrak{D}(x^*)$ is an upper bound of M. Since $x^* \in N$, there exists $\mathfrak{D}(x) \in M$ and $D(x) \in \mathfrak{D}(x)$ such that $x^* \in D(x)$, which implies that $x^* \leq w(x)$. On the other hand, since $w(x) \in N$, we have $w(x) \leq x^*$. This compels $x^* = w(x)$. Taking empty set as a link of linking $\mathfrak{D}(x)$ with $\mathfrak{D}(w(x))$, we have that $\mathfrak{D}(x) \leq_2 \mathfrak{D}(w(x)) = \mathfrak{D}(x^*)$. Given $\mathfrak{D}(u) \in M$, in virtue of M being totally ordered, or $\mathfrak{D}(u) \leq_2 \mathfrak{D}(x)$ which implies $\mathfrak{D}(u) \leq_2 \mathfrak{D}(x^*)$; or $\mathfrak{D}(x) <_2 \mathfrak{D}(u)$, which, applying (b$_2$), yields $w(x) \leq u$. Therefore, $x^* = w(x) \leq u$. Noting that $u \in N$, we have that $u \leq x^*$. Conclusively, $u = x^*$, so, by (a) we have $\mathfrak{D}(x^*) = \mathfrak{D}(u)$. This shows that $\mathfrak{D}(x^*)$ is an upper bound of M. This proof is completed. □

Remark 3.3. We observe that the result of Theorem 3.2 is true under assumptions of Theorem 3.2 if all " cl" are written as "wcl." The following corollary shows that Theorem 3.2 extends and improves the results of [8].

Corollary 3.4. *Let E be an ordered Banach space, $A : \Omega \subset E \to 2^E$ be a multivalued operator having nonempty and weakly closed values. Assume that there exists $u_0 \in E$ such that conditions (H2), (H3) and (H) hold, then A has at least a fixed point.*

Proof. Lemma 2.1 shows that there exists $x^* \in \mathrm{wcl}(C)$ such that $x \leq x^*$ for all $x \in C$. By means of Eberlein's theorem and Lemma 2.2 we have that x^* is the supremum of C, that is, (H1) is satisfied. Moreover, this implies that $w(x) \in \mathrm{wcl}(A(D))$, that is, A is upper sequentially order closed in the sense of "weak." Since A has weakly closed values, A has sup-closed values. From Remark 3.3 A has a fixed point. □

Corollary 3.5. *Let E be a weakly sequently completed ordered Banach space, P a normal cone. If the operator A is bounded and satisfies conditions (H2) and (H3), then A has at least one fixed point in K_{u_0}.*

Proof. It is suffice to prove that condition (H1) holds. Under these hypotheses, every bounded subset is weakly relatively compact (see [4]), which implies that (H1) is true. □

In what follows, we shall consider the existence of maximal and minimal fixed points.

Theorem 3.6. *Under assumptions of Theorem 3.2, A has a minimal fixed point in K_{u_0}.*

Proof. Let $Fix(A)$ denote the set consisting of fixed points of A. From Theorem 3.2 it follows that $Fix(A)$ is nonempty. Set $S_1 = \{\mathfrak{D}(x) : x \in \mathcal{R}, x \leq y \text{ for } y \in Fix(A)\}$. Clearly, $S_1 \subset S$ and $(S_1, \leq_2)$ is a partially ordered set. By the same methods as to prove Theorem 3.2, we can prove that S_1 has a maximal element $\mathfrak{D}(x^*)$ and x^* is a fixed point of A in S_1. It is easy to see that x^* is minimal fixed point of A. This completes the proof of Theorem 3.6. □

The next result is dual to that of Theorem 3.6.

Theorem 3.7. *Assume that E is an ordered topological space. Let $v_0 \in E$, $\Omega \subset K^{v_0} =: \{x \in E : x \leq v_0\}$ be nonempty and the multivalued operator $A : \Omega \multimap 2^E$ have inf-closed values such that the following hypotheses are satisfied.*

- (h1) *If $C = \{x_n\} \subset K^{v_0}$ is a countable totally ordered set and $C \subset \text{cl}(\{x_1\} \cup A(C))$, then C has a infimum.*
- (h2) *A is increasing downwards and $Av_0 \leq v_0$.*
- (h3) *$\mathcal{W}(v_0) \subset \text{cl}(A(X(v_0)))$, where $\mathcal{W}(x)$ stands for a set which consists of all infimums of $C(x)$ (its definition is similar to $\mathcal{W}(x)$).*

Then A admits at least one fixed point in K^{v_0}.

