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Research Article

Strong and Weak Convergence Theorems for Common Solutions of Generalized Equilibrium Problems and Zeros of Maximal Monotone Operators

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The purpose of this paper is to introduce and study two modified hybrid proximal-point algorithms for finding a common element of the solution set *EP* of a generalized equilibrium problem and the set $T^{-1}0\cap\widetilde{T}^{-1}0$ for two maximal monotone operators T and $\widetilde{T}$ defined on a Banach space X. Strong and weak convergence theorems for these two modified hybrid proximal-point algorithms are established.

1. Introduction

Let X be a real Banach space with its dual X^*. The mapping $J : X \to 2^{X^*}$ defined by

$$J(x) := \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in X, \tag{1.1}$$

is called the normalized duality mapping. From the Hahn-Banach theorem, it follows that $J(x) \neq \emptyset$ for each $x \in X$.

A Banach space X is said to be strictly convex, if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in X : \|z\| = 1\}$ with $x \neq y$. X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists

$\delta > 0$ such that $\|x+y\|/2 \leq 1-\delta$ for all $x, y \in U$ with $\|x-y\| \geq \epsilon$. Recall that each uniformly convex Banach space has the Kadec-Klee property, that is,

$$\begin{array}{c} x_n \rightharpoonup x \\ \|x_n\| \longrightarrow \|x\| \end{array} \Longrightarrow x_n \longrightarrow x. \tag{1.2}$$

It is well known that if X^* is strictly convex, then J is single-valued. In the sequel, we shall still denote the single-valued normalized duality mapping by J. Let C be a nonempty closed convex subset of X, $f : C \times C \to \mathbb{R}$ a bifunction, and $A : C \to X^*$ a nonlinear mapping. Very recently, Zhang [1] considered and studied the generalized equilibrium problem of finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by EP. Problem (1.3) and related problems have been studied and investigated extensively in the literature; See, for example, [2–12] and references therein. Whenever $A \equiv 0$, problem (1.3) reduces to the equilibrium problem of finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(f)$. Whenever $f \equiv 0$, problem (1.3) reduces to the variational inequality problem of finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $VI(C, A)$.

Whenever $X = H$ a Hilbert space, problem (1.3) was very recently introduced and considered by S. Takahashi and W. Takahashi [13]. Problem (1.3) is very general in the sense that it includes, as spacial cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; See, for example, [1, 2, 4, 6–9, 14–17] which are references therein.

A mapping $S : C \to X$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. Very recently, W. Takahashi and K. Zembayashi [18] proposed an iterative algorithm for finding a common element of the solution set of the equilibrium problem (1.4) and the set of fixed points of a relatively nonexpansive mapping S in a Banach space X. They also studied the strong and weak convergence of the sequences generated by their algorithm. In particular, they proposed the following iterative algorithm:

$$x_0 \in C,$$

$$y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n),$$

$$u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,$$
$$H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},$$
$$W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\},$$
$$x_{n+1} = \Pi_{H_n \cap W_n} x, \quad n \geq 0, \tag{1.6}$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in X$, $\{\alpha_n\} \subset [0,1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. They proved that the sequence $\{x_n\}$ generated by the above algorithm converges strongly to $\Pi_{F(S)\cap EP(f)} x_0$, where $\Pi_{F(S)\cap EP(f)}$ is the generalized projection of X onto $F(S) \cap EP(f)$. They have also studied the weak convergence of the sequence $\{x_n\}$ generated by the following algorithm:

$$u_0 \in X,$$
$$x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \tag{1.7}$$
$$u_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \quad n \geq 0,$$

to $z \in F(S) \cap EP(f)$, where $z = \lim_{n\to\infty} \Pi_{F(S)\cap EP(f)} x_n$.

Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space X. Let $A : C \to X^*$ be an α-inverse-strongly monotone mapping and $f : C \times C \to \mathbb{R}$ a bifunction satisfying the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Let $S_1, S_2 : C \to C$ be two relatively nonexpansive mappings such that $F(S_1) \cap F(S_2) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by

$$x_0 \in C, \quad C_0 = C;$$
$$z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JS_1 x_n),$$
$$y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS_2 z_n),$$
$$u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \tag{1.8}$$
$$C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1 - \beta_n)\phi(v, z_n) \leq \phi(v, x_n)\};$$
$$x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0.$$

Zhang [1] proved the strong convergence of the sequence $\{x_n\}$ to $\Pi_{F(S_1)\cap F(S_2)\cap EP} x_0$ under appropriate conditions.

On the other hand, a classic method of solving $0 \in Tx$ in a Hilbert space H is the proximal point algorithm which generates, for any starting point $x_0 \in H$, a sequence $\{x_n\}$ in H by the iterative scheme

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, \ldots, \tag{1.9}$$

where $\{r_n\}$ is a sequence in $(0, \infty)$, $J_r = (I + rT)^{-1}$ for each $r > 0$ is the resolvent operator for T, and I is the identity operator on H. This algorithm was first introduced by Martinet [19] and further studied by Rockafellar [20] in the framework of a Hilbert space H. Later several authors studied (1.9) and its variants in the setting of a Hilbert space H or in a Banach space X; See, for example, [15, 21–25] and references therein. Very recently, Li and Song [24] introduced and studied the following iterative scheme:

$$\begin{gathered} x_0 \in X \text{ chosen arbitrarily,} \\ y_n = J^{-1}\big(\beta_n J x_n + (1 - \beta_n) J J_{r_n} x_n\big), \\ x_{n+1} = J^{-1}\big(\alpha_n J x_0 + (1 - \alpha_n) J y_n\big), \quad n = 0, 1, 2, \ldots, \end{gathered} \tag{1.10}$$

where $J_r = (J + rT)^{-1} J$ and J is the duality mapping on X.

Algorithm (1.10) covers, as special cases, the algorithms introduced by Kohsaka and Takahashi [23] and Kamimura et al. [22] in a smooth and uniformly convex Banach space X.

Let X be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of X. Let $T : X \to 2^{X^*}$ be a maximal monotone operator such that:

(A5) $T^{-1}0 \cap EP(f) \neq \emptyset$.

In addition, for each $r > 0$, define a mapping $T_r : X \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0,\ \forall y \in C \right\} \tag{1.11}$$

for all $x \in X$.

Very recently, utilizing the ideas of the above algorithms in [15, 16, 18, 21, 22, 24], we [17] introduced two iterative methods for finding an element of $T^{-1}0 \cap EP(f)$ and established the following strong and weak convergence theorems.

Theorem 1.1 (see [17]). *Suppose that conditions (A1)–(A5) are satisfied and let $x_0 \in X$ be chosen arbitrarily. Consider the sequence*

$$x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \tag{1.12}$$

where

$$H_n = \{z \in C : \phi(z, T_{r_n} y_n) \leq \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, x_n)\},$$
$$W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \tag{1.13}$$
$$y_n = J^{-1}(\alpha_n J x_0 + (1 - \alpha_n)(\beta_n J x_n + (1 - \beta_n) J J_{r_n} x_n)),$$

T_r is defined by (1.11)*, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0\cap EP(f)}x_0$, where $\Pi_{T^{-1}0\cap EP(f)}$ is the generalized projection of X onto $T^{-1}0 \cap EP(f)$.*

Theorem 1.2 (see [17]). *Suppose that conditions (A1)–(A5) are satisfied and let $x_0 \in X$ be chosen arbitrarily. Consider the sequence*

$$x_{n+1} = J^{-1}(\alpha_n J x_0 + (1 - \alpha_n)(\beta_n J T_{r_n} x_n + (1 - \beta_n) J J_{r_n} T_{r_n} x_n)), \quad n = 0, 1, 2, \ldots, \tag{1.14}$$

where T_r is defined by (1.11)*, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfy the conditions $\sum_{n=0}^{\infty}\alpha_n < \infty$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to an element $z \in T^{-1}0 \cap EP(f)$, where $z = \lim_{n\to\infty}\Pi_{T^{-1}0\cap EP(f)}x_n$.*

The purpose of this paper is to introduce and study two new iterative methods for finding a common element of the solution set EP of generalized equilibrium problem (1.3) and the set $T^{-1}0 \cap \widetilde{T}^{-1}0$ for maximal monotone operators T and $\widetilde{T}$ in a uniformly smooth and uniformly convex Banach space X. Firstly, motivated by Theorem 1.1 and a result of Zhang [1], we introduce a sequence $\{x_n\}$ that converges strongly to $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$ under some appropriate conditions.

Secondly, inspired by Theorem 1.2 and a result of Zhang [1], we define a sequence that converges weakly to an element $z \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$, where $z = \lim_{n\to\infty}\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_n$ (Section 4).

Our results represent a generalization of known results in the literature, including those in [16–18, 24]. Our Theorems 3.1 and 4.2 are the extension and improvements of Theorems 1.1 and 1.2 in the following way:

(i) the problem of finding an element of $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$ includes the one of finding an element of $T^{-1}0 \cap EP(f)$ as a special case;

(ii) the algorithms in this paper are very different from those in [17] because of considering the complexity involving the problem of finding an element of $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$.

