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Research Article

Weak Convergence Theorems for a Countable Family of Strict Pseudocontractions in Banach Spaces

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We investigate the convergence of Mann-type iterative scheme for a countable family of strict pseudocontractions in a uniformly convex Banach space with the Fréchet differentiable norm. Our results improve and extend the results obtained by Marino-Xu, Zhou, Osilike-Udomene, Zhang-Guo and the corresponding results. We also point out that the condition given by Chidume-Shahzad (2010) is not satisfied in a real Hilbert space. We show that their results still are true under a new condition.

1. Introduction

Let E and E^* be a real Banach space and the dual space of E, respectively. Let K be a nonempty subset of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. If E is smooth or E^* is strictly convex, then J is single-valued.

Throughout this paper, we denote the single valued duality mapping by j and denote the set of fixed points of a nonlinear mapping $T : K \to E$ by

$$F(T) = \{x \in K : Tx = x\}. \tag{1.1}$$

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called

(i) *pseudocontractive* [1] if, for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \tag{1.2}$$

(ii) *λ-strictly pseudocontractive* [2] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2, \tag{1.3}$$

or equivalently

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda\|(I-T)x - (I-T)y\|^2, \tag{1.4}$$

(iii) *L-Lipschitzian* if, for all $x, y \in D(T)$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|. \tag{1.5}$$

Remark 1.2. It is obvious by the definition that

(1) every strictly pseudocontractive mapping is pseudocontractive,

(2) every λ-strictly pseudocontractive mapping is $((1+\lambda)/\lambda)$-Lipschitzian; see [3].

Remark 1.3. Let K be a nonempty subset of a real Hilbert space and $T : K \to K$ a mapping. Then T is said to be κ-strictly pseudocontractive [2] if, for all $x, y \in D(T)$, there exists $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \kappa\|(I-T)x - (I-T)y\|^2. \tag{1.6}$$

It is well know that (1.6) is equivalent to the following:

$$\langle Tx - Ty, x-y\rangle \leq \|x-y\|^2 - \frac{1-\kappa}{2}\|(I-T)x - (I-T)y\|^2. \tag{1.7}$$

It is worth mentioning that the class of strict pseudocontractions includes properly the class of nonexpansive mappings. Moreover, we know from [4] that the class of pseudocontractions also includes properly the class of strict pseudocontractions. A mapping $A : E \to E$ is called *accretive* if, for all $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that $\langle Ax - Ay, j(x-y)\rangle \geq 0$. It is also known that A is accretive if and only if $T := I - A$ is pseudocontractive. Hence, a solution of the equation $Au = 0$ is a solution of the fixed point of $T := I - A$. Note that if $T := I - A$, then A is λ-strictly accretive if and only if T is λ-strictly pseudocontractive.

In 1953, Mann [5] introduced the iteration as follows: a sequence $\{x_n\}$ defined by $x_0 \in K$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.8}$$

where $\alpha_n \in [0,1]$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.8) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [6]). However, if T is a Lipschitzian pseudocontractive mapping, then Mann iteration defined by (1.8) may fail to converge in a Hilbert space; see [4].

In 1967, Browder-Petryshyn [2] introduced the class of strict pseudocontractions and proved existence and weak convergence theorems in a real Hilbert setting by using Mann's iteration (1.8) with a constant sequence $\alpha_n = \alpha$ for all n. Respectively, Marino-Xu [7] and Zhou [8] extended the results of Browder-Petryshyn [2] to Mann's iteration process (1.8). To be more precise, they proved the following theorem.

Theorem 1.4 (see [7]). *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a κ-strict pseudocontraction for some $0 \leq \kappa < 1$, and assume that T admits a fixed point in K. Let a sequence $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm* (1.8). *Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - \kappa)(1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Meanwhile, Marino, and Xu raised the open question: whether Theorem 1.4 can be extended to Banach spaces which are uniformly convex and have a Fréchet differentiable norm. Later, Zhou [9] and Zhang-Su [10], respectively, extended the result above to 2-uniformly smooth and q-uniformly smooth Banach spaces which are uniformly convex or satisfy Opial's condition.

