

Research Article

Convergence of the Sequence of Successive Approximations to a Fixed Point

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If (X, d) is a complete metric space and T is a contraction on X , then the conclusion of the Banach-Caccioppoli contraction principle is that the sequence of successive approximations $\{T^n x\}$ of T starting from any point $x \in X$ converges to a unique fixed point. In this paper, using the concept of τ -distance, we obtain simple, sufficient, and necessary conditions of the above conclusion.

1. Introduction

The following famous theorem is referred to as the *Banach-Caccioppoli contraction principle*. This theorem is very forceful and simple, and it became a classical tool in nonlinear analysis.

Theorem 1.1 (see Banach [1] and Caccioppoli [2]). *Let (X, d) be a complete metric space and let T be a self contraction on X , that is, there exists $r \in [0, 1)$ such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then the following holds.*

(A) *T has a unique fixed point z , and $\{T^n x\}$ converges to z for any $x \in X$.*

We note that the conclusion of Kannan's fixed point theorem [3] is also (A). See Kirk's survey [4]. Recently, we obtained that (A) holds if and only if T is a strong Leader mapping [5, 6].

Theorem 1.2 (see [6]). *Let T be a mapping on a complete metric space (X, d) . Then the following are equivalent.*

- (i) *(A) holds.*
- (ii) *T is a strong Leader mapping, that is, the following hold.*

(a) For $x, y \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that

$$d(T^i x, T^j y) < \varepsilon + \delta \implies d(T^{i+\nu} x, T^{j+\nu} y) < \varepsilon, \quad (1.1)$$

for all $i, j \in \mathbb{N} \cup \{0\}$, where T^0 is the identity mapping on X .

(b) For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that

$$d(T^i x, T^j y) < \alpha_n \implies d(T^{i+\nu} x, T^{j+\nu} y) < \frac{1}{n}, \quad (1.2)$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

The following theorem is proved in [7, 8].

Theorem 1.3 (see Rus [7] and Subrahmanyam [8]). *Let (X, d) be a complete metric space and let T be a continuous mapping on X . Assume that there exists $r \in [0, 1)$ satisfying $d(Tx, T^2x) \leq rd(x, Tx)$ for all $x \in X$. Then the following holds.*

(B) $\{T^n x\}$ converges to a fixed point for every $x \in X$.

We obtained a condition equivalent to (B) in [9].

Theorem 1.4 (see [9]). *Let T be a mapping on a complete metric space (X, d) . Then the following are equivalent.*

(i) (B) holds.

(ii) The following hold.

(a) For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that

$$d(T^i x, T^j x) < \varepsilon + \delta \implies d(T^{i+\nu} x, T^{j+\nu} x) < \varepsilon, \quad (1.3)$$

for all $i, j \in \mathbb{N} \cup \{0\}$.

(b) For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that

$$d(T^i x, T^j y) < \alpha_n \implies d(T^{i+\nu} x, T^{j+\nu} y) < \frac{1}{n}, \quad (1.4)$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

We sometimes call a mapping satisfying (A) a *Picard operator* [10]. We also call a mapping satisfying (B) a *weakly Picard operator* [11–13].

We cannot tell that the conditions (ii) of Theorems 1.2 and 1.4 are simple. Motivated by this, we obtain simpler conditions which are equivalent to Conditions (A) and (B).

2. Preliminaries

Throughout this paper, we denote by \mathbb{N} , \mathbb{Z} , and \mathbb{R} the sets of positive integers, integers and real numbers, respectively.

In 2001, Suzuki introduced the concept of τ -distance in order to improve results in Tataru [14], Zhong [15, 16], and others. See also [17].

Definition 2.1 (see [18]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a τ -distance on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

- ($\tau 1$) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$,
- ($\tau 2$) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable,
- ($\tau 3$) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$,
- ($\tau 4$) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply that $\lim_n \eta(y_n, t_n) = 0$,
- ($\tau 5$) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ -distance on X . Many useful examples and propositions are stated in [9, 18–23] and references therein. The following fixed point theorems are proved in [18].

Theorem 2.2 (see [18]). *Let X be a complete metric space and let T be a mapping on X . Assume that there exist a τ -distance p and $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ for all $x \in X$. Assume the following.*

- (i) *If $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$, then $Ty = y$.*

Then (B) holds. Moreover, if $Tz = z$, then $p(z, z) = 0$.