2. Preliminaries

Throughout the paper, we denote the strong convergence, weak convergence, and weak* convergence of a sequence $\{x_n\}$ to a point $x \in X$ by $x_n \to x$, $x_n \rightharpoonup x$ and $x_n \overset{*}{\rightharpoonup} x$, respectively.

Assumption 2.1. Let X be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let $A : C \to X^*$ be an α-inverse-strongly monotone mapping and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)–(A4). Let $T, \widetilde{T} : X \to 2^{X^*}$ be two maximal monotone operators such that:

(A5)′ $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP \neq \emptyset$.

Recall that if C is a nonempty closed convex subset of a Hilbert space H, then the metric projection $P_C : H \to C$ of H onto C is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. In this connection, Alber [26] recently introduced a generalized projection operator Π_C in a Banach space X which is an analogue of the metric projection in Hilbert spaces.

Consider the functional defined as in [26] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{2.1}$$

It is clear that in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$.

The generalized projection $\Pi_C : X \to C$ is a mapping that assigns to an arbitrary point $x \in X$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J; See, for example, [27]. In a Hilbert space, $\Pi_C = P_C$. From [26], in a smooth, strictly convex and reflexive Banach space X, we have

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in X. \tag{2.3}$$

Moreover, by the property of subdifferential of convex functions, we easily get the following inequality:

$$\phi(x, y) \leq \phi\left(x, J^{-1}(Jy + Jz)\right) - 2\langle y - x, Jz\rangle, \quad \forall x, y, z \in X. \tag{2.4}$$

Let S be a mapping from C into itself. A point p in C is called an asymptotic fixed point of S [28] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\|Sx_n - x_n\| \to 0$. The set of asymptotic fixed points of S is denoted by $\widehat{F}(S)$. A mapping S from S into itself is called relatively nonexpansive [18, 29, 30] if $\widehat{F}(S) = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$, for all $x \in C$ and $p \in F(S)$.

Observe that, if X is a reflexive, strictly convex and smooth Banach space, then for any $x, y \in X$, $\phi(x, y) = 0$ if and only if $x = y$. To this end, it is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. Actually, from (2.3), we have $\|x\| = \|y\|$, which implies that $\langle x, Jy\rangle = \|x\|^2 = \|y\|^2$. From the definition of J, we have $Jx = Jy$ and therefore, $x = y$. For further details, we refer to [31].

We need the following lemmas for the proof of our main results.

Lemma 2.2 (see [32]). *Let X be a smooth and uniformly convex Banach space and let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of X. If* $\phi(x_n, y_n) \to 0$ *and either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded, then* $\|x_n - y_n\| \to 0$.

Lemma 2.3 (see [26, 32]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space X,* $x \in X$ *and* $z \in C$. *Then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C. \tag{2.5}$$

Lemma 2.4 (see [26, 32]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space X. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C,\ y \in X. \tag{2.6}$$

Lemma 2.5 (see [33]). *Let X be a reflexive, strictly convex and smooth Banach space and let* $T : X \to 2^{X^*}$ *be a multivalued operator. Then*

(i) $T^{-1}0$ *is closed and convex if T is maximal monotone such that* $T^{-1}0 \neq \emptyset$;

(ii) *T is maximal monotone if and only if T is monotone with* $R(J + rT) = X^*$ *for all* $r > 0$.

Lemma 2.6 (see [34]). *Let X be a uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous and convex function* $g : [0, 2r] \to \mathbb{R}$ *such that* $g(0) = 0$ *and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|), \tag{2.7}$$

for all $x, y \in B_r$ *and* $t \in [0,1]$, *where* $B_r = \{z \in X : \|z\| \leq r\}$.

Lemma 2.7 (see [32]). *Let X be a smooth and uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous, and convex function* $g : [0, 2r] \to \mathbb{R}$ *such that* $g(0) = 0$ *and*

$$g(\|x - y\|) \leq \phi(x, y), \quad \forall x, y \in B_r. \tag{2.8}$$

The following result is due to Blum and Oettli [14].

Lemma 2.8 (see [14]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space X,* $f : C \times C \to \mathbb{R}$ *a bifunction satisfying conditions (A1)–(A4), and* $r > 0$ *and* $x \in X$. *Then, there exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{2.9}$$

Motivated by a result in [35] in a Hilbert space setting, Takahashi and Zembayashi [18] established the following lemma.

Lemma 2.9 (see [18]). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space X, and $f : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4). For $r > 0$ and $x \in X$, define a mapping $T_r : X \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\} \tag{2.10}$$

for all $x \in X$. Then

(i) *T_r is single-valued;*

(ii) *T_r is a firmly nonexpansive-type mapping, that is, for all $x, y \in X$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \tag{2.11}$$

(iii) *$F(T_r) = \widehat{F}(T_r) = EP(f)$;*

(iv) *$EP(f)$ is closed and convex.*

Using Lemma 2.9, we have the following result.

Lemma 2.10 (see [18]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space X, $f : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $r > 0$. Then, for $x \in X$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.12}$$

Utilizing Lemmas 2.8, 2.9, and 2.10, Zhang [1] derived the following result.

Proposition 2.11 (see [1]). *Let X be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of X. Let $A : C \to X^*$ be an α-inverse-strongly monotone mapping, $f : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $r > 0$. Then*

(I) *for $x \in X$, there exists $u \in C$ such that*

$$f(u, y) + \langle Au, y - u \rangle + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C; \tag{2.13}$$

(II) *if X is additionally uniformly smooth and $K_r : C \to C$ is defined as*

$$K_r(x) = \left\{ u \in C : f(u, y) + \langle Au, y - u \rangle + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \ \forall y \in C \right\}, \quad \forall x \in C, \tag{2.14}$$

then the mapping K_r has the following properties:

(i) *K_r is single-valued,*

(ii) *K_r is a firmly nonexpansive-type mapping, that is,*

$$\langle K_r x - K_r y, JK_r x - JK_r y \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle, \quad \forall x, y \in X, \tag{2.15}$$

(iii) $F(K_r) = \widehat{F}(K_r) = EP$,

(iv) *EP is a closed convex subset of C*,

(v) $\phi(p, K_r x) + \phi(K_r x, x) \le \phi(p, x)$, *for all* $p \in F(K_r)$.

Proof. Define a bifunction $F : C \times C \to \mathbb{R}$ by

$$F(x, y) = f(x, y) + \langle Ax, y - x \rangle, \quad \forall x, y \in C. \tag{2.16}$$

It is easy to verify that F satisfies the conditions (A1)–(A4). Therefore, the conclusions (I) and (II) follow immediately from Lemmas 2.8, 2.9, and 2.10. □

Let $T, \widetilde{T} : X \to 2^{X^*}$ be two maximal monotone operators in a smooth Banach space X. We denote the resolvent operators of T and $\widetilde{T}$ by $J_r = (J + rT)^{-1} J$ and $\widetilde{J}_r = (J + r\widetilde{T})^{-1} J$ for each $r > 0$, respectively. Then $J_r : X \to D(T)$ and $\widetilde{J}_r : X \to D(\widetilde{T})$ are two single-valued mappings. Also, $T^{-1}0 = F(J_r)$ and $\widetilde{T}^{-1}0 = F(\widetilde{J}_r)$ for each $r > 0$, where $F(J_r)$ and $F(\widetilde{J}_r)$ are the sets of fixed points of J_r and $\widetilde{J}_r$, respectively. For each $r > 0$, the Yosida approximations of T and $\widetilde{T}$ are defined by $A_r = (J - JJ_r)/r$ and $\widetilde{A}_r = (J - J\widetilde{J}_r)/r$, respectively. It is known that

$$A_r x \in T(J_r x), \quad \widetilde{A}_r x \in \widetilde{T}\left(\widetilde{J}_r x\right), \quad \text{for each } r > 0, \ x \in X. \tag{2.17}$$

Lemma 2.12 (see [23]). *Let X be a reflexive, strictly convex and smooth Banach space, and let $T : X \to 2^{X^*}$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$. Then,*

$$\phi(z, J_r x) + \phi(J_r x, x) \le \phi(z, x), \quad \forall r > 0, \ z \in T^{-1}0, \ x \in X. \tag{2.18}$$

Lemma 2.13 (see [36]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $a_{n+1} \le a_n + b_n$ for all $n \ge 0$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem and the set $T^{-1}0 \cap \widetilde{T}^{-1}0$ for two maximal monotone operators T and $\widetilde{T}$.

