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Research Article

Some Convergence Theorems of a Sequence in Complete Metric Spaces and Its Applications

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The concept of weakly quasi-nonexpansive mappings with respect to a sequence is introduced. This concept generalizes the concept of quasi-nonexpansive mappings with respect to a sequence due to Ahmed and Zeyada (2002). Mainly, some convergence theorems are established and their applications to certain iterations are given.

1. Introduction

In 1916, Tricomi [1] introduced originally the concept of quasi-nonexpansive for real functions. Subsequently, this concept has studied for mappings in Banach and metric spaces (see, e.g., [2–7]). Recently, some generalized types of quasi-nonexpansive mappings in metric and Banach spaces have appeared. For example, see Ahmed and Zeyada [8], Qihou [9–11] and others.

Unless stated to the contrary, we assume that (X, d) is a metric space. Let $T : D \subseteq X \to X$ be any mapping and let $F(T)$ be the set of all fixed points of T. If $F : X \to R$ where R is the set of all real numbers and if $c \in R$, set $L_c := \{x \in X : F(x) \leq c\}$. We use the symbol μ to denote the usual Kuratowski measure of noncompactness. For some properties of μ, see Zeidler [12, pages 493–495]. For a given $x_0 \in D$, the Picard iteration (x_n) is determined by:

(I) $x_n = T(x_{n-1}) = T^n(x_0),\ n \in N$

where N is the set of all positive integers.

If X is a normed space, D is a convex set, and $T : D \to D$, Ishikawa [13] gave the following iteration:

(II) $x_n = T_{\alpha,\beta}(x_{n-1}) = T^n_{\alpha,\beta}(x_0),\ T_{\alpha,\beta} = (1-\alpha)I + \alpha T[(1-\beta)I + \beta T],$

for each $n \in N$, where $\alpha \in (0,1)$ and $\beta \in [0,1)$. When $\beta = 0$, it yields that $T_{\alpha,0} = (1-\alpha)I + \alpha T = T_\alpha$. Therefore, the iteration scheme (II) becomes

$$x_n = T_\alpha(x_{n-1}) = T_\alpha^n(x_0). \tag{1.1}$$

This iteration is called Mann iteration [14].

The concepts of quasi-nonexpansive mappings, with respect to a sequence and asymptotically regular mappings at a point were given in metric spaces as follows.

Definition 1.1 (see [6]). $T : D \to X$ is said to be quasi-nonexpansive mapping if for each $x \in D$ and for every $p \in F(T)$, $d(T(x), p) \leq d(x, p)$.

Definition 1.2 (see [8]). The map $T : D \to X$ is said to be quasi-nonexpansive with respect to $(x_n) \subseteq D$ if for all $n \in N \cup \{0\}$ and for every $p \in F(T)$, $d(x_{n+1}, p) \leq d(x_n, p)$.

Lemma 2.1 in [8] stated that quasi-nonexpansiveness converts to quasi-nonexpansiveness with respect to $(T^n(x_0))$ (resp., $(T_\alpha^n(x_0))$, $(T_{\alpha,\beta}^n(x_0))$) for each $x_0 \in D$. The reverse implication is not true (see, [8, Example 2.1]). Also, the authors [8] showed that the continuity of $T : D \to X$ leads to the closedness of $F(T)$ and the converse is not true (see, [8, Example 2.2]).

Definition 1.3 (see [15]). The mapping $T : X \to X$ is called an asymptotically regular at a point $x_0 \in X$ if $\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = 0$.

The following definition is given by Angrisani and Clavelli.

Definition 1.4 (see [16]). Let X be a topological space. The function $F : X \to R$ is said to be a regular-global-inf (r.g.i) at $x \in X$ if $F(x) > \inf_X(F)$ implies that there exists $\epsilon > 0$ such that $\epsilon < F(x) - \inf_X(F)$ and a neighborhood N_x of x such that $F(y) > F(x) - \epsilon$ for each $y \in N_x$. If this condition holds for each $x \in X$, then F is said to be an r.g.i on X.