Theorem 2.3 (see [18]). *Let X be a complete metric space and let T be a mapping on X . Assume that T is a contraction with respect to some τ -distance p , that is, there exist a τ -distance p and $r \in [0, 1)$ such that*

$$p(Tx, Ty) \leq rp(x, y), \quad (2.1)$$

for all $x, y \in X$. Then (A) and $p(z, z) = 0$ hold.

The following lemmas are useful in our proofs.

Lemma 2.4 (see [18]). *Let (X, d) be a metric space and let p be a τ -distance on X . If sequences $\{x_n\}$ and $\{y_n\}$ in X satisfy $\lim_n p(z, x_n) = 0$ and $\lim_n p(z, y_n) = 0$ for some $z \in X$, then $\lim_n d(x_n, y_n) = 0$. In particular for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply that $x = y$.*

Lemma 2.5 (see [18]). *Let (X, d) be a metric space and let p be a τ -distance on X . If a sequence $\{x_n\}$ in X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence. Moreover if a sequence $\{y_n\}$ in X satisfies $\lim_n p(x_n, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$.*

The following lemmas are easily deduced from Lemmas 2.4 and 2.5.

Lemma 2.6. Let (X, d) be a metric space and let p be a τ -distance on X . Then for every $z \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply that $d(x, y) \leq \varepsilon$.

Lemma 2.7. Let X be a metric space and let p be a τ -distance on X . Assume that a sequence $\{x_n\}$ in X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, y) = 0$, and $\lim_n p(x_n, z) = 0$. Then $y = z$.

The following is proved at Page 442 of [18]. However we give a proof because we use reductio ad absurdum in [18].

Lemma 2.8 (see [18]). Let g be a nondecreasing function from $[0, \infty)$ into itself satisfying $\inf\{g(t) : t > 0\} = 0$. Define a function f from $[0, \infty)$ into itself by

$$f(t) = t + \sup \left\{ \sum_{i=1}^n \alpha_i \min\{g(s_i), 1\} : t = \sum_{i=1}^n \alpha_i s_i, s_i \geq 0, \alpha_i > 0, \sum_{i=1}^n \alpha_i = 1 \right\}. \quad (2.2)$$

Then $f(0) = 0$, $f(t) \geq t + g(t)$ for all $t \in [0, \infty)$; and f is concave and continuous.

Proof. It is clear that $f(0) = 0$, $f(t) \geq t + g(t)$, and f is concave. We shall prove that f is continuous at 0. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $g(\delta) \leq \varepsilon$. Choose $\tau > 0$ with $\tau + \tau/\delta \leq \varepsilon$. Fix $t \in (0, \tau)$. Let $\alpha_1, \alpha_2, \dots, \alpha_n > 0$ and $s_1, s_2, \dots, s_n \geq 0$ such that $t = \sum_{i=1}^n \alpha_i s_i$ and $\sum_{i=1}^n \alpha_i = 1$. Since $\delta \sum\{\alpha_i : s_i \geq \delta\} \leq t$, we have

$$\begin{aligned} t + \sum_{i=1}^n \alpha_i \min\{g(s_i), 1\} &\leq t + \sum_{s_i < \delta} \alpha_i g(s_i) + \sum_{s_i \geq \delta} \alpha_i \leq t + \sum_{s_i < \delta} \alpha_i \varepsilon + \sum_{s_i \geq \delta} \alpha_i \\ &\leq t + \varepsilon + \frac{t}{\delta} \leq \tau + \varepsilon + \frac{\tau}{\delta} \leq 2\varepsilon. \end{aligned} \quad (2.3)$$

Since $\alpha_1, \alpha_2, \dots, \alpha_n > 0$ and $s_1, s_2, \dots, s_n \geq 0$ are arbitrary, we obtain $f(t) \leq 2\varepsilon$. Thus, $\lim_{t \rightarrow +0} f(t) = 0 = f(0)$. \square

The following is obvious.

Lemma 2.9. Let T be a mapping on a set X . Let A_0 be a subset of X such that $T(A_0) \subset A_0$. Define a sequence $\{A_n\}$ of subsets of X by

$$A_1 = T^{-1}(A_0) \setminus A_0, \quad A_{n+1} = T^{-1}(A_n). \quad (2.4)$$

Then the following hold.

- (i) For every $n \in \mathbb{N}$ and $x \in X$, $x \in A_n$ if and only if $T^j x \notin A_0$ for $j = 0, 1, \dots, n-1$ and $T^n x \in A_0$.
- (ii) $A_m \cap A_n = \emptyset$ for $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$.
- (iii) $T(A_{n+1}) = A_n$ for every $n \in \mathbb{N}$.

3. Condition (B)

In this section, we discuss Condition (B).