Theorem 3.1. *Suppose that Assumption 2.1 is satisfied. Let $x_0 \in X$ be chosen arbitrarily. Consider the sequence*

$$x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

where

$$
\begin{gathered}
H_n=\left\{z\in C:\phi\left(z,K_{r_n}y_n\right)\le\left(\alpha_n+\widetilde{\alpha}_n-\alpha_n\widetilde{\alpha}_n\right)\phi\left(z,x_0\right)+\left(1-\alpha_n\right)\left(1-\widetilde{\alpha}_n\right)\phi\left(z,x_n\right)\right\},\\
W_n=\left\{z\in C:\langle x_n-z,Jx_0-Jx_n\rangle\ge 0\right\},\\
\widetilde{x}_n=J^{-1}\left(\alpha_nJx_0+\left(1-\alpha_n\right)\left(\beta_nJx_n+\left(1-\beta_n\right)JJ_{r_n}x_n\right)\right),\\
y_n=J^{-1}\left(\widetilde{\alpha}_nJx_0+\left(1-\widetilde{\alpha}_n\right)\left(\widetilde{\beta}_nJ\widetilde{x}_n+\left(1-\widetilde{\beta}_n\right)J\widetilde{J}_{r_n}\widetilde{x}_n\right)\right),
\end{gathered}
\tag{3.2}
$$

K_r is defined by (2.14), $\{\alpha_n\},\{\beta_n\},\{\widetilde{\alpha}_n\},\{\widetilde{\beta}_n\}\subset[0,1]$ *satisfy*

$$
\lim_{n\to\infty}\alpha_n=0,\qquad \lim_{n\to\infty}\widetilde{\alpha}_n=0,\qquad \liminf_{n\to\infty}\beta_n\left(1-\beta_n\right)>0,\qquad \liminf_{n\to\infty}\widetilde{\beta}_n\left(1-\widetilde{\beta}_n\right)>0,
\tag{3.3}
$$

and $\{r_n\}\subset(0,\infty)$ *satisfies* $\liminf_{n\to\infty}r_n>0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$, *where* $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}$ *is the generalized projection of* X *onto* $T^{-1}0\cap\widetilde{T}^{-1}0\cap EP$.

Proof. For the sake of simplicity, we define

$$
u_n:=K_{r_n}y_n,\qquad z_n:=J^{-1}\left(\beta_nJx_n+\left(1-\beta_n\right)JJ_{r_n}x_n\right),\qquad \widetilde{z}_n:=J^{-1}\left(\widetilde{\beta}_nJ\widetilde{x}_n+\left(1-\widetilde{\beta}_n\right)J\widetilde{J}_{r_n}\widetilde{x}_n\right),
\tag{3.4}
$$

so that

$$
\widetilde{x}_n=J^{-1}\left(\alpha_nJx_0+\left(1-\alpha_n\right)Jz_n\right),\qquad y_n=J^{-1}\left(\widetilde{\alpha}_nJx_0+\left(1-\widetilde{\alpha}_n\right)J\widetilde{z}_n\right).
\tag{3.5}
$$

We divide the proof into several steps.

Step 1. We claim that $H_n\cap W_n$ is closed and convex for each $n\ge 0$.

Indeed, it is obvious that H_n is closed and W_n is closed and convex for each $n\ge 0$. Let us show that H_n is convex. For $z_1,z_2\in H_n$ and $t\in(0,1)$, put $z=tz_1+(1-t)z_2$. It is sufficient to show that $z\in H_n$. We first write $\gamma_n=\alpha_n+\widetilde{\alpha}_n-\alpha_n\widetilde{\alpha}_n$ for each $n\ge 0$. Next, we prove that

$$
\phi\left(z,u_n\right)\le\gamma_n\phi\left(z,x_0\right)+\left(1-\gamma_n\right)\phi\left(z,x_n\right)
\tag{3.6}
$$

is equivalent to

$$
2\gamma_n\langle z,Jx_0\rangle+2\left(1-\gamma_n\right)\langle z,Jx_n\rangle-2\langle z,Ju_n\rangle\le\gamma_n\|x_0\|^2+\left(1-\gamma_n\right)\|x_n\|^2-\|u_n\|^2.
\tag{3.7}
$$

Indeed, from (2.1) we deduce that there hold the following:

$$
\begin{aligned}
\phi\left(z,x_0\right)&=\|z\|^2-2\langle z,Jx_0\rangle+\|x_0\|^2,\\
\phi\left(z,x_n\right)&=\|z\|^2-2\langle z,Jx_n\rangle+\|x_n\|^2,\\
\phi\left(z,u_n\right)&=\|z\|^2-2\langle z,Ju_n\rangle+\|u_n\|^2,
\end{aligned}
\tag{3.8}
$$

which combined with (3.6) yield that (3.6) is equivalent to (3.7). Thus we have

$$\begin{aligned}
&2\gamma_n\langle z, Jx_0\rangle + 2(1-\gamma_n)\langle z, Jx_n\rangle - 2\langle z, Ju_n\rangle \\
&\quad = 2\gamma_n\langle tz_1 + (1-t)z_2, Jx_0\rangle + 2(1-\gamma_n)\langle tz_1 + (1-t)z_2, Jx_n\rangle \\
&\qquad - 2\langle tz_1 + (1-t)z_2, Ju_n\rangle \\
&\quad = 2t\gamma_n\langle z_1, Jx_0\rangle + 2(1-t)\gamma_n\langle z_2, Jx_0\rangle + 2(1-\gamma_n)t\langle z_1, Jx_n\rangle \\
&\qquad + 2(1-\gamma_n)(1-t)\langle z_2, Jx_n\rangle - 2t\langle z_1, Ju_n\rangle - 2(1-t)\langle z_2, Ju_n\rangle \\
&\quad \leq \gamma_n\|x_0\|^2 + (1-\gamma_n)\|x_n\|^2 - \|u_n\|^2.
\end{aligned} \tag{3.9}$$

This implies that $z \in H_n$. Therefore, H_n is closed and convex.

Step 2. We claim that $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$ for each $n \geq 0$ and that $\{x_n\}$ is well defined.

Indeed, take $w \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$ arbitrarily. Note that $u_n = K_{r_n}y_n$ is equivalent to

$$u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C. \tag{3.10}$$

Then from Lemma 2.12, we obtain

$$\begin{aligned}
\phi(w, z_n) &= \phi\Big(w, J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n)\Big) \\
&= \|w\|^2 - 2\langle w, \beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\rangle + \big\|\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\big\|^2 \\
&\leq \|w\|^2 - 2\beta_n\langle w, Jx_n\rangle - 2(1-\beta_n)\langle w, JJ_{r_n}x_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n}x_n\|^2 \\
&= \beta_n\phi(w, x_n) + (1-\beta_n)\phi(w, J_{r_n}x_n) \\
&\leq \beta_n\phi(w, x_n) + (1-\beta_n)\phi(w, x_n) = \phi(w, x_n), \\
\phi(w, \widetilde{x}_n) &= \phi\Big(w, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n)\Big) \\
&= \|w\|^2 - 2\langle w, \alpha_n Jx_0 + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)Jz_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_n\langle w, Jx_0\rangle - 2(1-\alpha_n)\langle w, Jz_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, z_n) \\
&\leq \alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, x_n).
\end{aligned} \tag{3.11}$$

Moreover, we have

$$\begin{aligned}
\phi(w,\tilde{z}_n) &= \phi\Big(w, J^{-1}\Big(\tilde{\beta}_n J\tilde{x}_n + \Big(1-\tilde{\beta}_n\Big) J\tilde{J}_{r_n}\tilde{x}_n\Big)\Big) \\
&\le \tilde{\beta}_n\phi(w,\tilde{x}_n) + \Big(1-\tilde{\beta}_n\Big)\phi\Big(w,\tilde{J}_{r_n}\tilde{x}_n\Big) \\
&\le \tilde{\beta}_n\phi(w,\tilde{x}_n) + \Big(1-\tilde{\beta}_n\Big)\phi(w,\tilde{x}_n) = \phi(w,\tilde{x}_n), \\
\phi(w,y_n) &= \phi\Big(w, J^{-1}(\tilde{\alpha}_n Jx_0 + (1-\tilde{\alpha}_n)J\tilde{z}_n)\Big) \\
&\le \|w\|^2 - 2\tilde{\alpha}_n\langle w, Jx_0\rangle - 2(1-\tilde{\alpha}_n)\langle w, J\tilde{z}_n\rangle + \tilde{\alpha}_n\|x_0\|^2 + (1-\tilde{\alpha}_n)\|\tilde{z}_n\|^2 \\
&= \tilde{\alpha}_n\phi(w,x_0) + (1-\tilde{\alpha}_n)\phi(w,\tilde{z}_n) \\
&\le \tilde{\alpha}_n\phi(w,x_0) + (1-\tilde{\alpha}_n)\phi(w,\tilde{x}_n) \\
&\le \tilde{\alpha}_n\phi(w,x_0) + (1-\tilde{\alpha}_n)\big[\alpha_n\phi(w,x_0) + (1-\alpha_n)\phi(w,x_n)\big] \\
&= [\tilde{\alpha}_n + (1-\tilde{\alpha}_n)\alpha_n]\phi(w,x_0) + (1-\tilde{\alpha}_n)(1-\alpha_n)\phi(w,x_n) \\
&\le (\alpha_n + \tilde{\alpha}_n - \alpha_n\tilde{\alpha}_n)\phi(w,x_0) + (1-\alpha_n)(1-\tilde{\alpha}_n)\phi(w,x_n),
\end{aligned} \tag{3.12}$$

and hence by Proposition 2.11,

$$\begin{aligned}
\phi(w,u_n) &= \phi(w, K_{r_n}y_n) \le \phi(w,y_n) \\
&\le (\alpha_n + \tilde{\alpha}_n - \alpha_n\tilde{\alpha}_n)\phi(w,x_0) + (1-\alpha_n)(1-\tilde{\alpha}_n)\phi(w,x_n).
\end{aligned} \tag{3.13}$$

So $w \in H_n$ for all $n \ge 0$. Now, let us show that

$$T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset W_n \quad \forall n \ge 0. \tag{3.14}$$

We prove this by induction. For $n = 0$, we have $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset C = W_0$. Assume that $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset W_n$. Since x_{n+1} is the projection of x_0 onto $H_n \cap W_n$, by Lemma 2.3 we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall z \in H_n \cap W_n. \tag{3.15}$$

As $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP$. This, together with the definition of W_{n+1} implies that $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset W_{n+1}$. Hence (3.14) holds for all $n \ge 0$. So, $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$ for all $n \ge 0$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We claim that $\{x_n\}$ is bounded and that $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$.