Definition 1.5 (see [17]). Let D be a convex subset of a normed space X. A mapping $T : D \to D$ is called directionally nonexpansive if $\|T(x) - T(m)\| \leq \|x - m\|$ for each $x \in D$ and for all $m \in [x, T(x)]$ where $[x, y]$ denotes the segment joining x and y; that is, $[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$.

Our objective in this paper is to introduce the concept of weakly quasi-nonexpansive mappings with respect to a sequence. Mainly, we establish some convergence theorems of a sequence in complete metric spaces. These theorems generalize and improve [8, Theorems 2.1 and 2.2], of [7, Theorems 1.1 and 1.1′], [5, Theorem 3.1], and [6, Proposition 1.1].

2. Main Result

In this section, we introduce the concept of weak quasi-nonexpansiveness of a mapping with respect to a sequence that generalizes quasi-nonexpansiveness of a mapping with respect to a sequence in [8]. We give a lemma and a counterexample to show the relation between our new concept; the previous one appeared in [8] and a monotonically decreasing sequence $(d(x_n, F(T)))$.

Definition 2.1. Let (X, d) be a metric space and let (x_n) be a sequence in $D \subseteq X$. Assume that $T : D \to X$ is a mapping with $F(T) \neq \phi$ satisfying $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus, for a given $\epsilon > 0$ there is a $n_1(\epsilon) \in N$ such that $d(x_n, F(T)) < \epsilon$ for all $n \geq n_1(\epsilon)$. T is called weakly quasi-nonexpansive with respect to $(x_n) \subseteq D$ if for each $\epsilon > 0$ there exists a $p(\epsilon) \in F(T)$ such that for all $n \in N$ with $n \geq n_1(\epsilon)$, $d(x_n, p(\epsilon)) < \epsilon$.

We state the following lemma without proof.

Lemma 2.2. *Let (X, d) be a metric space and, (x_n) be a sequence in $D \subseteq X$. Assume that $T : D \to X$ is a mapping with $F(T) \neq \phi$ satisfying $\lim_{n\to\infty} d(x_n, F(T)) = 0$. If T is quasi-nonexpansive with respect to (x_n), then*

(A) *T is weakly quasi-nonexpansive with respect to (x_n);*

(B) *$(d(x_n, F(T)))$ is a monotonically decreasing sequence in $[0, \infty)$.*

The following example shows that the converse of Lemma 2.2 may not be true.

Example 2.3. Let $X = [0, 1]$ be endowed with the Euclidean metric d. We define the map $T : X \to X$ by $T(x) = (3/4)x^2 + (1/4)x$ for each $x \in X$. Assume that $x_n = 1/n$ for all $n \in N - \{1, 2, 3\}$. Then

$$F(T) = \{0, 1\}, \quad \lim_{n\to\infty} d(x_n, F(T)) = \lim_{n\to\infty} d\left(\frac{1}{n}, F(T)\right) = 0. \tag{2.1}$$

Given $\epsilon > 0$, there exists $n_1(\epsilon) \in N - \{1, 2, 3\}$ such that for all $n \in N - \{1, 2, 3\}$ with $n \geq n_1(\epsilon)$, there exists $p = 0 \in F(T)$,

$$d(x_n, 0) = \left|\frac{1}{n} - 0\right| < \epsilon. \tag{2.2}$$

Thus, T is weakly quasi-nonexpansive with respect to (x_n). But, T is not quasi-nonexpansive with respect to (x_n) (Indeed, there exists $1 \in F(T)$ such that for all $n \in N - \{1, 2, 3\}$, $d(x_{n+1}, 1) > d(x_n, 1)$). Furthermore, the sequence $(d(x_n, F(T))) = (1/n)$ is monotonically decreasing in $[0, \infty)$.

Before stating the main theorem, let us introduce the following lemma without proof.