Theorem 3.1. *Let X be a complete metric space and let T be a mapping on X . Assume that there exist a τ -distance p , $r \in [0, 1)$, and $M \in [0, \infty)$ such that*

$$p(Tx, T^2x) \leq rp(x, Tx), \quad p(Tx, Ty) \leq Mp(x, y), \quad (3.1)$$

for all $x, y \in X$. Then (B) holds. Moreover, if $Tz = z$, then $p(z, z) = 0$.

Proof. Assume that $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$. Then we have

$$p(x_n, Ty) \leq p(x_n, Tx_n) + p(Tx_n, Ty) \leq p(x_n, Tx_n) + Mp(x_n, y), \quad (3.2)$$

and hence, $\lim_n p(x_n, Ty) = 0$. By Lemma 2.7, we obtain $Ty = y$. By Theorem 2.2, we obtain the desired result. \square

As a direct consequence of Theorem 3.1, we obtain the following.

Corollary 3.2. *Let X be a complete metric space and let T be a mapping on X . Assume that there exist a τ -distance p and $r \in (0, 1)$ such that*

$$p(Tx, T^2x) \leq rp(x, Tx), \quad p(Tx, Ty) \leq p(x, y), \quad (3.3)$$

for all $x, y \in X$. Then (B) holds.

Corollary 3.2 characterizes Condition (B).

Theorem 3.3. *Let T be a mapping on a metric space (X, d) such that (B) holds. Then there exist a τ -distance p and $r \in (0, 1)$ satisfying (3.3).*

Proof. Let $r \in (0, 1)$ be fixed. We note that every periodic point is a fixed point. That is, if $x \in X$ satisfies $T^n x = x$ for some $n \in \mathbb{N}$, then $Tx = x$. Define a mapping T^∞ from X onto $F(T)$ by $T^\infty x = \lim_n T^n x$ for $x \in X$, where $F(T)$ is the set of all fixed points of T . Define a mapping C from X into the set of subsets of X by

$$Cx = \{Tx, T^2x, T^3x, \dots, T^\infty x\}. \quad (3.4)$$

Since $T^\infty x$ is a fixed point of T , we have

$$y \in Cx \implies Cy \subset Cx. \quad (3.5)$$

Next, we define a function f from X into $\mathbb{Z} \cup \{\infty\}$ satisfying

$$f(Tx) \geq f(x) + 1, \quad f(x) = \infty \iff Tx = x, \quad (3.6)$$

for all $x \in X$. We put $f(x) = \infty$ for $x \in F(T)$. It is obvious that $f(Tx) = f(x) = \infty = f(x) + 1$ for $x \in F(T)$. Define a sequence $\{A_n\}$ of subsets of X by

$$A_1 = T^{-1}(F(T)) \setminus F(T), \quad A_{n+1} = T^{-1}(A_n). \quad (3.7)$$

Then by Lemma 2.9,

$$F(T) \cap A_n = \emptyset, \quad A_m \cap A_n = \emptyset, \quad (3.8)$$

for $m, n \in \mathbb{N}$ with $m \neq n$. We put $f(x) = -n$ for $x \in A_n$. We note that

$$f(Tx) = \begin{cases} \infty & \text{if } x \in A_1, \\ f(x) + 1 & \text{if } x \in \bigsqcup_{n=2}^{\infty} A_n. \end{cases} \quad (3.9)$$

Put

$$Y = X \setminus \left(F(T) \sqcup \left(\bigsqcup_{n \in \mathbb{N}} A_n \right) \right). \quad (3.10)$$

It is obvious that $T(Y) \subset Y$, $T^{-1}(Y) = Y$, and $Y \cap F(T) = \emptyset$. So,

$$T^m x = T^n x \iff m = n, \quad (3.11)$$

for $x \in Y$ and $m, n \in \mathbb{N} \cup \{0\}$. Define an equivalence relation \sim on Y as follows: $x \sim y$ if and only if there exist $m, n \in \mathbb{N} \cup \{0\}$ such that $T^m x = T^n y$. By Axiom of Choice, there exists a mapping B on Y such that

$$Bx \sim x, \quad x \sim y \iff Bx = By. \quad (3.12)$$

Let $u \in Y$ with $Bu = u$. Then we put $f(T^n u) = n$ for $n \in \mathbb{N} \cup \{0\}$. Define a sequence $\{D_n\}$ of subsets of Y by

$$D_0 = \{u, Tu, T^2u, T^3u, \dots\}, \quad D_1 = T^{-1}(D_0) \setminus D_0, \quad D_{n+1} = T^{-1}(D_n). \quad (3.13)$$

Then we have $D_m \cap D_n = \emptyset$ for $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$; and

$$\{x \in Y : x \sim u\} = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} D_n. \quad (3.14)$$

We put $f(x) = -n$ for $x \in Y$ with $n \in \mathbb{N}$ and $x \in D_n$. We have defined f . We note that $f(x) \in \mathbb{N}$ implies that $x \in Y$.