Theorem 3.8. *Assume that the operator A is increasing and satisfies conditions (H1)–(H3) and (h1)–(h3), then A has maximal and minimal fixed points on $[u_0, v_0]$.*

Remark 3.9. If E has the limit ordinal property, A is increasing and has nonempty closed values. Assume that $A([u_0, v_0])$ is relatively sequentially compact and conditions (H3) and (h3) hold, then A has sup-closed and inf-closed values and satisfies conditions (H1) and (h1) on $[u_0, v_0]$. Thereby, A has maximal and minimal fixed points on $[u_0, v_0]$. In this sense, we extend and improve the corresponding results of Theorem 2.1 in [10].

Corollary 3.10. *Let E be a partially ordered Banach space. If there exist $u_0, v_0 \in E$ with $u_0 \leq v_0$ such that $u_0 \leq Au_0$, $Av_0 \leq v_0$. Assume that A is increasing, has nonempty closed values, and satisfies one of the following hypotheses, then A has maximal and minimal fixed points on $[u_0, v_0]$.*

- (s1) *P is a regular cone.*
- (s2) *If $C = \{x_n\} \subset [u_0, v_0]$ is countable totally ordered subset and $C \subset \text{cl}(\{x_1\} \cup A(C))$, then C is relatively compact subset.*
- (s3) *$A([u_0, v_0])$ is a weakly relatively compact set.*
- (s4) *$[u_0, v_0]$ is a bounded ordered interval, and for any countable noncompact subset $C \subset [u_0, v_0]$ with $\alpha(C) \neq 0$, one has $\alpha(A(C)) < \alpha(C)$, where $\alpha(\cdot)$ denotes Kuratowskii's noncompactness measure.*

Proof. (s2) implies (H1) and (h1) holds. The rest is clear. □

Remark 3.11. (s2) is main condition of [12] for single-valued operators, (s4) is main condition of [13]. Hence the results presented here extend and improve the corresponding results of the above mentioned papers.

4. Application

In this section we assume that $(E, \|\cdot\|)$ is a Banach space with partial ordering derived by the continuous bounded function $\varphi : E \to \mathbf{R}$ as follows (see [14]):

$$x \leq y \quad \text{iff } \|x - y\| \leq \varphi(x) - \varphi(y). \tag{4.1}$$

To illustrate the ideas involved in Theorem 3.8 we discuss the Hammerstein integral inclusions of the form

$$u(t) \in \int_0^T k(t,s) G(s, u(s)) ds \quad \text{on } [0, T]. \tag{4.2}$$

Here k is a real single-valued function, while $G : [0, T] \times E \to 2^E$ is a multivalued map with nonempty closed values.

Let $0 < T < \infty$, $I = [0, T]$, $p \in [1, \infty]$, $q \in [0, \infty]$ and $r \in [1, \infty]$ be the conjugate exponent of q, that is, $1/q + 1/r = 1$. Let $\|u\|_p = (\int_0^T \|u(s)\|^p ds)^{1/p}$ denote the norm of the space $L^p(I, E)$. For $u, v \in L^p(I, E)$ stipulate that $u \leq v$ if and only if $u(t) \leq v(t)$ with all $t \in I$.

In order to prove the existence of solutions to (4.2) in $L^p(I, E)$ we assume the following.

(S1) The function $k : I^2 \to \mathbf{R}_+$ satisfies that $k(t, \cdot) \in L^r(I)$ and $t \to \|k(t, \cdot)\|_r$ belongs to $L^p(I)$.

(S2) $G(t, u)$ is increasing with regard to u for fixed $t \in [0, T]$.

(S3) There exist $u_0, v_0 \in C(I, E)$ with $u_0 \leq v_0$ such that $u_0(t) \leq G(t, u_0(t))$ and $G(t, v_0(t)) \leq v_0(t)$ for every $t \in I$.

(S4) $G(\cdot, x)$ has a strongly measurable selection on I for each $x \in E$.

(S5) $\sup\{\|u(t)\| : u(t) \in G(t, x)\} \leq h(t)$ a.e. on I for all $x \in E$. Here $h \in L^q(I, \mathbf{R}_+)$.