In 2001, Osilike-Udomene [11] proved the convergence theorems of the Mann [5] and Ishikawa [12] iteration methods in the framework of q-uniformly smooth and uniformly convex Banach spaces. They also obtained that a sequence $\{x_n\}$ defined by (1.8) converges weakly to a fixed point of T under suitable control conditions. However, the sequence $\{\alpha_n\} \subset [0,1]$ excluded the canonical choice $\alpha_n = 1/n$, $n \geq 1$. This was a motivation for Zhang-Guo [13] to improve the results in the same space. Observe that the results of Osilike-Udomene [11] and Zhang-Guo [13] hold under the assumption that

$$C_q < \frac{q^{\lambda}}{b^{q-1}}, \tag{1.9}$$

for some $b \in (0,1)$ and C_q is a constant depending on the geometry of the space.

Lemma 1.5 (see [14–16]). *Let E be a uniformly smooth real Banach space. Then there exists a nondecreasing continuous function $\beta : [0,\infty) \to [0,\infty)$ with $\lim_{t \to 0^+}\beta(t) = 0$ and $\beta(ct) \leq c\beta(t)$ for $c \geq 1$ such that, for all $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|). \tag{1.10}$$

Recently, Chidume-Shahzad [17] extended the results of Osilike-Udomene [11] and Zhang-Guo [13] by using Reich's inequality (1.10) to the much more general real Banach spaces which are uniformly smooth and uniformly convex. Under the assumption that

$$\beta(t) \leq \frac{\lambda t}{\max\{2r, 1\}}, \tag{1.11}$$

for some $r > 0$, they proved the following theorem.

Theorem 1.6 (see [17]). *Let E be a uniformly smooth real Banach space which is also uniformly convex and K a nonempty closed convex subset of E. Let $T : K \to K$ be a λ-strict pseudocontraction for some $0 \leq \lambda < 1$ with $x^* \in F(T) := \{x \in K : Tx = x\} \neq \emptyset$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \tag{1.12}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Then, $\{x_n\}$ converges weakly to a fixed point of T.

However, we would like to point out that the results of Chidume-Shahzad [17] do not hold in real Hilbert spaces. Indeed, we know from Chidume [14] that

$$\beta(t) = \sup\left\{ \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x)\rangle : \|x\| \leq 1, \|y\| \leq 1 \right\}. \tag{1.13}$$

If E is a real Hilbert space, then we have

$$\begin{aligned}
\beta(t) &= \sup\left\{ \frac{\|x + ty\|^2 - \|x\|^2}{t} - 2\langle y, x\rangle : \|x\| \leq 1, \|y\| \leq 1 \right\} \\
&= \sup\left\{ \frac{\|x\|^2 + 2t\langle x, y\rangle + t^2\|y\|^2 - \|x\|^2}{t} - 2\langle y, x\rangle : \|x\| \leq 1, \|y\| \leq 1 \right\} \\
&= \sup\left\{ t\|y\|^2 : \|y\| \leq 1 \right\} = t.
\end{aligned} \tag{1.14}$$

On the other hand, by assumption (1.11), we see that

$$\beta(t) \leq \frac{\lambda t}{\max\{2r, 1\}} < t, \tag{1.15}$$

which is a contradiction.

It is known that one can extend his result from a single strict pseudocontraction to a finite family of strict pseudocontractions by replacing the convex combination of these mappings in the iteration under suitable conditions. The construction of fixed points for pseudocontractions via the iterative process has been extensively investigated by many authors; see also [18–22] and the references therein.

Our motivation in this paper is the following:

(1) to modify the normal Mann iteration process for finding common fixed points of an infinitely countable family of strict pseudocontractions,

(2) to improve and extend the results of Chidume-Shahzad [17] from a real uniformly smooth and uniformly convex Banach space to a real uniformly convex Banach space which has the Fréchet differentiable norm.

Motivated and inspired by Marino-Xu [7], Osilike-Udomene [11], Zhou [8], Zhang-Guo [13], and Chidume-Shahzad [17], we consider the following Mann-type iteration: $x_1 \in K$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \tag{1.16}$$

where α_n is a real sequence in $[0,1]$ and $\{T_n\}_{n=1}^{\infty}$ is a countable family of strict pseudocontractions on a closed and convex subset K of a real Banach space E.

In this paper, we prove the weak convergence of a Mann-type iteration process (1.16) in a uniformly convex Banach space which has the Fréchet differentiable norm for a countable family of strict pseudocontractions under some appropriate conditions. The results obtained in this paper improve and extend the results of Chidume-Shahzad [17], Marino-Xu [7], Osilike-Udomene [11], Zhou [8], and Zhang-Guo [13] in some aspects.