Next, we define a τ -distance p by

$$p(x, y) = \begin{cases} r^{f(x)} + r^{f(y)} & \text{if } y \in Cx, \\ r^{f(x)} + r^{f(y)} + 1 & \text{if } y \notin Cx, \end{cases} \quad (3.15)$$

where $r^\infty = 0$. We note that $p(x, y) < 1$ implies either of the following.

- (i) $Tx = x = y$.
- (ii) There exist $u \in Y$, $k \in \mathbb{N}$, and $\ell \in \mathbb{N} \cup \{\infty\}$ such that $Bu = u$, $k < \ell$, $x = T^k u$, and $y = T^\ell u$. (In this case, $x \in Y$, $u = Bx$, $f(x) = k$, and $f(y) = \ell$ hold.)

We shall show that p is a τ -distance. Let $x, y, z \in X$. If $y \in Cx$ and $z \in Cy$, then $z \in Cx$. So we have

$$p(x, z) = r^{f(x)} + r^{f(z)} \leq r^{f(x)} + r^{f(y)} + r^{f(y)} + r^{f(z)} = p(x, y) + p(y, z). \quad (3.16)$$

If $y \notin Cx$ or $z \notin Cy$, then

$$p(x, z) \leq r^{f(x)} + r^{f(z)} + 1 \leq r^{f(x)} + r^{f(y)} + r^{f(y)} + r^{f(z)} + 1 \leq p(x, y) + p(y, z). \quad (3.17)$$

These imply $(\tau 1)$. We shall define a function η from $X \times [0, \infty)$ into $[0, \infty)$. For $x \in X \setminus Y$, we put $\eta(x, t) = t$. For $x \in Y$, we put $u = Bx$. Since $\{T^n u\}$ converges to $T^\infty u$, there exists a strictly increasing sequence $\{h_u(n)\}$ in \mathbb{N} such that $j \geq h_u(n)$ implies that $d(T^j u, T^\infty u) \leq 1/n$ for $j \in \mathbb{N} \cup \{\infty\}$. Since $\lim_n h_u(n) = \infty$, we can define a nondecreasing function g_u from $[0, \infty)$ into $[0, 1]$ such that $g_u(r^{h_u(n)}) = 1/n$. It is obvious that $g_u(0) = \lim_{t \rightarrow +0} g_u(t) = 0$. Put

$$\eta(x, t) = t + \sup \left\{ \sum_{i=1}^n \alpha_i g_u(s_i) : t = \sum_{i=1}^n \alpha_i s_i, s_i \geq 0, \alpha_i > 0, \sum_{i=1}^n \alpha_i = 1 \right\}. \quad (3.18)$$

Then $\eta(x, t)$ satisfies $(\tau 2)$ and $\eta(x, t) \geq t + g_u(t)$ by Lemma 2.8. In order to show $(\tau 3)$, we assume that $\lim_n x_n = x$ and $\lim_n \sup \{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$. Then without loss of generality, we may assume that $\sup \{\eta(z_n, p(z_n, x_m)) : m \geq n\} < 1$. Thus $\sup \{p(z_n, x_m) : m \geq n\} < 1$ for $n \in \mathbb{N}$. It is obvious that $x_m \in Cz_n$ for $m, n \in \mathbb{N}$ with $m \geq n$. We consider the following two cases.

- (i) There exists $\nu \in \mathbb{N}$ such that $x_n \in F(T)$ for $n \geq \nu$.
- (ii) There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \notin F(T)$.

In the first case, since $F(T) \cap Cz_\nu$ exactly consists of one element and $x_n \in F(T) \cap Cz_\nu$ for $n \geq \nu$, $x_n = x_\nu$ holds for all $n \geq \nu$. So $x = x_\nu$. Thus, $p(w, x) = \lim_n p(w, x_n)$ holds for every $w \in X$. In the second case, we note that $z_n \notin F(T)$ for all $n \in \mathbb{N}$. Hence $z_n \in Y$. Put $u = Pz_1$. Since $x_n \in Cz_1$, there exists a sequence $\{\ell_n\}$ in $\mathbb{N} \cup \{\infty\}$ such that $x_n = T^{\ell_n} u$. Since $\ell_{n_j} \in \mathbb{N}$ for all $j \in \mathbb{N}$, there exists a sequence $\{k_n\}$ in \mathbb{N} such that $z_n = T^{k_n} u$. Since $\lim_n p(z_n, x_n) = 0$, we

