Research Article

On Fixed Points of Maximalizing Mappings in Posets

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We use chain methods to prove fixed point results for maximalizing mappings in posets. Concrete examples are also presented.

1. Introduction

According to Bourbaki's fixed point theorem (cf. [1, 2]) a mapping G from a partially ordered set $X = (X, \leq)$ into itself has a fixed point if G is *extensive*, that is, $x \leq G(x)$ for all $x \in X$, and if every nonempty chain of X has the supremum in X. In [3, Theorem 3] the existence of a fixed point is proved for a mapping $G : X \to X$ which is *ascending*, that is, $G(x) \leq y$ implies $G(x) \leq G(y)$. It is easy to verify that every extensive mapping is ascending. In [4] the existence of a fixed point of G is proved if $a \leq G(a)$ for some $a \in X$, and if G is *semi-increasing upward*, that is, $G(x) \leq G(y)$ whenever $x \leq y$ and $G(x) \leq y$. This property holds, for instance, if G is ascending or *increasing*, that is, $G(x) \leq G(y)$ whenever $x \leq y$.

In this paper we prove further generalizations to Bourbaki's fixed point theorem by assuming that a mapping $G : X \to X$ is *maximalizing*, that is, $G(x)$ is a maximal element of $\{x, G(x)\}$ for all $x \in X$. Concrete examples of maximalizing mappings G which have or do not have fixed points are presented. Chain methods introduced in [5, 6] are used in the proofs. These methods are also compared with three other chain methods.

2. Preliminaries

A nonempty set X, equipped with a reflexive, antisymmetric, and transitive relation $\leq$ in $X \times X$, is called a *partially ordered set* (poset). An element b of a poset X is called an *upper*

bound of a subset A of X if $x \le b$ for each $x \in A$. If $b \in A$, we say that b is the *greatest element* of A, and denote $b = \max A$. A lower bound of A and the least element, $\min A$, of A are defined similarly, replacing $x \le b$ above by $b \le x$. If the set of all upper bounds of A has the least element, we call it the *supremum of A* and denote it by $\sup A$. We say that y is a *maximal element* of A if $y \in A$, and if $z \in A$ and $y \le z$ imply that $y = z$. The infimum of A, $\inf A$, and a minimal element of A are defined similarly. A subset W of X is called a *chain* if $x \le y$ or $y \le x$ for all $x, y \in W$. We say that W is *well ordered* if nonempty subsets of W have least elements. Every well-ordered set is a chain.

Let X be a nonempty poset. A basis to our considerations is the following chain method (cf. [6, Lemma 2]).

Lemma 2.1. *Given $G : X \to X$ and $a \in X$, there exists a unique well-ordered chain C in X, called a w-o chain of aG-iterations, satisfying*

$$x \in C \quad \text{iff } x = \sup\{a, G[C^{<x}]\}, \quad \text{where } C^{<x} = \{y \in C : y < x\}. \tag{2.1}$$

If $x_* = \sup\{a, G[C]\}$ exists in X, then $x_* = \max C$, and $G(x_*) \le x_*$.

The following result helps to analyze the w-o chain of aG-iterations.

Lemma 2.2. *Let A and B be nonempty subsets of X. If $\sup A$ and $\sup B$ exist, then the equation*

$$\sup(A \cup B) = \sup\{\sup A, \sup B\} \tag{2.2}$$

is valid whenever either of its sides is defined.

Proof. The sets $A \cup B$ and $\{\sup A, \sup B\}$ have same upper bounds, which implies the assertion. □

A subset W of a chain C is called an *initial segment* of C if $x \in W$, $y \in C$, and $y < x$ imply $y \in W$. If W is well ordered, then every element x of W which is not the possible maximum of W has a *successor*: $Sx = \min\{y \in W : x < y\}$, in W. The next result gives a characterization of elements of the w-o chain of aG-iterations.