We will use the following notation:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

(ii) $\omega_\omega(x_n) = \{x : \ x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that, for $x, y \in E$ with $\|x\|, \|y\| \leq 1$, and $\|x-y\| \geq \epsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\right\}, \tag{2.1}$$

for all $\epsilon \in [0,2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in S(E)$. In this case E is called *smooth*. The norm of E is said to be *Fréchet differentiable* or E is *Fréchet smooth* if, for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. In other words, there exists a function $\varepsilon_x(s)$ with $\varepsilon_x(s) \to 0$ as $s \to 0$ such that

$$\big| \|x+ty\| - \|x\| + t\langle y, j(x)\rangle \big| \leq |t|\varepsilon_x(|t|) \tag{2.3}$$

for all $y \in S(E)$. In this case the norm is *Gâteaux differentiable* and

$$\lim_{t\to 0} \sup_{y\in S(E)} \left| \frac{1/2\|x+ty\|^2 - 1/2\|x\|^2}{t} - \langle y, j(x)\rangle \right| = 0 \tag{2.4}$$

for all $x \in E$. On the other hand,

$$\frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle \le \frac{1}{2}\|x+h\|^2 \le \frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle + b(\|h\|) \tag{2.5}$$

for all $x, h \in E$, where b is a function defined on $\mathbb{R}^+$ such that $\lim_{t\to 0^+}(b(t)/t) = 0$. The norm of E is called *uniformly Fréchet differentiable* if the limit is attained uniformly for $x, y \in S(E)$.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E),\ \|y\| \le t \right\}. \tag{2.6}$$

A Banach space E is said to be *uniformly smooth* if $\rho_E(t)/t \to 0$ as $t \to 0$. Let $q > 1$, then E is said to be *q-uniformly smooth* if there exists $c > 0$ such that $\rho_E(t) \le ct^q$. It is easy to see that if E is q-uniformly smooth, then E is uniformly smooth. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable, and hence the norm of E is Fréchet differentiable, and it is also known that if E is Fréchet smooth, then E is smooth. Moreover, every uniformly smooth Banach space is reflexive. For more details, we refer the reader to [14, 23]. A Banach space E is said to satisfy *Opial's condition* [24] if $x \in E$ and $x_n \rightharpoonup x$; then

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|, \quad \forall y \in E,\ x \ne y. \tag{2.7}$$

In the sequel, we will need the following lemmas.

Lemma 2.1 (see [23]). *Let E be a Banach space and $J : E \to 2^{E^*}$ the duality mapping. Then one has the following:*

(i) $\|x+y\|^2 \ge \|x\|^2 + 2\langle y, j(x)\rangle$ *for all $x, y \in E$, where $j(x) \in J(x)$;*

(ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$ *for all $x, y \in E$, where $j(x+y) \in J(x+y)$.*

Lemma 2.2 (see [25]). *Let E be a real uniformly convex Banach space, K a nonempty, closed, and convex subset of E, and $T : K \to K$ a continuous pseudocontractive mapping. Then, $I - T$ is demiclosed at zero, that is, for all sequence $\{x_n\} \subset K$ with $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$ it follows that $p = Tp$.*

Lemma 2.3 (see [25]). *Let E be a real reflexive Banach space which satisfies Opial's condition, K a nonempty, closed and convex subset of E and $T : K \to K$ a continuous pseudocontractive mapping. Then, $I - T$ is demiclosed at zero.*

Lemma 2.4 (see [26]). *Let E be a real uniformly convex Banach space with a Fréchet differentiable norm. Let K be a closed and convex subset of E and $\{S_n\}_{n=1}^{\infty}$ a family of L_n-Lipschitzian self-mappings* *on K such that $\sum_{n=1}^{\infty}(L_n-1)<\infty$ and $F=\bigcap_{n=1}^{\infty}F(S_n)\neq\emptyset$. For arbitrary $x_1\in K$, define $x_{n+1}=S_nx_n$ for all $n\geq 1$. Then for every $p,q\in F$, $\lim_{n\to\infty}\langle x_n, j(p-q)\rangle$ exists, in particular, for all $u,v\in\omega_\omega(x_n)$ and $p,q\in F$, $\langle u-v, j(p-q)\rangle=0$.*

Lemma 2.5 (see [17, 27]). *Let $\{a_n\},\{b_n\}$ and $\{\delta_n\}$, be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1}\leq(1+\delta_n)a_n+b_n,\quad n\geq 0. \tag{2.8}$$

If $\sum_{n=0}^{\infty}\delta_n<\infty$ and $\sum_{n=0}^{\infty}b_n<\infty$, then $\lim_{n\to\infty}a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty}a_n=0$.