Lemma 2.4. *Let (X, d) be a metric space and let (x_n) be a sequence in $D \subseteq X$. Assume that $T : D \to X$ is weakly quasi-nonexpansive with respect to (x_n) with $F(T) \neq \phi$ satisfying $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Then, (x_n) is a Cauchy sequence.*

Now, we give the main theorem without proof in the following way.

Theorem 2.5. *Let (x_n) be a sequence in a subset D of a metric space (X, d) and let $T : D \to X$ be a map such that $F(T) \neq \phi$. Then*

(a) *$\lim_{n\to\infty} d(x_n, F(T)) = 0$ if (x_n) converges to a point in $F(T)$;*

(b) *(x_n) converges to a point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0$, $F(T)$ is a closed set, T is weakly quasi-nonexpansive with respect to (x_n), and X is complete.*

As corollaries of Theorem 2.5, we have the following.

Corollary 2.6. *For each $x_0 \in D$, let $(T^n(x_0))$ be a sequence in a subset D of a metric space (X, d) and let $T : D \to X$ be a map such that $F(T) \neq \phi$. Then*

(a) $\lim_{n\to\infty} d(T^n(x_0), F(T)) = 0$ *if $(T^n(x_0))$ converges to a point in $F(T)$;*

(b) $(T^n(x_0))$ *converges to a point in $F(T)$ if $\lim_{n\to\infty} d(T^n(x_0), F(T)) = 0$, $F(T)$ is a closed set, T is weakly quasi-nonexpansive with respect to $(T^n(x_0))$ and X is complete.*

Corollary 2.7. *For each $x_0 \in D$, let $(T_\alpha^n(x_0))$ be a sequence in a subset D of a normed space $(X, \|\cdot\|)$ and let $T : D \to X$ be a map such that $F(T) \neq \phi$. Then*

(a) $\lim_{n\to\infty} d(T_\alpha^n(x_0), F(T)) = 0$ *if $(T_\alpha^n(x_0))$ converges to a point in $F(T)$;*

(b) $(T_\alpha^n(x_0))$ *converges to a point in $F(T)$ if $\lim_{n\to\infty} d(T_\alpha^n(x_0), F(T)) = 0$, $F(T)$ is a closed set, T is weakly quasi-nonexpansive with respect to $(T_\alpha^n(x_0))$, and X is a Banach space.*

Corollary 2.8. *For each $x_0 \in D$, let $(T_{\alpha,\beta}^n(x_0))$ be a sequence in a subset D of a normed space $(X, \|\cdot\|)$ and let $T : D \to X$ be a map such that $F(T) \neq \phi$. Then*

(a) $\lim_{n\to\infty} d(T_{\alpha,\beta}^n(x_0), F(T)) = 0$ *if $(T_{\alpha,\beta}^n(x_0))$ converges to a point in $F(T)$;*

(b) $(T_{\alpha,\beta}^n(x_0))$ *converges to a point in $F(T)$ if $\lim_{n\to\infty} d(T_{\alpha,\beta}^n(x_0), F(T)) = 0$, $F(T)$ is a closed set, T is weakly quasi-nonexpansive with respect to $(T_{\alpha,\beta}^n(x_0))$, and X is a Banach space.*

Remark 2.9. (I) Theorem 2.5 generalizes and improves [8, Theorem 2.1] since T is weakly quasi-nonexpansive with respect to (x_n) instead of T being quasi-nonexpansive with respect to (x_n).

(II) Corollary 2.6 generalizes and improves [7, Theorem 1.1 page 462] for some reasons. These reasons are the following:

(1) the closedness of D is superfluous;

(2) $F(T)$ is closed instead of T being continuous;

(3) X is a complete metric space instead of X is a Banach space;

(4) T is weakly quasi-nonexpansive with respect to $(T^n(x_0))$ in lieu of T being quasi-nonexpansive.

(III) Corollary 2.7 (resp. Corollary 2.8) generalizes and improves [7, Theorem 1.1′ page 469] (resp. of [5, Theorem 3.1 page 98]) since the reasons (1) and (2) in (II) hold and

(1)′ the convexity of D in Theorem 1.1′ is superfluous;

(2)′ T is weakly quasi-nonexpansive with respect to $(T_\alpha^n(x_0))$ (resp. $(T_{\alpha,\beta}^n(x_0))$ instead of T being quasi-nonexpansive.