Lemma 2.3. *Given $G : X \to X$ and $a \in X$, let C be the w-o chain of aG-iterations. Then the elements of C have the following properties.*

(a) $\min C = a$.

(b) *An element x of C has a successor in C if and only if $\sup\{x, G(x)\}$ exists and $x < \sup\{x, G(x)\}$, and then $Sx = \sup\{x, G(x)\}$.*

(c) *If W is an initial segment of C and $y = \sup W$ exists, then $y \in C$.*

(d) *If $a < y \in C$ and y is not a successor, then $y = \sup C^{<y}$.*

(e) *If $y = \sup C$ exists, then $y = \max C$.*

Proof. (a) $\min C = \sup\{a, G[C^{<\min C}]\} = \sup\{a, G[\emptyset]\} = \sup\{a, \emptyset\} = a$.

(b) Assume first that $x \in C$, and that Sx exists in C. Applying (2.1), Lemma 2.2, and the definition of Sx we obtain

$$Sx = \sup\left\{a, G\left[C^{<Sx}\right]\right\} = \sup\{a, G[C^{<x}] \cup \{G(x)\}\} = \sup\{x, G(x)\}. \tag{2.3}$$

Moreover, $x < Sx$, by definition, whence $x < \sup\{x, G(x)\}$.

Assume next that $x \in C$, that $y = \sup\{x, G(x)\}$ exists, and that $x < \sup\{x, G(x)\}$. The previous proof implies the following

(i) There is no element $w \in C$ which satisfies $x < w < \sup\{x, G(x)\}$.

Then $\{z \in C : z \leq x\} = C^{<y}$, so that

$$
\begin{aligned}
x < \sup\{x, G(x)\} &= \sup\{\sup\{a, G[C^{<x}]\}, G(x)\} \\
&= \sup\{\{a\} \cup G[C^{<x}] \cup \{G(x)\}\} \\
&= \sup\{a, G[\{z \in C : z \leq x\}]\} \\
&= \sup\{a, G[C^{<y}]\}.
\end{aligned}
\tag{2.4}
$$

Thus $y = \sup\{x, G(x)\} \in C$ by (2.1). This result and (i) imply that $y = \sup\{x, G(x)\} = \min\{z \in C : x < z\} = Sx$.

(c) Assume that W is an initial segment of C, and that $y = \sup W$ exists. If there is $x \in W$ such that $Sx \notin W$, then $x = \max W = y$, so that $y \in C$. Assume next that every element x of W has the successor Sx in W. Since $Sx = \sup\{x, G(x)\}$ by (b), then $G(x) \leq Sx < y$. This holds for all $x \in W$. Since $a = \min C = \min W < y$, then y is an upper bound of $\{a\} \cup G[W]$. If z is an upper bound of $\{a\} \cup G[W]$, then $x = \sup\{a, G[C^{<x}]\} = \sup\{a, G[W^{<x}]\} \leq z$ for every $x \in W$. Thus z is an upper bound of W, whence $y = \sup W \leq z$. But then $y = \sup\{a, G[W]\} = \sup\{a, G[C^{<y}]\}$, so that $y \in C$ by (2.1).

(d) Assume that $a < y \in C$, and that y is not a successor of any element of C. Obviously, y is an upper bound of $C^{<y}$. Let z be an upper bound of $C^{<y}$. If $x \in C^{<y}$, then also $Sx \in C^{<y}$ since y is not a successor. Because $Sx = \sup\{x, G(x)\}$ by (b), then $G(x) \leq Sx \in C^{<y}$. This holds for every $x \in C^{<y}$. Since also $a \in C^{<y}$, then z is an upper bound of $\{a\} \cup G[C^{<y}]$. Thus $y = \sup\{a, G[C^{<y}]\} \leq z$. This holds for every upper bound z of $C^{<y}$, whence $y = \sup C^{<y}$.