Indeed, it follows from the definition of W_n that $x_n = \Pi_{W_n}x_0$. Since $x_n = \Pi_{W_n}x_0$ and $x_{n+1} = \Pi_{H_n \cap W_n}x_0 \in W_n$, so $\phi(x_n, x_0) \le \phi(x_{n+1}, x_0)$ for all $n \ge 0$, that is, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from $x_n = \Pi_{W_n}x_0$ and Lemma 2.4 that

$$\phi(x_n, x_0) = \phi(\Pi_{W_n}x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0) \tag{3.16}$$

for each $p \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP \subset W_n$ for each $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is bounded, which implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Since

$$(\|x_n\| - \|x_0\|)^2 \leq \phi(x_n, x_0) \leq (\|x_n\| + \|x_0\|)^2, \quad \forall n \geq 0, \tag{3.17}$$

so $\{x_n\}$ is bounded. From Lemma 2.4, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{W_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0),\end{aligned} \tag{3.18}$$

for each $n \geq 0$. This implies that

$$\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.19}$$

Step 4. We claim that $\lim_{n\to\infty}\|x_n - u_n\| = 0$, $\lim_{n\to\infty}\|x_n - J_{r_n} x_n\| = 0$, and $\lim_{n\to\infty}\|\widetilde{x}_n - \widetilde{J}_{r_n}\widetilde{x}_n\| = 0$.

Indeed, from $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in H_n$, we have

$$\phi(x_{n+1}, u_n) \leq (\alpha_n + \widetilde{\alpha}_n - \alpha_n \widetilde{\alpha}_n)\phi(x_{n+1}, x_0) + (1 - \alpha_n)(1 - \widetilde{\alpha}_n)\phi(x_{n+1}, x_n), \quad \forall n \geq 0. \tag{3.20}$$

Therefore, from $\alpha_n \to 0\widetilde{\alpha}_n \to 0$ and $\phi(x_{n+1}, x_n) \to 0$, it follows that $\lim_{n\to\infty}\phi(x_{n+1}, u_n) = 0$.

Since $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = \lim_{n\to\infty}\phi(x_{n+1}, u_n) = 0$ and X is uniformly convex and smooth, we have from Lemma 2.2 that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \|x_{n+1} - u_n\| = 0, \tag{3.21}$$

and, therefore, $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded subsets of X and $\|x_n - u_n\| \to 0$, then $\lim_{n\to\infty}\|Jx_n - Ju_n\| = 0$.

Let us set $\Omega := T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$. Then, according to Lemma 2.5 and Proposition 2.11, we know that Ω is a nonempty closed convex subset of X such that $\Omega \subset C$. Fix $u \in \Omega$ arbitrarily. As in the proof of Step 2, we can show that $\phi(u, z_n) \leq \phi(u, x_n)$,

$$\begin{gathered}\phi(u, \widetilde{x}_n) \leq \alpha_n \phi(u, x_0) + (1 - \alpha_n)\phi(u, x_n), \\ \phi(u, \widetilde{z}_n) \leq \phi(u, \widetilde{x}_n), \\ \phi(u, y_n) \leq (\alpha_n + \widetilde{\alpha}_n - \alpha_n \widetilde{\alpha}_n)\phi(u, x_0) + (1 - \alpha_n)(1 - \widetilde{\alpha}_n)\phi(u, x_n), \\ \phi(u, u_n) \leq (\alpha_n + \widetilde{\alpha}_n - \alpha_n \widetilde{\alpha}_n)\phi(u, x_0) + (1 - \alpha_n)(1 - \widetilde{\alpha}_n)\phi(u, x_n).\end{gathered} \tag{3.22}$$

Hence it follows from the boundedness of $\{x_n\}$ that $\{z_n\}, \{\tilde{x}_n\}, \{\tilde{z}_n\}, \{y_n\}$, and $\{u_n\}$ are also bounded. Let $r = \sup\{\|x_n\|, \|\tilde{x}_n\|, \|J_{r_n}x_n\|, \|\tilde{J}_{r_n}\tilde{x}_n\| : n \geq 0\}$. Since X is a uniformly smooth Banach space, we know that X^* is a uniformly convex Banach space. Therefore, by Lemma 2.6 there exists a continuous, strictly increasing, and convex function g, with $g(0) = 0$, such that

$$\left\|\alpha x^* + (1-\alpha)y^*\right\|^2 \leq \alpha\|x^*\|^2 + (1-\alpha)\left\|y^*\right\|^2 - \alpha(1-\alpha)g\left(\left\|x^* - y^*\right\|\right), \tag{3.23}$$

for $x^*, y^* \in B_r^*$ and $\alpha \in [0,1]$. So, we have that

$$\begin{aligned}
\phi(u, z_n) &= \phi\left(u, J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n)\right) \\
&= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\rangle + \left\|\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\right\|^2 \\
&\leq \|u\|^2 - 2\beta_n\langle u, Jx_n\rangle - 2(1-\beta_n)\langle u, JJ_{r_n}x_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n}x_n\|^2 - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&= \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, J_{r_n}x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&\leq \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&= \phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|), \\
\phi(u, \tilde{z}_n) &= \phi\left(u, J^{-1}\left(\tilde{\beta}_n J\tilde{x}_n + \left(1-\tilde{\beta}_n\right)J\tilde{J}_{r_n}\tilde{x}_n\right)\right) \\
&= \|u\|^2 - 2\left\langle u, \tilde{\beta}_n J\tilde{x}_n + \left(1-\tilde{\beta}_n\right)J\tilde{J}_{r_n}\tilde{x}_n\right\rangle + \left\|\tilde{\beta}_n J\tilde{x}_n + (1-\tilde{\beta}_n)J\tilde{J}_{r_n}\tilde{x}_n\right\|^2 \\
&\leq \|u\|^2 - 2\tilde{\beta}_n\langle u, J\tilde{x}_n\rangle - 2\left(1-\tilde{\beta}_n\right)\left\langle u, J\tilde{J}_{r_n}\tilde{x}_n\right\rangle \\
&\quad + \tilde{\beta}_n\|\tilde{x}_n\|^2 + \left(1-\tilde{\beta}_n\right)\left\|\tilde{J}_{r_n}\tilde{x}_n\right\|^2 - \tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\right\|\right) \\
&= \tilde{\beta}_n\phi(u, \tilde{x}_n) + \left(1-\tilde{\beta}_n\right)\phi\left(u, \tilde{J}_{r_n}\tilde{x}_n\right) - \tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\right\|\right) \\
&\leq \tilde{\beta}_n\phi(u, \tilde{x}_n) + \left(1-\tilde{\beta}_n\right)\phi(u, \tilde{x}_n) - \tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\right\|\right) \\
&= \phi(u, \tilde{x}_n) - \tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\right\|\right),
\end{aligned} \tag{3.24}$$

and hence

$$\begin{aligned}
\phi(u,\tilde{x}_n) &= \phi\Big(u, J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n)\Big) \\
&= \|u\|^2 - 2\langle u, \alpha_n J x_0 + (1-\alpha_n) J z_n\rangle + \|\alpha_n J x_0 + (1-\alpha_n) J z_n\|^2 \\
&\le \|u\|^2 - 2\alpha_n \langle u, J x_0\rangle - 2(1-\alpha_n)\langle u, J z_n\rangle + \alpha_n \|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n \phi(u, x_0) + (1-\alpha_n)\phi(u, z_n) \\
&\le \alpha_n \phi(u, x_0) + (1-\alpha_n)\left[\phi(u, x_n) - \beta_n(1-\beta_n) g(\|J x_n - J J_{r_n} x_n\|)\right] \\
&= \alpha_n \phi(u, x_0) + (1-\alpha_n)\phi(u, x_n) - (1-\alpha_n)\beta_n(1-\beta_n) g(\|J x_n - J J_{r_n} x_n\|), \\
\phi(u, u_n) &= \phi(u, K_{r_n} y_n) \le \phi(u, y_n) \quad \text{(using Proposition 2.10)} \\
&= \phi\Big(u, J^{-1}(\tilde{\alpha}_n J x_0 + (1-\tilde{\alpha}_n) J \tilde{z}_n)\Big) \\
&= \|u\|^2 - 2\langle u, \tilde{\alpha}_n J x_0 + (1-\tilde{\alpha}_n) J \tilde{z}_n\rangle + \|\tilde{\alpha}_n J x_0 + (1-\tilde{\alpha}_n) J \tilde{z}_n\|^2 \\
&\le \|u\|^2 - 2\tilde{\alpha}_n \langle u, J x_0\rangle - 2(1-\tilde{\alpha}_n)\langle u, J \tilde{z}_n\rangle + \tilde{\alpha}_n \|x_0\|^2 + (1-\tilde{\alpha}_n)\|\tilde{z}_n\|^2 \\
&= \tilde{\alpha}_n \phi(u, x_0) + (1-\tilde{\alpha}_n)\phi(u, \tilde{z}_n) \\
&\le \tilde{\alpha}_n \phi(u, x_0) + (1-\tilde{\alpha}_n)\Big[\phi(u, \tilde{x}_n) - \tilde{\beta}_n\Big(1-\tilde{\beta}_n\Big) g\Big(\Big\|J \tilde{x}_n - J \tilde{J}_{r_n} \tilde{x}_n\Big\|\Big)\Big] \\
&= \tilde{\alpha}_n \phi(u, x_0) + (1-\tilde{\alpha}_n)\phi(u, \tilde{x}_n) - (1-\tilde{\alpha}_n)\tilde{\beta}_n\Big(1-\tilde{\beta}_n\Big) g\Big(\Big\|J \tilde{x}_n - J \tilde{J}_{r_n} \tilde{x}_n\Big\|\Big) \\
&\le \tilde{\alpha}_n \phi(u, x_0) + \phi(u, \tilde{x}_n),
\end{aligned} \tag{3.25}$$