(IV) If we take $T : D \to X$ instead of $T : X \to X$, $F(T)$ is closed in lieu of $T : X \to X$ being continuous and T is weakly quasi-nonexpansive with respect to $(T^n(x_0))$ in lieu of T being quasi-nonexpansive, then Corollary 2.6 generalizes and improves Kirk [6, Proposition 1.1].

In the light of Lemma 2.2 and Example 2.3, we state the following theorem.

Theorem 2.10. *Let (x_n) be a sequence in a subset D of a complete metric space (X, d) and $T : D \to X$ be a map such that $F(T) \neq \phi$ is a closed set. Assume that*

(i) *T is weakly quasi-nonexpansive with respect to (x_n);*

(ii) *$(d(x_n, F(T)))$ is a monotonically decreasing sequence in $[0, \infty)$;*

(iii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*;*

(iv) *if the sequence (y_n) satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, then*

$$\liminf_{n} d(y_n, F(T)) = 0 \qquad \text{or} \qquad \limsup_{n} d(y_n, F(T)) = 0. \tag{2.3}$$

Then (x_n) converges to a point in $F(T)$.

Proof. From the boundedness from below by zero of the sequence $(d(x_n, F(T)))$ and (ii), we obtain that $\lim_{n\to\infty} d(x_n, F(T))$ exists. So, from (iii) and (iv), we have that $\liminf_n d(x_n, F(T)) = 0$ or $\limsup_n d(x_n, F(T)) = 0$. Then $\lim_{n\to\infty} d(x_n, F(T)) = 0$ (see, [18, page 37]). Therefore, by Theorem 2.5(b), the sequence (x_n) converges to a point in $F(T)$. □

Corollary 2.11. *For each $x_0 \in D$, let $(T^n(x_0))$ be a sequence in a subset D of a complete metric space (X, d) and let $T : D \to X$ be a map such that $F(T) \neq \phi$ is a closed set. Assume that*

(i) *T is weakly quasi-nonexpansive with respect to $(T^n(x_0))$;*

(ii) *$(d(T^n(x_0), F(T)))$ is a monotonically decreasing sequence in $[0, \infty)$;*

(iii) $\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = 0$*;*

(iv) *if the sequence (y_n) satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, then*

$$\liminf_{n} d(y_n, F(T)) = 0 \qquad \text{or} \qquad \limsup_{n} d(y_n, F(T)) = 0. \tag{2.4}$$

Then $(T^n(x_0))$ converges to a point in $F(T)$.

Corollary 2.12. *For each $x_0 \in D$, let $(T_\alpha^n(x_0))$ be a sequence in a subset D of a Banach space X and let $T : D \to X$ be a map such that $F(T) \neq \phi$ is a closed set. Assume that*

(i) *T is weakly quasi-nonexpansive with respect to $(T_\alpha^n(x_0))$;*

(ii) *$(d(T_\alpha^n(x_0), F(T)))$ is a monotonically decreasing sequence in $[0, \infty)$;*

(iii) $\lim_{n\to\infty} \|T_\alpha^n(x_0) - T_\alpha^{n+1}(x_0)\| = 0$*;*

(iv) *if the sequence (y_n) satisfies $\lim_{n\to\infty} \|y_n - y_{n+1}\| = 0$, then*

$$\liminf_{n} d(y_n, F(T)) = 0 \qquad \text{or} \qquad \limsup_{n} d(y_n, F(T)) = 0. \tag{2.5}$$

Then $(T_\alpha^n(x_0))$ converges to a point in $F(T)$.