(e) If $y = \sup C$ exists, then $y \in C$ by (c) when $W = C$, whence $y = \max C$. $\qquad\square$

In the case when $a \leq G(a)$ we obtain the following result (cf. [7, Proposition 1]).

Lemma 2.4. *Given $G : X \to X$ and $a \in X$, there exists a unique well-ordered chain $C(a)$ in X, calleda w-o chain of G-iterations of a, satisfying*

$$a = \min C, \qquad x \in C \setminus \{a\} \text{ iff } x = \sup G[C^{<x}]. \tag{2.5}$$

If $a \leq G(a)$, and if $x_* = \sup G[C(a)]$ exists, then $a \leq x_* = \max C(a)$, and $G(x_*) \leq x_*$.

Lemma 2.4 is in fact a special case of Lemma 2.1, since the assumption $a \leq G(a)$ implies that $C(a)$ equals to the w-o chain of aG-iterations. As for the use of $C(a)$ in fixed point theory and in the theory of discontinuous differential and integral equations, see, for example, [8, 9] and the references therein.

3. Main Results

Let $X = (X, \leq)$ be a nonempty poset. As an application of Lemma 2.1 we will prove our first existence result.

Theorem 3.1. *A mapping $G : X \to X$ has a fixed point if G is maximalizing, that is, $G(x)$ is a maximal element of $\{x, G(x)\}$ for all $x \in X$, and if $x_* = \sup\{a, G[C]\}$ exists in X for some $a \in X$ where C is the w-o chain of aG-iterations.*

Proof. If C is the w-o chain of aG-iterations, and if $x_* = \sup\{a, G[C]\}$ exists in X, then $x_* = \max C$ and $G(x_*) \leq x_*$ by Lemma 2.1. Since G is maximalizing, then $G(x_*) = x_*$, that is, x_* is a fixed point of G.   □

The next result is a consequence of Theorem 3.1. and Lemma 2.3(e).

Proposition 3.2. *Assume that $G : X \to X$ is maximalizing. Given $a \in X$, let C be the w-o chain of aG-iterations. If $z = \sup C$ exists, it is a fixed point of G if and only if $x_* = \sup\{z, G(z)\}$ exists.*

Proof. Assume that $z = \sup C$ exists. It follows from Lemma 2.3(e) that $z = \max C$. If z is a fixed point of G, that is, $z = G(z)$, then $x_* = \sup\{z, G(z)\} = z$, and $x_* = G(x_*)$.

Assume conversely that $x_* = \sup\{z, G(z)\}$ exist. Applying (2.1) and Lemma 2.2 we obtain

$$x_* = \sup\{z, G(z)\} = \sup\{\sup\{a, G[C^{<z}]\}, \sup\{G(z)\}\}$$
$$= \sup\{\{a\} \cup G[C^{<z}] \cup \{G(z)\}\} = \sup\{a, G[C]\}. \tag{3.1}$$

Thus, by Theorem 3.1, $x_* = \max C = z$ is a fixed point of G.   □

As a consequence of Proposition 3.2 we obtain the following result.

Corollary 3.3. *If nonempty chains of X have supremums, if $G : X \to X$ is maximalizing, and if $\sup\{x, G(x)\}$ exists for all $x \in X$, then for each $a \in X$ the maximum of the w-o chain of aG-iterations exists and is a fixed point of G.*

Proof. Let C be the w-o chain of aG-iterations. The given hypotheses imply that both $z = \sup C$ and $x_* = \sup\{z, G(z)\}$ exist. Thus the hypotheses of Proposition 3.2 are valid.   □

The results of Lemma 2.3 are valid also when C is replaced by the w-o chain $C(a)$ of G-iterations of a. As a consequence of these results and Lemma 2.4 we obtain the following generalizations to Bourbaki's fixed point theorem.