have $\lim_n k_n = \infty$ and $\lim_n \ell_n = \infty$. So we obtain $x = T^\infty u$. We note that $x \in Cx_n$ for all $n \in \mathbb{N}$. Let $w \in X$. In the case where $x \in Cw$, we have

$$p(w, x) = r^{f(w)} = \lim_{n \rightarrow \infty} \left(r^{f(w)} + r^{f(x_n)} \right) \leq \liminf_{n \rightarrow \infty} p(w, x_n). \quad (3.19)$$

In the other case, where $x \notin Cw$, we have $x_n \notin Cw$, and hence,

$$p(w, x) = r^{f(w)} + 1 = \lim_{n \rightarrow \infty} \left(r^{f(w)} + r^{f(x_n)} + 1 \right) = \lim_{n \rightarrow \infty} p(w, x_n). \quad (3.20)$$

Therefore we have shown $(\tau 3)$. Let us prove $(\tau 4)$. We assume that $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$. Without loss of generality, we may assume that $\sup\{p(x_n, y_m) : m \geq n\} < 1$. We consider the following two cases.

- (i) There exists $\nu \in \mathbb{N}$ such that $y_n \in F(T)$ for $n \geq \nu$.
- (ii) There exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \notin F(T)$.

In the first case, we have

$$\lim_{n \rightarrow \infty} \eta(y_n, t_n) = \lim_{n \rightarrow \infty} t_n \leq \lim_{n \rightarrow \infty} \eta(x_n, t_n) = 0. \quad (3.21)$$

In the second case, as in the proof of $(\tau 3)$, there exist $u \in Y$, a sequence $\{k_n\}$ in \mathbb{N} , and a sequence $\{\ell_n\}$ in $\mathbb{N} \cup \{\infty\}$ such that $Bu = u$, $x_n = T^{k_n}u$, and $y_n = T^{\ell_n}u$. We note that $\eta(x_n, t) = \eta(u, t)$. If $y_n \in F(T)$, then $\eta(y_n, t) = t \leq \eta(u, t)$. If $y_n \notin F(T)$, then $\eta(y_n, t) = \eta(u, t)$. Therefore

$$\lim_{n \rightarrow \infty} \eta(y_n, t_n) \leq \lim_{n \rightarrow \infty} \eta(u, t_n) = \lim_{n \rightarrow \infty} \eta(x_n, t_n) = 0. \quad (3.22)$$

Let us prove $(\tau 5)$. We assume that $\eta(z, p(z, x)) < 1/n$. We note that $p(z, x) < 1$. In the case where $Tz = z = x$, we have $d(z, x) = 0 < 1/n$. In the other case, where there exist $u \in Y$, $k \in \mathbb{N}$, and $\ell \in \mathbb{N} \cup \{\infty\}$ such that $Bu = u$, $k < l$, $z = T^k u$, and $x = T^\ell u$, we have

$$\eta(z, p(z, x)) < \frac{1}{n} = g_u \left(r^{h_u(n)} \right) \leq \eta \left(z, r^{h_u(n)} \right). \quad (3.23)$$

Hence

$$r^k + r^\ell = p(z, x) < r^{h_u(n)}. \quad (3.24)$$

Thus, we obtain $k > h_u(n)$ and $\ell > h_u(n)$. So we have

$$d(z, x) = d \left(T^k u, T^\ell u \right) \leq d \left(T^k u, T^\infty u \right) + d \left(T^\ell u, T^\infty u \right) \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}. \quad (3.25)$$

Therefore

$$\eta(z, p(z, x)) < \frac{1}{n}, \quad \eta(z, p(z, y)) < \frac{1}{n} \implies d(x, y) \leq \frac{4}{n}, \quad (3.26)$$

which imply $(\tau 5)$. Therefore we have shown that p is a τ -distance on X .