Theorem 4.1. *Assume that conditions (S1)–(S5) hold, then* (4.2) *has maximal and minimal solutions in* $[u_0, v_0]$.

Proof. Define a multivalued operator A as follows:

$$(Ax)(t) = \int_0^T k(t,s) G(s, x(s)) ds. \tag{4.3}$$

(S4) guarantees that A makes sense. For any $v \in Ax$ with $x \in L^p(I, E)$, there exists $u \in G(\cdot, x)$

such that $v(t) = \int_0^T k(t,s)u(s)ds$. From (S5) and Hölder inequality, it follows that

$$\|v(t)\| \leq \int_0^T k(t,s)\|u(s)\|ds \leq \int_0^T h(s)k(t,s)ds \leq \|h\|_q \|k(t,\cdot)\|_r =: a(t). \tag{4.4}$$

This implies that $v \in L^p(I,E)$, that is, A maps $L^p(I,E)$ into itself. We seek to apply Theorem 3.8. Note that (S1) and (S3) guarantee that $u_0 \leq A(u_0)$, $A(v_0) \leq v_0$. For every given $t \in I$ and any $C(u_0) \in \mathcal{C}(u_0)$, set $C(u_0)(t) = \{x_n(t)\}$ with $x_n(t) \leq x_{n+1}(t)$ for $n = 1,2,\ldots$. Thus, $\{\varphi(x_n(t))\}$ is a decreasing sequence. Note that φ is a bounded function, we obtain that the sequence $\{\varphi(x_n(t))\}$ is convergent. Hence, for any $\varepsilon > 0$, there exists a natural number n_0 such that

$$\|x_m(t) - x_n(t)\| \leq \varphi(x_n(t)) - \varphi(x_m(t)) < \varepsilon, \tag{4.5}$$

whenever $m > n \geq n_0$. This shows that $\{x_n(t)\}$ is a Cauchy sequence, thereby $\{x_n(t)\}$ is convergent. Lemma 2.2 guarantees $\sup(C(u_0)(t)) = w(u_0)(t) = \lim_{n\to\infty} x_n(t)$, which yields $w(u_0)(t) \in \operatorname{cl}(A(C(u_0))(t)) \subset \operatorname{cl}(A(X(u_0))(t))$. From the arbitrariness of t it follows that $w(u_0)$ is supremum of $C(u_0)$. From (4.4) and the dominated convergence theorem, it follows that $w(u_0) \in L^p(I,E)$. Moreover, $\|x_n - w(u_0)\|_p \to 0$ for $n \to \infty$. Consequently, $\mathcal{W}(u_0) \subset \operatorname{cl}(A(X(u_0)))$. Similarly, we have $\mathcal{W}(v_0) \subset \operatorname{cl}(A(X(v_0)))$. This shows that (H3) and (h3) are satisfied for $t \in I$. (S2) guarantees that A is increasing. It is easy to see that A has closed values. This yields that A has sup-closed and inf-closed values.

Finally, we check conditions (H1) and (h1). Suppose that the set $C = \{x_n\} \subset K_{u_0}$ is countable, totally ordered, and satisfies $C(t) \subset \operatorname{cl}(\{x_1(t)\} \cup (AC)(t))$ for all $t \in I$. We have to prove that the set $C(t)$ has a supremum. Since $C(t)$ is countable totally ordered, we can assume $C(t) = \{x_n(t) : n \geq 1\}$ with $x_n(t) \leq x_{n+1}(t)$ for $n = 1,2,\ldots$. This implies that the sequence $\{\varphi(x_n(t))\}$ is decreasing. In the same way, we can prove that the sequence $\{x_n(t)\}$ is convergent. Again, Lemma 2.2 guarantees that $C(t)$ has a supremum, which implies that condition (H1) is satisfied. Similarly, we can prove that condition (h1) holds. All conditions of Theorem 3.8 are satisfied, consequently, the operator A has minimum and maximum fixed points in $[u_0, v_0]$ and this proof is completed. □

Remark 4.2. By comparing the results of Theorem 4.2 in [15] in which Couchouron and Precup have proved that (4.2) has at least one solution, we omit the conditions that $G(t,x)$ is continuous and has compact values in Theorem 4.1.

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