Theorem 3.4. *Assume that $G : X \rightarrow X$ is maximalizing, and that $a \leq G(a)$ for some $a \in X$, and let $C(a)$ be the w-o chain of G-iterations of a.*

(a) *If $x_* = \sup G[C(a)]$ exists, then $x_* = \max C(a)$, and x_* is a fixed point of G.*

(b) *If $z = \sup C(a)$ exists, it is a fixed point of G if and only if $x_* = \sup\{z, G(z)\}$ exists.*

(c) *If nonempty chains of X have supremums, and if $\sup\{x, G(x)\}$ exists for all $x \in X$, then $x_* = \max C(a)$ exists, and x_* is a fixed point of G.*

The previous results have obvious duals, which imply the following results.

Theorem 3.5. *A mapping $G : X \rightarrow X$ has a fixed point if G is minimalizing, that is, $G(x)$ is a minimal element of $\{x, G(x)\}$ for all $x \in X$, and if $\inf\{a, G[W]\}$ exists in X for some $a \in X$ whenever W is a nonempty chain in X.*

Theorem 3.6. *A minimalizing mapping $G : X \rightarrow X$ has a fixed point if $\inf G[W]$ exists whenever W is a nonempty chain in X, and if $G(a) \leq a$ for some $a \in X$.*

Proposition 3.7. *A minimalizing mapping $G : X \rightarrow X$ has a fixed point if every nonempty chain X has the infimum in X, and if $\inf\{x, G(x)\}$ exists for all $x \in X$.*

Remark 3.8. The hypothesis that $G : X \rightarrow X$ is maximalizing can be weakened in Theorems 3.1 and 3.4 and in Proposition 3.2 to the form: $G \mid \{x_*\}$ is maximalizing, that is, $G(x_*)$ is a maximal element of $\{x_*, G(x_*)\}$.

4. Examples and Remarks

We will first present an example of a maximalizing mapping whose fixed point is obtained as the maximum of the w-o chain of aG-iterations.

Example 4.1. Let X be a closed disc $X = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 2\}$, ordered coordinate-wise. Let $[u]$ denote the greatest integer $\leq u$ when $u \in \mathbb{R}$. Define a function $G : X \rightarrow \mathbb{R}^2$ by

$$G(u, v) = \left(\min\{1, 1 - [u] + [v]\}, \frac{1}{2}\left([u] + v^2\right) \right), \quad (u, v) \in X. \tag{4.1}$$

It is easy to verify that $G[X] \subset X$, and that G is maximalizing. To find a fixed point of G, choose $a = (1, 0)$. It follows from **Lemma 2.3**(b) that the first elements of the w-o chain of aG-iterations are successive approximations

$$x_0 = a, \qquad x_{n+1} = Sx_n = \sup\{x_n, G(x_n)\}, \quad n = 0, 1, \ldots, \tag{4.2}$$

as long as Sx_n is defined. Denoting $x_n = (u_n, v_n)$, these successive approximations can be rewritten in the form

$$u_0 = 1, \qquad u_{n+1} = \max\{u_n, \min\{1, 1 - [u_n] + [v_n]\}\},$$

$$v_0 = 0, \qquad v_{n+1} = \max\left\{v_n, \frac{1}{2}\left([u_n] + v_n^2\right)\right\}, \quad n = 0, 1, \ldots, \tag{4.3}$$

as long as $u_n \leq u_{n+1}$ and $v_n \leq v_{n+1}$, and at least one of these inequalities is strict. Elementary calculations show that $u_n = 1$, for every $n \in \mathbb{N}_0$. Thus (4.3) can be rewritten as

$$u_n = 1, \quad v_0 = 0, \quad v_{n+1} = \max\left\{ v_n, \frac{1}{2}\left(1 + v_n^2\right) \right\}, \quad n = 0, 1, \ldots. \tag{4.4}$$