We shall show (3.3). Let $x, y \in X$. Since $Tx \in Cx, T^2x \in C(Tx), f(Tx) \geq f(x) + 1$, and $f(T^2x) \geq f(Tx) + 1$, we have

$$p(Tx, T^2x) = r^{f(Tx)} + r^{f(T^2x)} \leq r^{f(x)+1} + r^{f(Tx)+1} = rp(x, Tx). \quad (3.27)$$

If $y \in Cx$, then $Ty \in C(Tx)$ holds. So we have

$$p(Tx, Ty) = r^{f(Tx)} + r^{f(Ty)} \leq r^{f(x)+1} + r^{f(y)+1} = rp(x, y) \leq p(x, y). \quad (3.28)$$

If $y \notin Cx$, then we have

$$p(Tx, Ty) \leq r^{f(Tx)} + r^{f(Ty)} + 1 \leq r^{f(x)} + r^{f(y)} + 1 = p(x, y). \quad (3.29)$$

Therefore (3.3) holds. \square

Remark 3.4. We have proved that, for every $r \in (0, 1)$, there exists a τ -distance p satisfying (3.3).

Combining Theorem 6 in [9], we obtain the following.

Corollary 3.5. *Let T be a mapping on a complete metric space (X, d) . Then the following are equivalent.*

- (i) *(B) holds.*
- (ii) *There exists a τ -distance p on X satisfying the following.*

- (a) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that*

$$p(T^i x, T^j x) < \varepsilon + \delta \implies p(T^{i+\nu} x, T^{j+\nu} x) < \varepsilon, \quad (3.30)$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.

- (b) *For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that*

$$p(T^i x, T^j y) < \alpha_n \implies p(T^{i+\nu} x, T^{j+\nu} y) < \frac{1}{n}, \quad (3.31)$$

for all $n \in \mathbb{N}$ and $i, j \in \mathbb{N} \cup \{0\}$ with $i > j$.

- (iii) *There exist a τ -distance p and $r \in (0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ and $p(Tx, Ty) \leq p(x, y)$ for all $x, y \in X$.*

4. Condition (A)

In this section, we discuss Condition (A).

Define a relation $\underset{0}{<}$ on $[0, \infty)$ as follows: $s \underset{0}{<} t$ if and only if either $s = t = 0$ or $s < t$ holds.

Theorem 4.1. *Let X be a complete metric space and let T be a mapping on X . Assume that there exist a τ -distance p and $r \in (0, 1)$ such that*

$$p(Tx, T^2x) \leq rp(x, Tx), \quad p(Tx, Ty) \underset{0}{<} p(x, y), \quad (4.1)$$

for all $x, y \in X$. Then (A) holds.

Proof. By Theorem 3.1, (B) holds. Moreover, if $Tx = x$, then $p(x, x) = 0$. Let $z, w \in X$ be fixed points of T . Then

$$p(z, w) = p(Tz, Tw) \underset{0}{<} p(z, w), \quad (4.2)$$

which implies that $p(z, w) = 0$. Since $p(z, z) = 0$, we obtain $z = w$ by Lemma 2.4. Thus the fixed point is unique. \square

Theorem 4.2. *Let X be a complete metric space and let T be a mapping on X . Assume that there exist a τ -distance p and $r \in (0, 1)$ such that*

$$p(Tx, T^2x) \leq rp(x, Tx), \quad p(Tx, Ty) < p(x, y), \quad (4.3)$$

for all $x, y \in X$ with $x \neq y$. Then (A) holds.

Proof. In the case where X consists of one element, the conclusion obviously holds. So we consider the other case. Assume that $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$. We consider the following two cases:

- (i) $x_n \neq y$ for sufficiently large $n \in \mathbb{N}$,
- (ii) there exists a sequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} = y$.

In the first case, we have

$$p(x_n, Ty) \leq p(x_n, Tx_n) + p(Tx_n, Ty) < p(x_n, Tx_n) + p(x_n, y) \quad (4.4)$$

for sufficiently large n , and hence, $\lim_n p(x_n, Ty) = 0$. By Lemma 2.7, we obtain $Ty = y$. In the second case, we have

$$p(y, Ty) = \lim_{j \rightarrow \infty} p(x_{n_j}, Tx_{n_j}) = 0, \quad p(y, y) = \lim_{j \rightarrow \infty} p(x_{n_j}, y) = 0. \quad (4.5)$$

By Lemma 2.4, we obtain $Ty = y$. By Theorem 2.2, (B) holds. Let $z, w \in X$ be distinct fixed points of T . Then

$$p(z, w) = p(Tz, Tw) < p(z, w), \quad (4.6)$$

which implies a contradiction. Thus the fixed point is unique. \square

We shall show that Theorems 4.1 and 4.2 characterize Condition (A).

Theorem 4.3. *Let T be a mapping on a metric space (X, d) such that (A) holds. Then there exist a τ -distance p and $r \in (0, 1)$ satisfying (4.1).*

Proof. Let p, r, f , and C be as in the proof of Theorem 3.3. Then $p(Tx, T^2x) \leq rp(x, Tx)$ holds. Fix $x, y \in X$. We consider the following two cases:

- (i) $Tx = x$ and $Ty = y$,
- (ii) either $Tx \neq x$ or $Ty \neq y$.