To deal with a family of mappings, the following conditions are introduced. Let K be a subset of a real Banach space E, and let $\{T_n\}$ be a family of mappings of K such that $\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. Then $\{T_n\}$ is said to satisfy the AKTT-condition [28] if for each bounded subset B of K,

$$\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|: z\in B\}<\infty. \tag{2.9}$$

Lemma 2.6 (see [28]). *Let K be a nonempty and closed subset of a Banach space E, and let $\{T_n\}$ be a family of mappings of K into itself which satisfies the AKTT-condition, then the mapping $T: K\to K$ defined by*

$$Tx=\lim_{n\to\infty}T_nx,\quad \forall x\in K \tag{2.10}$$

satisfies

$$\lim_{n\to\infty}\sup\{\|Tz-T_nz\|: z\in B\}=0 \tag{2.11}$$

for each bounded subset B of K.

So we have the following results proved by Boonchari-Saejung [29, 30].

Lemma 2.7 (see [29, 30]). *Let K be a closed and convex subset of a smooth Banach space E. Suppose that $\{T_n\}_{n=1}^{\infty}$ is a family of λ-strictly pseudocontractive mappings from K into E with $\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$ and $\{\beta_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\beta_n=1$. Then the following conclusions hold:*

(1) *$G:=\sum_{n=1}^{\infty}\beta_nT_n: K\to E$ is a λ-strictly pseudocontractive mapping;*

(2) *$F(G)=\bigcap_{n=1}^{\infty}F(T_n)$.*

Lemma 2.8 (see [30]). *Let K be a closed and convex subset of a smooth Banach space E. Suppose that $\{S_k\}_{k=1}^{\infty}$ is a countable family of λ-strictly pseudocontractive mappings of K into itself with $\bigcap_{k=1}^{\infty}F(S_k)\neq\emptyset$. For each $n\in\mathbb{N}$, define $T_n: K\to K$ by*

$$T_nx=\sum_{k=1}^{n}\beta_n^kS_kx,\quad x\in K, \tag{2.12}$$

where $\{\beta_n^k\}$ *is a family of nonnegative numbers satisfying*

(i) $\sum_{k=1}^{n} \beta_n^k = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\beta^k := \lim_{n\to\infty} \beta_n^k > 0$ *for all* $k \in \mathbb{N}$;

(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

Then

(1) *each* T_n *is a* λ*-strictly pseudocontractive mapping;*

(2) $\{T_n\}$ *satisfies AKTT-condition;*

(3) *If* $T : K \to K$ *is defined by*

$$Tx = \sum_{k=1}^{\infty} \beta^k S_k x, \quad x \in K, \tag{2.13}$$

then $Tx = \lim_{n\to\infty} T_n x$ *and* $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k)$.

For convenience, we will write that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.6 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

3. Main Results

Lemma 3.1. *Let* E *be a real Banach space, and let* K *be a nonempty, closed, and convex subset of* E*. Let* $\{T_n\}_{n=1}^{\infty} : K \to K$ *be a family of* λ*-strict pseudocontractions for some* $0 < \lambda < 1$ *such that* $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$*. Define a sequence* $\{x_n\}$ *by* $x_1 \in K$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1]$ *satisfying* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$*. If* $\{T_n\}$ *satisfies the AKTT-condition, then*

(i) $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in F$;

(ii) $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$.

Proof. Let $p \in F$, and put $L = (\lambda + 1)/\lambda$. First, we observe that

$$\begin{gathered} \|x_{n+1} - p\| \leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|T_n x_n - p\| \leq (1+L)\|x_n - p\|, \\ \|x_{n+1} - x_n\| = \alpha_n\|T_n x_n - x_n\| \leq \alpha_n(1+L)\|x_n - p\|. \end{gathered} \tag{3.2}$$