Corollary 2.13. *For each $x_0 \in D$, let $(T_{\alpha,\beta}^n(x_0))$ be a sequence in a subset D of a Banach space X and let $T : D \to X$ be a map such that $F(T) \neq \phi$ is a closed set. Assume that*

(i) *T is weakly quasi-nonexpansive with respect to* $(T^n_{\alpha,\beta}(x_0))$;

(ii) $(d(T^n_{\alpha,\beta}(x_0), F(T)))$ *is a monotonically decreasing sequence in* $[0,\infty)$;

(iii) $\lim_{n\to\infty}\|T^n_{\alpha,\beta}(x_0) - T^{n+1}_{\alpha,\beta}(x_0)\| = 0$;

(iv) *if the sequence* (y_n) *satisfies* $\lim_{n\to\infty}\|y_n - y_{n+1}\| = 0$, *then*

$$\liminf_n d(y_n, F(T)) = 0 \qquad or \qquad \limsup_n d(y_n, F(T)) = 0. \tag{2.6}$$

Then $(T^n_{\alpha,\beta}(x_0))$ *converges to a point in* $F(T)$.

Remark 2.14. From Lemma 2.2, we find that [8, Theorem 2.2] is a special case of Theorem 2.10. Also, Corollary 2.11 generalizes and improves [7, Theorem 1.2 page 464] for the same reasons in Remark 2.9(II).

We establish another consequence of Theorem 2.5 as follows.

Theorem 2.15. *Let* (x_n) *be a sequence in a subset* D *of a complete metric space* (X, d). *Furthermore, let* $T : D \to X$ *be a mapping such that* $F(T) \neq \phi$ *is a closed set. Assume that the conditions (i) and (ii) in Theorem 2.10 hold and*

(iii)′ *the sequence* (x_n) *contains a convergent subsequence* (x_{n_j}) *converging to* $x^* \in D$ *such that there exists a continuous mapping* $S : D \to D$ *satisfying* $S(x_{n_j}) = x_{n_{j+1}}$ *for all* $j \in N$ *and* $d(S(x^*), p) < d(x^*, p)$ *for some* $p \in F(T)$.

Then $x^* \in F(T)$ *and* $\lim_{n\to\infty} x_n = x^*$.

Proof. From (ii), one can deduce that $\lim_{n\to\infty} d(x_n, F(T))$ exists, say equal $r \in [0,\infty)$. Suppose that x^* does not belong to $F(T)$. So, we have from (iii)′ that for some $p \in F(T)$,

$$d(x^*, p) > d(S(x^*), p) = d\left(S\left(\lim_{j\to\infty} x_{n_j}\right), p\right) = d\left(\lim_{j\to\infty} S\left(x_{n_j}\right), p\right) = d\left(\lim_{j\to\infty} x_{n_j+1}, p\right) = d(x^*, p). \tag{2.7}$$

This contradiction implies that $x^* \in F(T)$. Then,

$$r = \lim_{n\to\infty} d(x_n, F(T)) = \lim_{j\to\infty} d\left(x_{n_j}, F(T)\right) = d\left(\lim_{j\to\infty} x_{n_j}, F(T)\right) = d(x^*, F(T)) = 0. \tag{2.8}$$

From Theorem 2.5(b), we obtain that $\lim_{n\to\infty} x_n = x^*$. □

Corollary 2.16. *For each* $x_0 \in D$, *let* $(T^n(x_0))$ *be a sequence in a subset* D *of a complete metric space* (X, d). *Furthermore, let* $T : D \to X$ *be a mapping such that* $F(T) \neq \phi$ *is a closed set. Assume that the conditions (i) and (ii) in Corollary 2.11 hold and*

(iii)′ *the sequence* $(T^n(x_0))$ *contains a convergent subsequence* $(T^{n_j}(x_0))$ *converging to* $x^* \in D$

such that there exists a continuous mapping $S : D \to D$ *satisfying* $S(T^{n_j}(x_0)) = T^{n_j+1}(x_0)$ *for all* $j \in N$ *and* $d(S(x^*), p) < d(x^*, p)$ *for some* $p \in F(T)$.

Then $x^* \in F(T)$ *and* $\lim_{n\to\infty} T^n(x_0) = x^*$.