In the first case, $x = y$ holds by (A). Since

$$p(x, x) = p(Tx, T^2x) \leq rp(x, Tx) = p(x, x), \quad (4.7)$$

we obtain $p(x, x) = 0$. Thus, $p(Tx, Ty) = p(x, y) = 0$. In the second case, we note that either $f(x) \in \mathbb{Z}$ or $f(y) \in \mathbb{Z}$ holds. Thus

$$r^{f(x)+1} + r^{f(y)+1} < r^{f(x)} + r^{f(y)}. \quad (4.8)$$

If $y \in Cx$, then $Ty \in C(Tx)$ holds. So we have

$$p(Tx, Ty) = r^{f(Tx)} + r^{f(Ty)} \leq r^{f(x)+1} + r^{f(y)+1} < r^{f(x)} + r^{f(y)} = p(x, y). \quad (4.9)$$

If $y \notin Cx$, then we have

$$p(Tx, Ty) \leq r^{f(Tx)} + r^{f(Ty)} + 1 < r^{f(x)} + r^{f(y)} + 1 = p(x, y). \quad (4.10)$$

Therefore (4.1) holds. \square

Theorem 4.4. *Let T be a mapping on a metric space (X, d) such that (A) holds. Then there exist a τ -distance p and $r \in (0, 1)$ satisfying (4.3) for all $x, y \in X$ with $x \neq y$.*

Proof. The proof of Theorem 4.3 works. \square

Combining Theorem 7 in [9], we obtain the following.

Corollary 4.5. *Let T be a mapping on a complete metric space (X, d) . Then the following are equivalent.*

- (i) (A) holds.
- (ii) There exists a τ -distance p on X satisfying the following.
 - (a) For $x, y \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that

$$p(T^i x, T^j y) < \varepsilon + \delta \implies p(T^{i+\nu} x, T^{j+\nu} y) < \varepsilon, \quad (4.11)$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.

- (b) For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that

$$p(T^i x, T^j y) < \alpha_n \implies p(T^{i+\nu} x, T^{j+\nu} y) < \frac{1}{n}, \quad (4.12)$$

for all $n \in \mathbb{N}$ and $i, j \in \mathbb{N} \cup \{0\}$ with $i > j$.

- (iii) There exist a τ -distance p and $r \in (0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ and $p(Tx, Ty) < p(x, y)$ for all $x, y \in X$.
- (iv) There exist a τ -distance p and $r \in (0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ and $p(Tx, Ty) < p(x, y)$ for all $x, y \in X$ with $x \neq y$.

5. Additional Result

Since Theorem 2.2 deduces Corollary 3.2, we can tell that Theorem 2.2 characterizes Condition (B). However, the following example tells that Theorem 2.3 does not characterize Condition (A).

Example 5.1. Let A be the set of all real sequences $\{a_n\}$ such that $a_n \in (0, \infty)$ for $n \in \mathbb{N}$, $\{a_n\}$ is strictly decreasing, and $\{a_n\}$ converges to 0. Let H be a Hilbert space consisting of all the functions x from A into \mathbb{R} satisfying $\sum_{a \in A} |x(a)|^2 < \infty$ with inner product $\langle x, y \rangle = \sum_{a \in A} x(a)y(a)$ for all $x, y \in H$. Define a subset X of H by

$$X = \{0\} \cup \left(\bigcup_{a \in A} \{a_n e_a : n \in \mathbb{N}\} \right), \quad (5.1)$$

where $e_a \in H$ is defined by $e_a(a) = 1$ and $e_a(b) = 0$ for $b \in A \setminus \{a\}$. Define a mapping T on X by

$$T0 = 0, \quad T(a_n e_a) = a_{n+1} e_a. \quad (5.2)$$

Then (A) holds. However, T is not a contraction with respect to any τ -distance p .