Since T_n is a λ-strict pseudocontraction, there exists $j(x_{n+1}-p) \in J(x_{n+1}-p)$. By Lemma 2.1 we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(x_n-p)+\alpha_n(T_nx_n-x_n)\|^2 \\
&\le \|x_n-p\|^2 + 2\alpha_n\langle T_nx_n-x_n, j(x_{n+1}-p)\rangle \\
&= \|x_n-p\|^2 + 2\alpha_n\langle T_nx_n-T_nx_{n+1}, j(x_{n+1}-p)\rangle \\
&\quad + 2\alpha_n\langle T_nx_{n+1}-x_{n+1}, j(x_{n+1}-p)\rangle + 2\alpha_n\langle x_{n+1}-x_n, j(x_{n+1}-p)\rangle \\
&\le \|x_n-p\|^2 + 2\alpha_n L\|x_n-x_{n+1}\|\|x_{n+1}-p\| \\
&\quad - 2\alpha_n\lambda\|T_nx_{n+1}-x_{n+1}\|^2 + 2\alpha_n\|x_n-x_{n+1}\|\|x_{n+1}-p\| \\
&\le \|x_n-p\|^2 + 2\alpha_n^2 L(1+L)^2\|x_n-p\|^2 \\
&\quad - 2\alpha_n\lambda\|T_nx_{n+1}-x_{n+1}\|^2 + 2\alpha_n^2(1+L)^2\|x_n-p\|^2 \\
&= \|x_n-p\|^2 + 2\alpha_n^2(1+L)^3\|x_n-p\|^2 - 2\alpha_n\lambda\|T_nx_{n+1}-x_{n+1}\|^2.
\end{aligned} \tag{3.3}$$

This implies that

$$\|x_{n+1}-p\|^2 \le \left(1+2\alpha_n^2(1+L)^3\right)\|x_n-p\|^2. \tag{3.4}$$

Hence, by $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$, we have from Lemma 2.5 that $\lim_{n\to\infty}\|x_n-p\|$ exists; consequently, $\{x_n\}$ is bounded. Moreover, by (3.3), we also have

$$\sum_{n=1}^{\infty}\alpha_n\lambda\|T_nx_{n+1}-x_{n+1}\|^2 \le \sum_{n=1}^{\infty}\left(\|x_n-p\|^2 - \|x_{n+1}-p\|^2\right) + 2(1+L)^3M_1^2\sum_{n=1}^{\infty}\alpha_n^2 < \infty, \tag{3.5}$$

where $M_1 = \sup_{n\ge1}\{\|x_n-p\|\}$. It follows that $\liminf_{n\to\infty}\|T_nx_{n+1}-x_{n+1}\| = 0$. Since $\{x_n\}$ is bounded,

$$\begin{aligned}
\|x_{n+1}-T_{n+1}x_{n+1}\| &\le \|x_{n+1}-T_nx_{n+1}\| + \|T_nx_{n+1}-T_{n+1}x_{n+1}\| \\
&\le \|x_{n+1}-T_nx_{n+1}\| + \sup_{z\in\{x_n\}}\|T_nz-T_{n+1}z\|.
\end{aligned} \tag{3.6}$$

Since $\{T_n\}$ satisfies the AKTT-condition, it follows that $\liminf_{n\to\infty}\|x_n-T_nx_n\| = 0$. This completes the proof of (i) and (ii). □

Lemma 3.2. *Let E be a real Banach space with the Fréchet differentiable norm. For $x \in E$, let $\beta^*(t)$ be defined for $0 < t < \infty$ by*

$$\beta^*(t) = \sup_{y\in S(E)}\left|\frac{\|x+ty\|^2-\|x\|^2}{t} - 2\langle y, j(x)\rangle\right|. \tag{3.7}$$

Then, $\lim_{t\to 0^+}\beta^*(t) = 0$, *and*

$$\|x+h\|^2 \le \|x\|^2 + 2\langle h, j(x)\rangle + \|h\|\beta^*(\|h\|) \tag{3.8}$$

for all $h \in E \setminus \{0\}$.