Corollary 2.17. *For each* $x_0 \in D$, *let* $(T_\alpha^n(x_0))$ *be a sequence in a subset* D *of a complete metric space* (X, d). *Furthermore, let* $T : D \to X$ *be a mapping such that* $F(T) \neq \phi$ *is a closed set. Assume that the conditions (i) and (ii) in* Corollary 2.12 *hold and*

(iii)′ *the sequence* $(T_\alpha^n(x_0))$ *contains a convergent subsequence* $(T_\alpha^{n_j}(x_0))$ *converging to* $x^* \in D$ *such that there exists a continuous mapping* $S : D \to D$ *satisfying* $S(T_\alpha^{n_j}(x_0)) = T_\alpha^{n_j+1}(x_0)$ *for all* $j \in N$ *and* $d(S(x^*), p) < d(x^*, p)$ *for some* $p \in F(T)$.

Then $x^* \in F(T)$ *and* $\lim_{n\to\infty} T_\alpha^n(x_0) = x^*$.

Corollary 2.18. *For each* $x_0 \in D$, *let* $(T_{\alpha,\beta}^n(x_0))$ *be a sequence in a subset* D *of a complete metric space* (X, d). *Furthermore, let* $T : D \to X$ *be a mapping such that* $F(T) \neq \phi$ *is a closed set. Assume that the conditions (i) and (ii) in* Corollary 2.13 *hold and*

(iii)′ *the sequence* $(T_{\alpha,\beta}^n(x_0))$ *contains a convergent subsequence* $(T_{\alpha,\beta}^{n_j}(x_0))$ *converging to* $x^* \in D$ *such that there exists a continuous mapping* $S : D \to D$ *satisfying* $S(T_{\alpha,\beta}^{n_j}(x_0)) = T_{\alpha,\beta}^{n_j+1}(x_0)$ *for all* $j \in N$ *and* $d(S(x^*), p) < d(x^*, p)$ *for some* $p \in F(T)$.

Then $x^* \in F(T)$ *and* $\lim_{n\to\infty} T_{\alpha,\beta}^n(x_0) = x^*$.

Remark 2.19. Theorem 1.3 in [7] is a special case of Corollary 2.16 for the same reasons in Remark 2.9(II) and for the generalization of the conditions (1.6) and (1.7) in [7, Theorem 1.3] to the condition (iii)′ in Corollary 2.16.

From [17, Corollary 2.4] and Theorem 2.5(b), one can prove the following theorem.

Theorem 2.20. *Let* $T : X \to X$ *be a mapping of a complete metric space* (X, d) *satisfying*

(i) $d(T(x), T^2(x)) \leq hd(x, T(x))$ *for some* $h \in (0, 1)$ *and for all* $x \in X$;

(ii) $\mu(T(L_c)) \leq k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* X;

(iv) (x_n) *is a sequence in* X *such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* T *is weakly quasi-nonexpansive with respect to* (x_n).

Then (x_n) *converges to a point in* $F(T)$.

Corollary 2.21. *Let* $T : X \to X$ *be a mapping of a complete metric space* (X, d) *satisfying*

(i) $d(T(x), T^2(x)) \leq hd(x, T(x))$ *for some* $h \in (0, 1)$ *and for all* $x \in X$;

(ii) $\mu(T(L_c)) \leq k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* X;

(iv) $(T^n(x_0))$ *is a sequence satisfying* $\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = 0$ *for each* $x_0 \in X$ *and* T *is weakly quasi-nonexpansive with respect to* $(T^n(x_0))$.

Then $(T^n(x_0))$ *converges to a point in* $F(T)$.

Corollary 2.22. *Let $T : X \to X$ be a mapping of a Banach space (X, d) satisfying*

(i) $\|T(x) - T^2(x)\| \le h\|x - T(x)\|$ *for some* $h \in (0,1)$ *and for all* $x \in X$;

(ii) $\mu(T(L_c)) \le k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* X;

(iv) $(T_\alpha^n(x_0))$ *is a sequence in X such that* $\lim_{n\to\infty}\|T_\alpha^n(x_0) - TT_\alpha^n(x_0)\| = 0$ *for each* $x_0 \in X$ *and* T *is weakly quasi-nonexpansive with respect to* $(T_\alpha^n(x_0))$.