for all $n \ge 0$. Consequently, we have

$$\begin{aligned}
&(1-\alpha_n)\beta_n(1-\beta_n) g(\|J x_n - J J_{r_n} x_n\|) \\
&\quad \le \alpha_n \varphi(u, x_0) + (1-\alpha_n)\varphi(u, x_n) - \varphi(u, \tilde{x}_n) \\
&\quad \le \alpha_n \varphi(u, x_0) + \varphi(u, x_n) - \varphi(u, \tilde{x}_n) \\
&\quad = \alpha_n \varphi(u, x_0) + \varphi(u, x_n) - \varphi(u, u_n) + \varphi(u, u_n) - \varphi(u, \tilde{x}_n) \\
&\quad = \alpha_n \varphi(u, x_0) + \|x_n\|^2 - \|u_n\|^2 - 2\langle u, J x_n - J u_n\rangle + \varphi(u, u_n) - \varphi(u, \tilde{x}_n) \\
&\quad \le \alpha_n \varphi(u, x_0) + \Big|\|x_n\|^2 - \|u_n\|^2\Big| + 2|\langle u, J x_n - J u_n\rangle| + \varphi(u, u_n) - \varphi(u, \tilde{x}_n) \\
&\quad \le \alpha_n \varphi(u, x_0) + \tilde{\alpha}_n \varphi(u, x_0) + |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|u\|\|J x_n - J u_n\| \\
&\quad \le (\alpha_n + \tilde{\alpha}_n)\varphi(u, x_0) + \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|J x_n - J u_n\|,
\end{aligned} \tag{3.26}$$

Since $\|x_n - u_n\| \to 0$ and J is uniformly norm-to-norm continuous on bounded subsets of X, we obtain $\|Jx_n - Ju_n\| \to 0$. From $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ and $\lim_{n\to\infty}(\alpha_n + \tilde{\alpha}_n) = 0$, we have

$$\lim_{n\to\infty} g(\|Jx_n - JJ_{r_n}x_n\|) = 0. \tag{3.27}$$

Therefore, from the properties of g, we get

$$\lim_{n\to\infty}\|Jx_n - JJ_{r_n}x_n\| = \lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0, \tag{3.28a}$$

recalling that J^{-1} is uniformly norm-to-norm continuous on bounded subsets of X^*. Next let us show that

$$\lim_{n\to\infty}\left\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\right\| = \lim_{n\to\infty}\left\|\tilde{x}_n - \tilde{J}_{r_n}\tilde{x}_n\right\| = 0. \tag{3.28b}$$

Observe first that

$$\begin{aligned}
&\phi(u_n, x_n) - \phi(x_{n+1}, u_n)\\
&\quad = \|x_n\|^2 - \|x_{n+1}\|^2 - 2\langle u_n, Jx_n\rangle + 2\langle x_{n+1}, Ju_n\rangle\\
&\quad = (\|x_n\| - \|x_{n+1}\|)(\|x_n\| + \|x_{n+1}\|) + 2\langle x_{n+1} - u_n, Jx_n\rangle + 2\langle x_{n+1}, Ju_n - Jx_n\rangle\\
&\quad \le \|x_n - x_{n+1}\|(\|x_n\| + \|x_{n+1}\|) + 2\|x_{n+1} - u_n\|\|x_n\| + 2\|x_{n+1}\|\|Ju_n - Jx_n\|.
\end{aligned} \tag{3.29}$$

Since $\phi(x_{n+1}, u_n) \to 0$, $\|x_{n+1} - x_n\| \to 0$, $\|x_{n+1} - u_n\| \to 0$, $\|Ju_n - Jx_n\| \to 0$, and $\{x_n\}$ is bounded, so it follows that $\phi(u_n, x_n) \to 0$. Also, observe that

$$\begin{aligned}
\phi(u_n, J_{r_n}x_n) - \phi(u_n, x_n) &= \|J_{r_n}x_n\|^2 - \|x_n\|^2 + 2\langle u_n, Jx_n - JJ_{r_n}x_n\rangle\\
&= (\|J_{r_n}x_n\| - \|x_n\|)(\|J_{r_n}x_n\| + \|x_n\|) + 2\langle u_n, Jx_n - JJ_{r_n}x_n\rangle\\
&\le \|J_{r_n}x_n - x_n\|(\|J_{r_n}x_n\| + \|x_n\|) + 2\|u_n\|\|Jx_n - JJ_{r_n}x_n\|.
\end{aligned} \tag{3.30}$$

Since $\phi(u_n, x_n) \to 0$, $\|J_{r_n}x_n - x_n\| \to 0$, $\|Jx_n - JJ_{r_n}x_n\| \to 0$, and the sequences $\{x_n\}, \{u_n\}, \{J_{r_n}x_n\}$ are bounded, so it follows that $\phi(u_n, J_{r_n}x_n) \to 0$. Meantime, observe that

$$\begin{aligned}
\phi(u_n, z_n) &= \phi\Big(u_n, J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n)\Big)\\
&= \|u_n\|^2 - 2\langle u_n, \beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\rangle + \left\|\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\right\|^2\\
&\le \|u_n\|^2 - 2\beta_n\langle u_n, Jx_n\rangle - 2(1-\beta_n)\langle u_n, JJ_{r_n}x_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n}x_n\|^2\\
&= \beta_n\phi(u_n, x_n) + (1-\beta_n)\phi(u_n, J_{r_n}x_n)\\
&\le \phi(u_n, x_n) + \phi(u_n, J_{r_n}x_n),
\end{aligned} \tag{3.31}$$

and hence

$$\begin{aligned}
\phi(u_n, \tilde{x}_n) &= \phi\Big(u_n, J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n)\Big) \\
&= \|u_n\|^2 - 2\langle u_n, \alpha_n J x_0 + (1-\alpha_n) J z_n\rangle + \|\alpha_n J x_0 + (1-\alpha_n) J z_n\|^2 \\
&\le \|u_n\|^2 - 2\alpha_n\langle u_n, J x_0\rangle - 2(1-\alpha_n)\langle u_n, J z_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n\phi(u_n, x_0) + (1-\alpha_n)\phi(u_n, z_n) \\
&\le \alpha_n\phi(u_n, x_0) + \phi(u_n, z_n) \\
&\le \alpha_n\phi(u_n, x_0) + \phi(u_n, x_n) + \phi(u_n, J_{r_n} x_n).
\end{aligned} \tag{3.32}$$

Since $\alpha_n \to 0$, $\phi(u_n, x_n) \to 0$ and $\phi(u_n, J_{r_n} x_n) \to 0$, it follows from the boundedness of $\{u_n\}$ that $\phi(u_n, \tilde{x}_n) \to 0$. Thus, in terms of Lemma 2.2, we have that $\|u_n - \tilde{x}_n\| \to 0$ and so $\|x_n - \tilde{x}_n\| \to 0$. Furthermore, it follows from (3.25) that

$$\begin{aligned}
\phi(u, u_n) &\le \tilde{\alpha}_n\phi(u, x_0) + (1-\tilde{\alpha}_n)\phi(u, \tilde{x}_n) - (1-\tilde{\alpha}_n)\tilde{\beta}_n\Big(1-\tilde{\beta}_n\Big) g\Big(\Big\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\Big\|\Big) \\
&\le \tilde{\alpha}_n\phi(u, x_0) + \phi(u, \tilde{x}_n) - (1-\tilde{\alpha}_n)\tilde{\beta}_n\Big(1-\tilde{\beta}_n\Big) g\Big(\Big\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\Big\|\Big),
\end{aligned} \tag{3.33}$$

and hence

$$\begin{aligned}
&(1-\tilde{\alpha}_n)\tilde{\beta}_n\Big(1-\tilde{\beta}_n\Big) g\Big(\Big\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\Big\|\Big) \\
&\quad\le \tilde{\alpha}_n\phi(u, x_0) + \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&\quad= \tilde{\alpha}_n\phi(u, x_0) + \|\tilde{x}_n\|^2 - \|u_n\|^2 + 2\langle u, J u_n - J\tilde{x}_n\rangle \\
&\quad= \tilde{\alpha}_n\phi(u, x_0) + (\|\tilde{x}_n\| - \|u_n\|)(\|\tilde{x}_n\| + \|u_n\|) + 2\langle u, J u_n - J\tilde{x}_n\rangle \\
&\quad\le \tilde{\alpha}_n\phi(u, x_0) + \|\tilde{x}_n - u_n\|(\|\tilde{x}_n\| + \|u_n\|) + 2\|u\|\|J u_n - J\tilde{x}_n\|.
\end{aligned} \tag{3.34}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of X, it follows from $\|\tilde{x}_n - u_n\| \to 0$ that $\|J u_n - J\tilde{x}_n\| \to 0$. Thus from $\tilde{\alpha}_n \to 0$, $\liminf_{n\to\infty}\tilde{\beta}_n(1-\tilde{\beta}_n) > 0$, and the boundedness of both $\{\tilde{x}_n\}$ and $\{u_n\}$, we deduce that $g(\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\|) \to 0$. Utilizing the properties of g, we have that $\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\| \to 0$. Since J^{-1} is uniformly norm-to-norm continuous on bounded subsets of X^*, it follows that $\|\tilde{x}_n - \tilde{J}_{r_n}\tilde{x}_n\| \to 0$.