Proof. Let $x \in E$. Since E has the Fréchet differentiable norm, it follows that

$$\lim_{t\to 0}\sup_{y\in S(E)}\left|\frac{1/2\|x+ty\|^2 - 1/2\|x\|^2}{t} - \langle y, j(x)\rangle\right| = 0. \tag{3.9}$$

Then $\lim_{t\to 0^+}\beta^*(t) = 0$, and hence

$$\left|\frac{\|x+ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x)\rangle\right| \le \beta^*(t), \quad \forall y \in S(E) \tag{3.10}$$

which implies that

$$\|x+ty\|^2 \le \|x\|^2 + 2t\langle y, j(x)\rangle + t\beta^*(t), \quad \forall y \in S(E). \tag{3.11}$$

Suppose that $h \neq 0$. Put $y = h/\|h\|$ and $t = \|h\|$. By (3.11), we have

$$\|x+h\|^2 \le \|x\|^2 + 2\langle h, j(x)\rangle + \|h\|\beta^*(\|h\|). \tag{3.12}$$

This completes the proof. □

Remark 3.3. In a real Hilbert space, we see that $\beta^*(t) = t$ for $t > 0$.

In our more general setting, throughout this paper we will assume that

$$\beta^*(t) \le 2t, \tag{3.13}$$

where β^* is a function appearing in (3.8).

So we obtain the following results.

Lemma 3.4. *Let E be a real Banach space with the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E. Let $\{T_n\}_{n=1}^{\infty} : K \to K$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in K$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \ge 1, \tag{3.14}$$

where $\{\alpha_n\} \subset [0,1]$ satisfying $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\lim_{n\to\infty}\|x_n - T_n x_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. Let $p \in F$, and put $M_2 = \sup_{n\geq 1}\{\|x_n - T_n x_n\|\} > 0$. Then by (3.8) and (3.13) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(x_n - p) + \alpha_n (T_n x_n - x_n)\|^2 \\
&\leq \|x_n - p\|^2 + 2\alpha_n \langle T_n x_n - x_n, j(x_n - p)\rangle \\
&\quad + \alpha_n \|T_n x_n - x_n\| \beta^*(\alpha_n \|T_n x_n - x_n\|) \\
&\leq \|x_n - p\|^2 - 2\alpha_n \lambda \|x_n - T_n x_n\|^2 + 2\alpha_n^2 \|x_n - T_n x_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\alpha_n \lambda \|x_n - T_n x_n\|^2 + 2\alpha_n^2 M_2^2.
\end{aligned} \tag{3.15}$$

It follows that

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - T_n x_n\|^2 < \infty. \tag{3.16}$$

Observe that

$$\begin{aligned}
\|x_n - T_{n+1} x_{n+1}\|^2 &= \|(x_n - T_n x_n) + (T_n x_n - T_{n+1} x_{n+1})\|^2 \\
&\leq \|x_n - T_n x_n\|^2 + 2\langle T_n x_n - T_{n+1} x_{n+1}, j(x_n - T_{n+1} x_{n+1})\rangle \\
&= \|x_n - T_n x_n\|^2 + 2\langle T_n x_n - T_n x_{n+1}, j(x_n - T_{n+1} x_{n+1})\rangle \\
&\quad + 2\langle T_n x_{n+1} - T_{n+1} x_{n+1}, j(x_n - T_{n+1} x_{n+1})\rangle \\
&\leq \|x_n - T_n x_n\|^2 + 2L\|x_n - x_{n+1}\| \|x_n - T_{n+1} x_{n+1}\| \\
&\quad + 2\|T_n x_{n+1} - T_{n+1} x_{n+1}\| \|x_n - T_{n+1} x_{n+1}\| \\
&\leq \|x_n - T_n x_n\|^2 + 2L\|x_n - x_{n+1}\| \|x_n - T_n x_n\| \\
&\quad + 2L\|x_n - x_{n+1}\| \|T_n x_n - T_n x_{n+1}\| \\
&\quad + 2L\|x_n - x_{n+1}\| \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
&\quad + 2\|T_n x_{n+1} - T_{n+1} x_{n+1}\| \|x_n - x_{n+1}\| \\
&\quad + 2\|T_n x_{n+1} - T_{n+1} x_{n+1}\| \|x_{n+1} - T_{n+1} x_{n+1}\| \\
&\leq \|x_n - T_n x_n\|^2 + \left(2L\alpha_n + 2L^2\alpha_n^2\right)\|x_n - T_n x_n\|^2 \\
&\quad + (2LM_2\alpha_n + 2M_2\alpha_n + 2M_2)\|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
&\leq \|x_n - T_n x_n\|^2 + 2L(1+L)\alpha_n \|x_n - T_n x_n\|^2 \\
&\quad + 2M_2(L+2)\|T_n x_{n+1} - T_{n+1} x_{n+1}\|.
\end{aligned} \tag{3.17}$$