Since the function $g(v) = (1/2)(1 + v^2)$ is increasing $\mathbb{R}_+$, then $v_n < g(v_n)$ for every $n = 0, 1, \ldots$. Thus (4.4) can be reduced to the form

$$u_n = 1, \quad v_0 = 0, \quad v_{n+1} = g(v_n) = \frac{1}{2}\left(1 + v_n^2\right), \quad n = 0, 1, \ldots. \tag{4.5}$$

The sequence $(g(v_n))_{n=0}^{\infty}$ is strictly increasing, whence also $(v_n)_{n=0}^{\infty}$ is strictly increasing by (4.5). Thus the set $W = \{(1, g(v_n))\}_{n \in \mathbb{N}_0}$ is an initial segment of C. Moreover, $v_0 = 0 < 1$, and if $0 \leq v_n < 1$, then $0 < g(v_n) < 1$. Since $(g(v_n))_{n=0}^{\infty}$ is bounded above by 1, then $v_* = \lim_n g(v_n)$ exists, and $0 < v_* \leq 1$. Thus $(1, v_*) = \sup W$, and it belongs to X, whence $(1, v_*) \in C$ by Lemma 2.3(c). To determine v_*, notice that $v_{n+1} \rightarrow v_*$ by (4.5). Thus $v_* = g(v_*)$, or equivalently, $v_*^2 - 2v_* + 1 = 0$, so that $v_* = 1$. Since $\sup W = (1, v_*) = (1, 1)$, then $(1, 1) \in C$ by Lemma 2.3(c). Because $(1, 1)$ is a maximal element of X, then $(1, 1) = \max C$. Moreover, $G(1, 1) = (1, 1)$, so that $(1, 1)$ is a fixed point of G.

The first $m + 1$ elements of the w-o chain C of aG-iterations can be estimated by the following Maple program (floor$(\cdot) = [\cdot]$):

x := min(1,1-floor(u) + floor(v)): y := (floor(u) + v²)/2: (u, v) := (1, 0) : c[0] := (u, v):
for n to m do (u, v) := (max(x, u), evalf(max(y, v)); c[n] := (u, v) end do;
For instance, c[100000] = (1, 0.99998).
The verification of the following properties are left to the reader.

(i) If $c = (u, v) \in X$, $u < 1$, and $v < 1$, then the elements of w-o chain C of aG-iterations, after two first terms if $u < 1$, are of the form $(1, w_n)$, $n = 0, 1, \ldots$, where $(w_n)_{n=0}^{\infty}$ is increasing and converges to 1. Thus $(1, 1)$ is the maximum of C and a fixed point of G.

(ii) If $a = (u, 1)$, $u < 1$, or $a = (1, -1)$, then $C = \{a, (1, 1)\}$.

(iii) If $a = (1, 0)$, then $G^{2k}a = (1, z_k)$ and $G^{2k+1}a = (0, y_k)$, $k \in \mathbb{N}_0$, where the sequences (z_k) and (y_k) are bounded and increasing. The limit z of (z_k) is the smaller real root of $z^4 - 8z + 4 = 0$; $z \approx 0.50834742498666121699$, and the limit y of (y_k) is $y = (1/2)z^2 \approx 0.12920855224528457650$. Moreover $G(1, y) = (0, z)$ and $G(0, z) = (1, y)$, whence no subsequence of the iteration $(G^n a)$ converges to a fixed point of G.

(iv) For any choice of $a = (u, v) \in P \setminus \{(1, 1)\}$ the iterations $G^n a$ and $G^{n+1} a$ are not order related when $n \geq 2$. The sequence $(G^n c)$ does not converge, and no subsequence of it converges to a fixed point of G.

(v) Denote $Y = \{(u, v) \in \mathbb{R}_+^2 : u^2 + v^2 \leq 2, v > 0\} \cup \{(1, 0)\}$. The function G, defined by (4.1), satisfies $G[Y] \subset Y$ and is maximalizing. The maximum of the w-o chain of aG-iterations with $a = (1, 0)$ is $x_* = (1, 1)$, and x_* is a fixed point of G. If $x \in Y \setminus \{x_*\}$, then x and $G(x)$ are not comparable.