Step 5. We claim that $\omega_w(\{x_n\}) \subset T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP$, where

$$\omega_w(\{x_n\}) := \big\{\hat{x} \in C : x_{n_k} \rightharpoonup \hat{x} \text{ for some subsequence } \{n_k\} \subset \{n\} \text{ with } n_k \uparrow \infty\big\}. \tag{3.35}$$

Indeed, since $\{x_n\}$ is bounded and X is reflexive, we know that $\omega_w(\{x_n\}) \neq \emptyset$. Take $\hat{x} \in \omega_w(\{x_n\})$ arbitrarily. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Hence it follows from $\|x_n - \tilde{x}_n\| \to 0$, $\|x_n - J_{r_n} x_n\| \to 0$, and $\|\tilde{x}_n - \tilde{J}_{r_n}\tilde{x}_n\| \to 0$ that $\{\tilde{x}_{n_k}\}$, $\{J_{r_{n_k}} x_{n_k}\}$

and $\{\widetilde{J}_{r_{n_k}}\widetilde{x}_{n_k}\}$ converge weakly to the same point $\widehat{x}$. On the other hand, from (3.28a), (3.28b) and $\liminf_{n\to\infty} r_n > 0$, we obtain that

$$\lim_{n\to\infty}\|A_{r_n}x_n\| = \lim_{n\to\infty}\frac{1}{r_n}\|Jx_n - JJ_{r_n}x_n\| = 0,$$
$$\lim_{n\to\infty}\left\|\widetilde{A}_{r_n}\widetilde{x}_n\right\| = \lim_{n\to\infty}\frac{1}{r_n}\left\|J\widetilde{x}_n - J\widetilde{J}_{r_n}\widetilde{x}_n\right\| = 0. \tag{3.36}$$

If $z^* \in Tz$ and $\widetilde{z}^* \in \widetilde{T}\widetilde{z}$, then it follows from (2.17) and the monotonicity of the operators T, $\widetilde{T}$ that for all $k \geq 1$

$$\left\langle z - J_{r_{n_k}}x_{n_k}, z^* - A_{r_{n_k}}x_{n_k}\right\rangle \geq 0, \qquad \left\langle \widetilde{z} - \widetilde{J}_{r_{n_k}}\widetilde{x}_{n_k}, \widetilde{z}^* - \widetilde{A}_{r_{n_k}}\widetilde{x}_{n_k}\right\rangle \geq 0. \tag{3.37}$$

Letting $k \to \infty$, we have that $\langle z - \widehat{x}, z^*\rangle \geq 0$ and $\langle \widetilde{z} - \widehat{x}, \widetilde{z}^*\rangle \geq 0$. Then the maximality of the operators T, $\widetilde{T}$ implies that $\widehat{x} \in T^{-1}0$ and $\widehat{x} \in \widetilde{T}^{-1}0$.

Next, let us show that $\widehat{x} \in EP$. Since

$$\phi(u, y_n) \leq (\alpha_n + \widetilde{\alpha}_n - \alpha_n\widetilde{\alpha}_n)\phi(u, x_0) + (1 - \alpha_n)(1 - \widetilde{\alpha}_n)\phi(u, x_n), \tag{3.38}$$

from $u_n = K_{r_n}y_n$ and Proposition 2.11 it follows that

$$\begin{aligned}\phi(u_n, y_n) = \phi(K_{r_n}y_n, y_n) &\leq \phi(u, y_n) - \phi(u, K_{r_n}y_n)\\ &\leq (\alpha_n + \widetilde{\alpha}_n - \alpha_n\widetilde{\alpha}_n)\phi(u, x_0) + (1 - \alpha_n)(1 - \widetilde{\alpha}_n)\phi(u, x_n) - \phi(u, K_{r_n}y_n)\\ &\leq (\alpha_n + \widetilde{\alpha}_n - \alpha_n\widetilde{\alpha}_n)\phi(u, x_0) + \phi(u, x_n) - \phi(u, u_n).\end{aligned} \tag{3.39}$$

Also, since

$$\begin{aligned}\left|\varphi(u, x_n) - \varphi(u, u_n)\right| &= \left|\|x_n\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_n\rangle\right|\\ &\leq \left|\|x_n\|^2 - \|u_n\|^2\right| + 2|\langle u, Ju_n - Jx_n\rangle|\\ &= |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\|\\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\|,\end{aligned} \tag{3.40}$$

so we get

$$\lim_{n\to\infty}(\phi(u, x_n) - \phi(u, u_n)) = 0. \tag{3.41}$$

So, from (3.39), $\alpha_n \to 0$, $\widetilde{\alpha}_n \to 0$, and $\phi(u, x_n) - \phi(u, u_n) \to 0$, we have $\lim_{n\to\infty}\phi(u_n, y_n) = 0$.

Since X is uniformly convex and smooth, we conclude from Lemma 2.2 that

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.42}$$

From $x_{n_k} \rightharpoonup \widehat{x}$, $\|x_n - u_n\| \to 0$, and (3.42), we have $y_{n_k} \rightharpoonup \widehat{x}$ and $u_{n_k} \rightharpoonup \widehat{x}$.

Since J is uniformly norm-to-norm continuous on bounded subsets of X, from (3.42) we derive

$$\lim_{n\to\infty}\|Ju_n - Jy_n\| = 0. \tag{3.43}$$

From $\liminf_{n\to\infty} r_n > 0$, it follows that

$$\lim_{n\to\infty}\frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.44}$$

By the definition of $u_n := K_{r_n} y_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \tag{3.45}$$

where

$$F(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle. \tag{3.46}$$

Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k}\rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C. \tag{3.47}$$

Since $y \mapsto f(x, y) + \langle Ax, y - x\rangle$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (3.44) and (A4), we have

$$F(y, \widehat{x}) \leq 0, \quad \forall y \in C. \tag{3.48}$$

For t, with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\widehat{x}$. Since $y \in C$ and $\widehat{x} \in C$, then $y_t \in C$ and hence $F(y_t, \widehat{x}) \leq 0$. So, from (A1) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \widehat{x}) \leq tF(y_t, y). \tag{3.49}$$

Dividing by t, we have

$$F(y_t, y) \geq 0, \quad \forall y \in C. \tag{3.50}$$

Letting $t \downarrow 0$, from (A3) it follows that

$$F(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{3.51}$$

So, $\hat{x} \in EP$. Therefore, we obtain that $\omega_w(\{x_n\}) \subset T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$ by the arbitrariness of $\hat{x}$.

Step 6. We claim that $\{x_n\}$ converges strongly to $w = \Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$.

Indeed, from $x_{n+1} = \Pi_{H_n \cap W_n} x_0$ and $w \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$, it follows that

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0). \tag{3.52}$$

Since the norm is weakly lower semicontinuous, then

$$\begin{aligned} \phi(\hat{x}, x_0) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx_0\rangle + \|x_0\|^2 \leq \liminf_{k\to\infty}\left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2\right) \\ &= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0). \end{aligned} \tag{3.53}$$

From the definition of $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}$, we have $\hat{x} = w$. Hence $\lim_{k\to\infty}\phi(x_{n_k}, x_0) = \phi(w, x_0)$, and

$$\begin{aligned} 0 &= \lim_{k\to\infty}\left(\phi(x_{n_k}, x_0) - \phi(w, x_0)\right) = \lim_{k\to\infty}\left(\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0\rangle\right) \\ &= \lim_{k\to\infty}\left(\|x_{n_k}\|^2 - \|w\|^2\right), \end{aligned} \tag{3.54}$$

which implies that $\lim_{k\to\infty}\|x_{n_k}\| = \|w\|$. Since X has the Kadec-Klee property, then $x_{n_k} \to w = \Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$. Therefore, $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$. □

Remark 3.2. In Theorem 3.1, let $A \equiv 0$, $\widetilde{T} \equiv 0$, and $\widetilde{\alpha}_n = 0$, $\forall n \geq 0$. Then, for all $\alpha, r \in (0,\infty)$ and $x, y \in C$, we have that

$$\begin{aligned} &\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \\ K_r(x) &= \left\{u \in C : f(u,y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\right\} \\ &= \left\{u \in C : f(u,y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\right\} = T_r(x). \end{aligned} \tag{3.55}$$