By (3.17), we have

$$
\begin{aligned}
\|x_{n+1}-T_{n+1}x_{n+1}\|^2 &\le (1-\alpha_n)\|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n\|T_nx_n-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_{n+1}x_{n+1}\|^2\\
&\quad+\alpha_n(\|T_nx_n-T_nx_{n+1}\|+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|)^2\\
&= \|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n\|T_nx_n-T_nx_{n+1}\|^2\\
&\quad+2\alpha_n\|T_nx_n-T_nx_{n+1}\|\|T_nx_{n+1}-T_{n+1}x_{n+1}\|\\
&\quad+\alpha_n\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n^2L^2\|x_n-T_nx_n\|^2\\
&\quad+2\alpha_n^2L\|x_n-T_nx_n\|\|T_nx_{n+1}-T_{n+1}x_{n+1}\|\\
&\quad+\alpha_n\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n^2L^2M_2^2\\
&\quad+2LM_2\|T_nx_{n+1}-T_{n+1}x_{n+1}\|+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_nx_n\|^2+2L(1+L)\alpha_n\|x_n-T_nx_n\|^2\\
&\quad+2M_2(L+2)\|T_nx_{n+1}-T_{n+1}x_{n+1}\|+\alpha_n^2L^2M_2^2\\
&\quad+2LM_2\|T_nx_{n+1}-T_{n+1}x_{n+1}\|+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_nx_n\|^2+2L(1+L)\alpha_n\|x_n-T_nx_n\|^2\\
&\quad+\alpha_n^2L^2M_2^2+2M_2(2L+2)\|T_nx_{n+1}-T_{n+1}x_{n+1}\|\\
&\quad+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_nx_n\|^2+2L(1+L)\alpha_n\|x_n-T_nx_n\|^2\\
&\quad+\ \alpha_n^2L^2M_2^2+2M_2(2L+2)\sup_{z\in\{x_n\}}\|T_nz-T_{n+1}z\|\\
&\quad+\ \sup_{z\in\{x_n\}}\|T_nz-T_{n+1}z\|^2.
\end{aligned}
\tag{3.18}
$$

Since $\sum_{n=1}^{\infty}\alpha_n\|x_n-T_nx_n\|^2<\infty$, $\sum_{n=1}^{\infty}\alpha_n^2<\infty$, and $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|:z\in\{x_n\}\}<\infty$, it follows from Lemma 2.5 that $\lim_{n\to\infty}\|x_n-T_nx_n\|$ exists. Hence, by Lemma 3.1(ii), we can conclude that $\lim_{n\to\infty}\|x_n-T_nx_n\|=0$. Since

$$
\begin{aligned}
\|x_n-Tx_n\| &\le \|x_n-T_nx_n\|+\|T_nx_n-Tx_n\|\\
&\le \|x_n-T_nx_n\|+\sup_{z\in\{x_n\}}\|T_nz-Tz\|,
\end{aligned}
\tag{3.19}
$$

it follows from Lemma 2.6 that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. This completes the proof. □

Now, we prove our main result.

Theorem 3.5. *Let E be a real uniformly convex Banach space with the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E. Let $\{T_n\}_{n=1}^{\infty} : K \to K$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in K$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \tag{3.20}$$

where $\{\alpha_n\} \subset [0,\lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}$.

Proof. Let $p \in F$, and define $S_n : K \to K$ by

$$S_n x = (1-\alpha_n)x + \alpha_n T_n x, \quad x \in K. \tag{3.21}$$

Then $\bigcap_{n=1}^{\infty} F(S_n) = F = F(T)$. By (3.8), we have for bounded $x, y \in K$ that

$$\begin{aligned}
\left\|S_n x - S_n y\right\|^2 &= \left\|x - y - \alpha_n[x - y - (T_n x - T_n y)]\right\|^2 \\
&\leq \left\|x-y\right\|^2 - 2\alpha_n \langle (I-T_n)x - (I-T_n)y, j(x-y)\rangle \\
&\quad + \alpha_n \left\|x - y - (T_n x - T_n y)\right\| \beta^*(\alpha_n \left\|x-y-(T_n x - T_n y)\right\|) \\
&\leq \left\|x-y\right\|^2 - 2\alpha_n \lambda \left\|x - y - (T_n x - T_n y)\right\|^2 \\
&\quad + 2\alpha_n^2 \left\|x - y - (T_n x - T_n y)\right\|^2 \\
&= \left\|x-y\right\|^2 - 2\alpha_n(\lambda - \alpha_n)\left\|x - y - (T_n x - T_n y)\right\|^2 \\
&\leq \left\|x-y\right\|^2.
\end{aligned} \tag{3.22}$$

This implies that S_n is nonexpansive. By Lemma 3.1(i), we know that $\{x_n\}$ is bounded. By Lemma 3.4, we also know that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Applying Lemma 2.2, we also have $\omega_\omega(x_n) \subset F(T)$.