Then $(T_\alpha^n(x_0))$ *converges to a point in* $F(T)$.

Corollary 2.23. *Let $T : X \to X$ be a mapping of a Banach space (X, d) satisfying*

(i) $\|T(x) - T^2(x)\| \le h\|x - T(x)\|$ *for some* $h \in (0,1)$ *and for all* $x \in X$;

(ii) $\mu(T(L_c)) \le k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* X;

(iv) $(T_{\alpha,\beta}^n(x_0))$ *is a sequence in X such that* $\lim_{n\to\infty}\|T_{\alpha,\beta}^n(x_0) - TT_{\alpha,\beta}^n(x_0)\| = 0$ *for each* $x_0 \in X$ *and* T *is weakly quasi-nonexpansive with respect to* $(T_{\alpha,\beta}^n(x_0))$.

Then $(T_{\alpha,\beta}^n(x_0))$ *converges to a point in* $F(T)$.

Theorem 2.24. *Let D be a bounded closed convex subset of a Banach space X. Suppose that $T : D \to D$ satisfies*

(i) T *is directionally nonexpansive on* D;

(ii) $\mu(T(L_c)) \le k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* D;

(iv) $(x_n) \subseteq D$ *satisfies* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ *and* T *is weakly quasi-nonexpansive with respect to* (x_n).

Then (x_n) *converges to a point in* $F(T)$.

Proof. The conclusion is obtained by combining [17, Theorem 3.3] and Theorem 2.5(b). □

Corollary 2.25. *Let D be a bounded closed convex subset of a Banach space X. Suppose that $T : D \to D$ satisfies*

(i) T *is directionally nonexpansive on* D;

(ii) $\mu(T(L_c)) \le k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* D;

(iv) $(T^n(x_0))$ *for each* $x_0 \in D$ *satisfies* $\lim_{n\to\infty}\|T^n(x_0) - T^{n+1}(x_0)\| = 0$ *and* T *is weakly quasi-nonexpansive with respect to* $(T^n(x_0))$.

Then $(T^n(x_0))$ *converges to a point in* $F(T)$.

Corollary 2.26. *Let D be a bounded closed convex subset of a Banach space X. Suppose that $T : D \to D$ satisfies*

(i) T *is directionally nonexpansive on* D;

(ii) $\mu(T(L_c)) \le k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* D;

(iv) $(T_\alpha^n(x_0))$ *for each* $x_0 \in D$ *satisfies* $\lim_{n\to\infty}\|T_\alpha^n(x_0) - TT_\alpha^n(x_0)\| = 0$ *and* T *is weakly quasi-nonexpansive with respect to* $(T_\alpha^n(x_0))$.

Then $(T_\alpha^n(x_0))$ *converges to a point in* $F(T)$.

Corollary 2.27. *Let* D *be a bounded closed convex subset of a Banach space* X. *Suppose that* $T : D \to D$ *satisfies*

(i) T *is directionally nonexpansive on* D;

(ii) $\mu(T(L_c)) \le k\mu(L_c)$ *for some* $k < 1$ *and for all* $c > 0$;

(iii) F *is an r.g.i. on* D;

(iv) $(T_{\alpha,\beta}^n(x_0))$ *for each* $x_0 \in D$ *satisfies* $\lim_{n\to\infty}\|T_{\alpha,\beta}^n(x_0) - TT_{\alpha,\beta}^n(x_0)\| = 0$ *and* T *is weakly quasi-nonexpansive with respect to* $(T_{\alpha,\beta}^n(x_0))$.

Then $(T_{\alpha,\beta}^n(x_0))$ *converges to a point in* $F(T)$.

Remark 2.28. It is worth to mention that Corollaries 2.12, 2.13, 2.17, 2.18, 2.21–2.23, 2.25–2.27 are new results.

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