The following example shows that G need not to have a fixed point if either of the hypothesis of Theorem 3.1 is not valid.

Example 4.2. Denote $a = (1, y)$ and $b = (0, z)$, where y and z are as in Example 4.1. Choose $X = \{a, b\}$, and let $G : X \to X$ be defined by (4.1). G is maximalizing, but G has no fixed points, since $G(a) = b$ and $G(b) = a$. The last hypothesis of Theorem 3.1 is not satisfied.

Denoting $c = (1, z)$, then the set $X = \{a, b, c\}$ is a complete join lattice, that is, every nonempty subset of X has the supremum in X. Let $G : X \to X$ satisfy $G(a) = b$ and $G(b) = G(c) = a$. G has no fixed points, but G is not maximalizing, since $G(c) < c$.

Example 4.3. The components $u = 1$, $v = 1$ of the fixed point of G in Example 4.1 form also a solution of the system

$$u = \min\{1, 1 - [u] + [v]\}, \quad v = \frac{[u] + v^2}{2}. \tag{4.6}$$

Moreover a Maple program introduced in Example 4.1 serves a method to estimate this solution. When $m = 100000$, the estimate is $u = 1$, $v = 0.99998$.

Remark 4.4. The standard "solve" and "fsolve" commands of Maple 12 do not give a solution or its approximation for the system of Example 4.3.

In Example 4.1 the mapping G is nonincreasing, nonextensive, nonascending, not semiincreasing upward, and noncontinuous.

Chain $C(a)$ is compared in [10] with three other chains which generalize the sequence of ordinary iterations $(G^n(a))_{n=0}^{\infty}$, and which are used to prove fixed point results for G. These chains are the generalized orbit $O(a)$ defined in [10] (being identical with the set $W(a)$ defined in [11]), the smallest admissible set $I(a)$ containing a (cf. [12–14]), and the smallest complete G-chain $B(a)$ containing a (cf. [10, 15]). If G is extensive, and if nonempty chains of X have supremums, then $C(a) = O(a) = I(a)$, and $B(a)$ is their cofinal subchain (cf. [10, Corollary 7]). The common maximum x_* of these four chains is a fixed point of G. This result implies Bourbaki's Fixed Point Theorem.

On the other hand, if the hypotheses of Theorem 3.4 hold and $x \in C(a) \setminus \{a, x_*\}$, then x and $G(x)$ are not necessarily comparable. The successor of such an x in $C(a)$ is $\sup\{x, G(x)\}$ by [14, Proposition 5]. In such a case the chains $O(a)$, $I(a)$ and $B(a)$ attain neither x nor any fixed point of G. For instance when $a = (0, 0)$ in Example 4.1, then $C(a) = \{(0, 0)\} \cup C$, where C is the w-o chain of $(1, 0)G$-iterations. Since $(G^n(0, 0))_{n=0}^{\infty} = \{(0, 0)\} \cup (G^n(1, 0))_{n=0}^{\infty}$, then $B(a)$ does not exist, $O(a) = I(a) = \{(0, 0), (1, 0)\}$ (see [10]). Thus only $C(a)$ attains a fixed point of G as its maximum. As shown in Example 4.1, the consecutive elements of the iteration sequence $(G^n(1, 0))_{n=0}^{\infty}$ are unordered, and their limits are not fixed points of G. Hence, in these examples also finite combinations of chains $W(a_i)$ used in [16, Theorem 4.2] to prove a fixed point result are insufficient to attain a fixed point of G.

Neither the above-mentioned four chains nor their duals are available to find fixed points of G if a and $G(a)$ are unordered. For instance, they cannot be applied to prove Theorems 3.1 and 3.5 or Propositions 3.2 and 3.7.

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