Finally, we will show that $\omega_\omega(x_n)$ is a singleton. Suppose that $x^*, y^* \in \omega_\omega(x_n) \subset F(T)$. Hence $x^*, y^* \in \bigcap_{n=1}^{\infty} F(S_n)$. By Lemma 2.4, $\lim_{n\to\infty}\langle x_n, j(x^* - y^*)\rangle$ exists. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. Then

$$\|x^* - y^*\|^2 = \langle x^* - y^*, j(x^* - y^*)\rangle = \lim_{k\to\infty}\langle x_{n_k} - x_{m_k}, j(x^* - y^*)\rangle = 0. \tag{3.23}$$

Hence $x^* = y^*$; consequently, $x_n \rightharpoonup x^* \in \bigcap_{n=1}^{\infty} F(S_n) = F$ as $n \to \infty$. This completes the proof. □

As a direct consequence of Theorem 3.5, Lemmas 2.7 and 2.8 we also obtain the following results.

Theorem 3.6. *Let E be a real uniformly convex Banach space with the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E. Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of λ_k-strict pseudocontractions of K into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define a sequence $\{x_n\}$ by $x_1 \in K$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \sum_{k=1}^{n} \beta_n^k S_k x_n, \quad n \geq 1, \tag{3.24}$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\{\beta_n^k\}$ satisfies conditions (i)–(iii) of Lemma 2.8. Then, $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.

Remark 3.7. (i) Theorems 3.5 and 3.6 extend and improve Theorems 3.3 and 3.4 of Chidume-Shahzad [17] in the following senses:

(i) from real uniformly smooth and uniformly convex Banach spaces to real uniformly convex Banach spaces with Fréchet differentiable norms;

(ii) from finite strict pseudocontractions to infinite strict pseudocontractions.

Using Opial's condition, we also obtain the following results in a real reflexive Banach space.

Theorem 3.8. *Let E be a real Fréchet smooth and reflexive Banach space which satisfies Opial's condition, and let K be a nonempty, closed, and convex subset of E. Let $\{T_n\}_{n=1}^{\infty}$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Define a sequence $\{x_n\}$ by $x_1 \in K$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \tag{3.25}$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}$.

Proof. Let $p \in F$. By Lemma 3.1(i), we know that $\lim_{n\to\infty}\|x_n - p\|$ exists. Since E has the Fréchet differentiable norm, by Lemma 3.4, we know that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. It follows from Lemma 2.3 that $\omega_\omega(x_n) \subset F(T) = F$. Finally, we show that $\omega_\omega(x_n)$ is a singleton. Let $x^*, y^* \in \omega_\omega(x_n)$, and let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be subsequences of $\{x_n\}$ chosen so that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. If $x^* \neq y^*$, then Opial's condition of E implies that

$$\begin{aligned} \lim_{n\to\infty}\|x_n - x^*\| &= \lim_{k\to\infty}\|x_{n_k} - x^*\| < \lim_{k\to\infty}\left\|x_{n_k} - y^*\right\| = \lim_{k\to\infty}\left\|x_{m_k} - y^*\right\| \\ &< \lim_{k\to\infty}\|x_{m_k} - x^*\| = \lim_{n\to\infty}\|x_n - x^*\|. \end{aligned} \tag{3.26}$$

This is a contradiction, and thus the proof is complete. □

Theorem 3.9. *Let E be a real Fréchet smooth and reflexive Banach space which satisfies Opial's condition, and let K be a nonempty, closed, and convex subset of E. Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of*

λ_k-strict pseudocontractions of K into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define a sequence $\{x_n\}$ by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \sum_{k=1}^{n} \beta_n^k S_k x_n, \quad n \geq 1, \tag{3.27}$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\{\beta_n^k\}$ satisfies conditions (i)–(iii) of Lemma 2.8. Then, $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.

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