Moreover, there hold the following

$$
\begin{aligned}
H_n &= \{z \in C : \phi(z, K_{r_n} y_n) \leq (\alpha_n + \widetilde{\alpha}_n - \alpha_n \widetilde{\alpha}_n)\phi(z, x_0) + (1-\alpha_n)(1-\widetilde{\alpha}_n)\phi(z, x_n)\} \\
&= \{z \in C : \phi(z, T_{r_n} y_n) \leq \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\
y_n &= J^{-1}\Big(\widetilde{\alpha}_n J x_0 + (1-\widetilde{\alpha}_n)\Big(\widetilde{\beta}_n J\widetilde{x}_n + \Big(1-\widetilde{\beta}_n\Big) J\widetilde{J}_{r_n}\widetilde{x}_n\Big)\Big) \\
&= J^{-1}\Big(\widetilde{\beta}_n J\widetilde{x}_n + \Big(1-\widetilde{\beta}_n\Big) J\widetilde{J}_{r_n}\widetilde{x}_n\Big) \\
&= J^{-1}\Big(\widetilde{\beta}_n J\widetilde{x}_n + \Big(1-\widetilde{\beta}_n\Big) J\widetilde{x}_n\Big) \\
&= J^{-1} J\widetilde{x}_n = \widetilde{x}_n,
\end{aligned}
\tag{3.56}
$$

and hence

$$
y_n = \widetilde{x}_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J J_{r_n} x_n)). \tag{3.57}
$$

In this case, Theorem 3.1 reduces to [17, Theorem 3.1].

4. Weak Convergence Theorem

In this section, we present the following algorithm for finding a common element of the solution set of a generalized equilibrium problem and the set $T^{-1}0 \cap \widetilde{T}^{-1}0$ for two maximal monotone operators T and $\widetilde{T}$.

Let $x_0 \in X$ be chosen arbitrarily and consider the sequence $\{x_n\}$ generated by

$$
\begin{aligned}
\widetilde{x}_n &= J^{-1}(\alpha_n J x_0 + (1-\alpha_n)(\beta_n J K_{r_n} x_n + (1-\beta_n) J J_{r_n} K_{r_n} x_n)), \\
x_{n+1} &= J^{-1}\Big(\widetilde{\alpha}_n J x_0 + (1-\widetilde{\alpha}_n)\Big(\widetilde{\beta}_n J K_{r_n}\widetilde{x}_n + \Big(1-\widetilde{\beta}_n\Big) J\widetilde{J}_{r_n} K_{r_n}\widetilde{x}_n\Big)\Big), \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{4.1}
$$

where $\{\alpha_n\}, \{\beta_n\}, \{\widetilde{\alpha}_n\}, \{\widetilde{\beta}_n\} \subset [0,1]$, $\{r_n\} \subset (0, \infty)$, and K_r, $r > 0$ is defined by (2.14).

Before proving a weak convergence theorem, we need the following proposition.

Proposition 4.1. *Suppose that Assumption 2.1 is fulfilled and let $\{x_n\}$ be a sequence defined by* (4.1)*, where $\{\alpha_n\}, \{\beta_n\}, \{\widetilde{\alpha}_n\}, \{\widetilde{\beta}_n\} \subset [0,1]$ satisfy the following conditions:*

$$
\sum_{n=0}^{\infty} \alpha_n < \infty, \qquad \sum_{n=0}^{\infty} \widetilde{\alpha}_n < \infty, \qquad \liminf_{n\to\infty} \beta_n (1-\beta_n) > 0, \qquad \liminf_{n\to\infty} \widetilde{\beta}_n \Big(1-\widetilde{\beta}_n\Big) > 0. \tag{4.2}
$$

Then, $\{\Pi_{T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP} x_n\}$ converges strongly to $z \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$, where $\Pi_{T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP}$ is the generalized projection of X onto $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$.

Proof. We set $\Omega := T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$ and

$$\begin{aligned} u_n &:= K_{r_n} x_n, \quad y_n := J^{-1}\left(\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\right), \\ \widetilde{u}_n &:= K_{r_n} \widetilde{x}_n, \quad \widetilde{y}_n := J^{-1}\left(\widetilde{\beta}_n J \widetilde{u}_n + \left(1-\widetilde{\beta}_n\right) J \widetilde{J}_{r_n} \widetilde{u}_n\right), \end{aligned} \tag{4.3}$$

so that

$$\begin{aligned} \widetilde{x}_n &= J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J y_n\right), \\ x_{n+1} &= J^{-1}\left(\widetilde{\alpha}_n J x_0 + (1-\widetilde{\alpha}_n) J \widetilde{y}_n\right), \quad n = 0,1,2,\ldots. \end{aligned} \tag{4.4}$$

Then, in terms of Lemma 2.5 and Proposition 2.11, Ω is a nonempty closed convex subset of X such that $\Omega \subset C$. We first prove that $\{x_n\}$ is bounded. Fix $u \in \Omega$. Note that by the first and third of (4.3), $u_n, \widetilde{u}_n \in C$ and

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J x_n\rangle &\geq 0, \quad \forall y \in C, \\ F(\widetilde{u}_n, y) + \frac{1}{r_n}\langle y - \widetilde{u}_n, J \widetilde{u}_n - J \widetilde{x}_n\rangle &\geq 0, \quad \forall y \in C. \end{aligned} \tag{4.5}$$

Here, each K_{r_n} is relatively nonexpansive. Then from Proposition 2.11, we obtain

$$\begin{aligned} \phi(u, y_n) &= \phi\left(u, J^{-1}\left(\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\right)\right) \\ &= \|u\|^2 - 2\left\langle u, \beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\right\rangle + \left\|\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\right\|^2 \\ &\leq \|u\|^2 - 2\beta_n\langle u, J u_n\rangle - 2(1-\beta_n)\langle u, J J_{r_n} u_n\rangle + \beta_n\|u_n\|^2 + (1-\beta_n)\|J_{r_n} u_n\|^2 \\ &= \beta_n \phi(u, u_n) + (1-\beta_n)\phi(u, J_{r_n} u_n) \\ &\leq \beta_n \phi(u, u_n) + (1-\beta_n)\phi(u, u_n) \\ &= \phi(u, u_n) = \phi(u, K_{r_n} x_n) \leq \phi(u, x_n), \end{aligned} \tag{4.6a}$$

$$\begin{aligned} \phi(u, \widetilde{y}_n) &= \phi\left(u, J^{-1}\left(\widetilde{\beta}_n J \widetilde{u}_n + \left(1-\widetilde{\beta}_n\right) J \widetilde{J}_{r_n} \widetilde{u}_n\right)\right) \\ &= \|u\|^2 - 2\left\langle u, \widetilde{\beta}_n J \widetilde{u}_n + \left(1-\widetilde{\beta}_n\right) J \widetilde{J}_{r_n} \widetilde{u}_n\right\rangle + \left\|\widetilde{\beta}_n J \widetilde{u}_n + (1-\widetilde{\beta}_n) J \widetilde{J}_{r_n} \widetilde{u}_n\right\|^2 \\ &\leq \|u\|^2 - 2\widetilde{\beta}_n\langle u, J \widetilde{u}_n\rangle - 2\left(1-\widetilde{\beta}_n\right)\left\langle u, J \widetilde{J}_{r_n} \widetilde{u}_n\right\rangle + \widetilde{\beta}_n\|\widetilde{u}_n\|^2 + \left(1-\widetilde{\beta}_n\right)\left\|\widetilde{J}_{r_n} \widetilde{u}_n\right\|^2 \\ &= \widetilde{\beta}_n \phi(u, \widetilde{u}_n) + \left(1-\widetilde{\beta}_n\right)\phi\left(u, \widetilde{J}_{r_n} \widetilde{u}_n\right) \\ &\leq \widetilde{\beta}_n \phi(u, \widetilde{u}_n) + \left(1-\widetilde{\beta}_n\right)\phi(u, \widetilde{u}_n) \\ &= \phi(u, \widetilde{u}_n) = \phi(u, K_{r_n} \widetilde{x}_n) \leq \phi(u, \widetilde{x}_n), \end{aligned} \tag{4.6b}$$

and hence by Proposition 2.11

$$\begin{aligned}
\phi(u,\widetilde{x}_n) &= \phi\Big(u, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jy_n)\Big) \\
&= \|u\|^2 - 2\langle u, \alpha_n Jx_0 + (1-\alpha_n)Jy_n\rangle + \big\|\alpha_n Jx_0 + (1-\alpha_n)Jy_n\big\|^2 \\
&\leq \|u\|^2 - 2\alpha_n\langle u, Jx_0\rangle - 2(1-\alpha_n)\langle u, Jy_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\big\|y_n\big\|^2 \\
&= \alpha_n\phi(u,x_0) + (1-\alpha_n)\phi(u,y_n) \\
&\leq \alpha_n\phi(u,x_0) + \phi(u,y_n) \\
&\leq \phi(u,x_n) + \alpha_n\phi(u,x_0),
\end{aligned} \tag{4.6c}$$