Proof. It is obvious that (A) holds. Arguing by contradiction, we assume that T is a contraction with respect to some τ -distance p . That is, there exist a τ -distance p and $r \in [0, 1)$ such that $p(Tx, Ty) \leq rp(x, y)$ for all $x, y \in X$. Since

$$p(0, 0) = p(T0, T0) \leq rp(0, 0), \quad (5.3)$$

we have $p(0, 0) = 0$. By Lemma 2.6, there exists a strictly increasing sequence $\{\kappa_n\}$ in \mathbb{N} such that

$$p(0, x) \leq r^{\kappa_n} \implies d(0, x) \leq \frac{1}{n}. \quad (5.4)$$

We choose $\alpha \in A$ such that $\alpha_{2\kappa_n+1} > 1/n$. Fix $\nu \in \mathbb{N}$ with $r^{\kappa_\nu} p(0, \alpha_1 e_\alpha) \leq 1$. Then we have

$$p(0, \alpha_{2\kappa_\nu+1} e_\alpha) = p(T^{2\kappa_\nu} 0, T^{2\kappa_\nu}(\alpha_1 e_\alpha)) \leq r^{2\kappa_\nu} p(0, \alpha_1 e_\alpha) \leq r^{\kappa_\nu}, \quad (5.5)$$

and hence,

$$\frac{1}{\nu} < \alpha_{2\kappa_\nu+1} = d(0, \alpha_{2\kappa_\nu+1} e_\alpha) \leq \frac{1}{\nu}. \quad (5.6)$$

This is a contradiction. □

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References

- [1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.
- [2] R. Caccioppoli, "Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale," *Rendiconti dell'Accademia Nazionale dei Lincei*, vol. 11, pp. 794–799, 1930.
- [3] R. Kannan, "Some results on fixed points. II," *The American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.
- [4] W. A. Kirk, "Contraction mappings and extensions," in *Handbook of Metric Fixed Point Theory*, W. A. Kirk and B. Sims, Eds., pp. 1–34, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.
- [5] S. Leader, "Equivalent Cauchy sequences and contractive fixed points in metric spaces," *Studia Mathematica*, vol. 76, no. 1, pp. 63–67, 1983.
- [6] T. Suzuki, "A sufficient and necessary condition for the convergence of the sequence of successive approximations to a unique fixed point," *Proceedings of the American Mathematical Society*, vol. 136, no. 11, pp. 4089–4093, 2008.
- [7] I. A. Rus, "The method of successive approximations," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 17, pp. 1433–1437, 1972.
- [8] P. V. Subrahmanyam, "Remarks on some fixed-point theorems related to Banach's contraction principle," *Journal of Mathematical and Physical Sciences*, vol. 8, pp. 445–457, 1974.

- [9] T. Suzuki, "Subrahmanyam's fixed point theorem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1678–1683, 2009.
- [10] I. A. Rus, "Picard operators and applications," *Scientiae Mathematicae Japonicae*, vol. 58, no. 1, pp. 191–219, 2003.
- [11] I. A. Rus, "The theory of a metrical fixed point theorem: theoretical and applicative relevances," *Fixed Point Theory*, vol. 9, no. 2, pp. 541–559, 2008.
- [12] I. A. Rus, A. S. Mureşan, and V. Mureşan, "Weakly Picard operators on a set with two metrics," *Fixed Point Theory*, vol. 6, no. 2, pp. 323–331, 2005.
- [13] I. A. Rus, A. Petruşel, and M. A. Şerban, "Weakly Picard operators: equivalent definitions, applications and open problems," *Fixed Point Theory*, vol. 7, no. 1, pp. 3–22, 2006.
- [14] D. Tataru, "Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms," *Journal of Mathematical Analysis and Applications*, vol. 163, no. 2, pp. 345–392, 1992.
- [15] C.-K. Zhong, "On Ekeland's variational principle and a minimax theorem," *Journal of Mathematical Analysis and Applications*, vol. 205, no. 1, pp. 239–250, 1997.
- [16] C.-K. Zhong, "A generalization of Ekeland's variational principle and application to the study of the relation between the weak P.S. condition and coercivity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 29, no. 12, pp. 1421–1431, 1997.
- [17] L.-J. Lin and W.-S. Du, "Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 360–370, 2006.
- [18] T. Suzuki, "Generalized distance and existence theorems in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 253, no. 2, pp. 440–458, 2001.
- [19] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.
- [20] T. Suzuki, "Several fixed point theorems concerning τ -distance," *Fixed Point Theory and Applications*, no. 3, pp. 195–209, 2004.
- [21] T. Suzuki, "Contractive mappings are Kannan mappings, and Kannan mappings are contractive mappings in some sense," *Commentationes Mathematicae. Prace Matematyczne*, vol. 45, no. 1, pp. 45–58, 2005.
- [22] T. Suzuki, "The strong Ekeland variational principle," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 787–794, 2006.
- [23] T. Suzuki, "On the relation between the weak Palais-Smale condition and coercivity given by Zhong," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 8, pp. 2471–2478, 2008.