$$\begin{aligned}
\phi(u,x_{n+1}) &= \phi\Big(u, J^{-1}(\widetilde{\alpha}_n Jx_0 + (1-\widetilde{\alpha}_n)J\widetilde{y}_n)\Big) \\
&= \|u\|^2 - 2\langle u, \widetilde{\alpha}_n Jx_0 + (1-\widetilde{\alpha}_n)J\widetilde{y}_n\rangle + \big\|\widetilde{\alpha}_n Jx_0 + (1-\widetilde{\alpha}_n)J\widetilde{y}_n\big\|^2 \\
&\leq \|u\|^2 - 2\widetilde{\alpha}_n\langle u, Jx_0\rangle - 2(1-\widetilde{\alpha}_n)\langle u, J\widetilde{y}_n\rangle + \widetilde{\alpha}_n\|x_0\|^2 + (1-\widetilde{\alpha}_n)\big\|\widetilde{y}_n\big\|^2 \\
&= \widetilde{\alpha}_n\phi(u,x_0) + (1-\widetilde{\alpha}_n)\phi(u,\widetilde{y}_n) \\
&\leq \widetilde{\alpha}_n\phi(u,x_0) + \phi(u,\widetilde{y}_n) \\
&\leq \phi(u,\widetilde{x}_n) + \widetilde{\alpha}_n\phi(u,x_0).
\end{aligned} \tag{4.6d}$$

Consequently, the last two inequalities yield that

$$\begin{aligned}
\phi(u,x_{n+1}) &\leq \phi(u,\widetilde{x}_n) + \widetilde{\alpha}_n\phi(u,x_0) \\
&\leq \phi(u,x_n) + \alpha_n\phi(u,x_0) + \widetilde{\alpha}_n\phi(u,x_0) \\
&= \phi(u,x_n) + (\alpha_n + \widetilde{\alpha}_n)\phi(u,x_0),
\end{aligned} \tag{4.6e}$$

for all $n \geq 0$. So, from $\sum_{n=0}^{\infty}\alpha_n < \infty$, $\sum_{n=0}^{\infty}\widetilde{\alpha}_n < \infty$, and Lemma 2.13, we deduce that $\lim_{n\to\infty}\phi(u,x_n)$ exists. This implies that $\{\phi(u,x_n)\}$ is bounded. Thus, $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{\widetilde{u}_n\}$, $\{J_{r_n}u_n\}$, and $\{\widetilde{J}_{r_n}\widetilde{u}_n\}$.

Define $z_n = \Pi_\Omega x_n$ for all $n \geq 0$. Let us show that $\{z_n\}$ is bounded. Indeed, observe that

$$\begin{aligned}
(\|z_n\| - \|x_n\|)^2 \leq \phi(z_n,x_n) = \phi(\Pi_\Omega x_n, x_n) &\leq \phi(p,x_n) - \phi(p,\Pi_\Omega x_n) \\
&= \phi(p,x_n) - \phi(p,z_n) \leq \phi(p,x_n),
\end{aligned} \tag{4.7}$$

for each $p \in \Omega$. This, together with the boundedness of $\{x_n\}$, implies that $\{z_n\}$ is bounded and so is $\phi(z_n,x_0)$. Furthermore, from $z_n \in \Omega$ and (4.6e), we have

$$\phi(z_n,x_{n+1}) \leq \phi(z_n,x_n) + (\alpha_n + \widetilde{\alpha}_n)\phi(z_n,x_0). \tag{4.8}$$

Since Π_Ω is the generalized projection, then, from Lemma 2.4 we obtain

$$\begin{aligned}\phi(z_{n+1}, x_{n+1}) &= \phi(\Pi_\Omega x_{n+1}, x_{n+1}) \leq \phi(z_n, x_{n+1}) - \phi(z_n, \Pi_\Omega x_{n+1}) \\ &= \phi(z_n, x_{n+1}) - \phi(z_n, z_{n+1}) \leq \phi(z_n, x_{n+1}).\end{aligned} \tag{4.9}$$

Hence, from (4.8), it follows that $\phi(z_{n+1}, x_{n+1}) \leq \phi(z_n, x_n) + (\alpha_n + \tilde{\alpha}_n)\phi(z_n, x_0)$.

Note that $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} \tilde{\alpha}_n < \infty$, and $\{\phi(z_n, x_0)\}$ is bounded, so that $\sum_{n=0}^{\infty}(\alpha_n + \tilde{\alpha}_n)\phi(z_n, x_0) < \infty$. Therefore, $\{\phi(z_n, x_n)\}$ is a convergent sequence. On the other hand, from (4.6e) we derive, for all $m \geq 0$,

$$\phi(u, x_{n+m}) \leq \phi(u, x_n) + \sum_{j=0}^{m-1}(\alpha_{n+j} + \tilde{\alpha}_{n+j})\phi(u, x_0). \tag{4.10}$$

In particular, we have

$$\phi(z_n, x_{n+m}) \leq \phi(z_n, x_n) + \sum_{j=0}^{m-1}(\alpha_{n+j} + \tilde{\alpha}_{n+j})\phi(z_n, x_0). \tag{4.11}$$

Consequently, from $z_{n+m} = \Pi_\Omega x_{n+m}$ and Lemma 2.4, we have

$$\phi(z_n, z_{n+m}) + \phi(z_{n+m}, x_{n+m}) \leq \phi(z_n, x_{n+m}) \leq \phi(z_n, x_n) + \sum_{j=0}^{m-1}(\alpha_{n+j} + \tilde{\alpha}_{n+j})\phi(z_n, x_0) \tag{4.12}$$

and hence

$$\phi(z_n, z_{n+m}) \leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + \sum_{j=0}^{m-1}(\alpha_{n+j} + \tilde{\alpha}_{n+j})\phi(z_n, x_0). \tag{4.13}$$

Let $r = \sup\{\|z_n\| : n \geq 0\}$. From Lemma 2.7, there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that

$$g(\|x - y\|) \leq \phi(x, y), \quad \forall x, y \in B_r. \tag{4.14}$$

So, we have

$$\begin{aligned}g(\|z_n - z_{n+m}\|) &\leq \phi(z_n, z_{n+m}) \\ &\leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + \sum_{j=0}^{m-1}(\alpha_{n+j} + \tilde{\alpha}_{n+j})\phi(z_n, x_0).\end{aligned} \tag{4.15}$$

Since $\{\phi(z_n, x_n)\}$ is a convergent sequence, $\{\phi(z_n, x_0)\}$ is bounded and $\sum_{n=0}^{\infty}(\alpha_n + \tilde{\alpha}_n)$ is convergent; from the property of g, we have that $\{z_n\}$ is a Cauchy sequence. Since Ω is closed, $\{z_n\}$ converges strongly to $z \in \Omega$. This completes the proof. □

Now, we are in a position to prove the following theorem.

Theorem 4.2. *Suppose that Assumption 2.1 is fulfilled and let* $\{x_n\}$ *be a sequence defined by* (4.1), *where* $\{\alpha_n\}, \{\beta_n\}, \{\widetilde{\alpha}_n\}, \{\widetilde{\beta}_n\} \subset [0,1]$ *satisfy the following conditions:*

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \sum_{n=0}^{\infty}\widetilde{\alpha}_n < \infty, \qquad \liminf_{n\to\infty}\beta_n(1-\beta_n) > 0, \qquad \liminf_{n\to\infty}\widetilde{\beta}_n\left(1-\widetilde{\beta}_n\right) > 0, \tag{4.16}$$

and $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *If* J *is weakly sequentially continuous, then* $\{x_n\}$ *converges weakly to* $z \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$, *where* $z = \lim_{n\to\infty}\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_n$.

Proof. We consider the notations (4.3). As in the proof of Proposition 4.1, we have that $\{x_n\}$, $\{u_n\}$, $\{J_{r_n}u_n\}$, $\{\widetilde{x}_n\}$, $\{\widetilde{u}_n\}$, and $\{\widetilde{J}_{r_n}\widetilde{u}_n\}$ are bounded sequences. Let

$$r = \sup\left\{\|u_n\|, \|J_{r_n}u_n\|, \|\widetilde{u}_n\|, \left\|\widetilde{J}_{r_n}\widetilde{u}_n\right\| : n \geq 0\right\}. \tag{4.17}$$

From Lemma 2.6 and as in the proof of Theorem 3.1, there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that

$$\|\alpha x^* + (1-\alpha)y^*\|^2 \leq \alpha\|x^*\|^2 + (1-\alpha)\|y^*\|^2 - \alpha(1-\alpha)g(\|x^* - y^*\|) \tag{4.18}$$

for $x^*, y^* \in B_r^*$ and $\alpha \in [0,1]$. Observe that for $u \in \Omega := T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$,

$$\begin{aligned}
\phi(u, y_{\mathrm{n}}) &= \phi\left(u, J^{-1}(\beta_n Ju_n + (1-\beta_n)JJ_{r_n}u_n)\right) \\
&= \|u\|^2 - 2\langle u, \beta_n Ju_n + (1-\beta_n)JJ_{r_n}u_n\rangle + \|\beta_n Ju_n + (1-\beta_n)JJ_{r_n}u_n\|^2 \\
&\leq \|u\|^2 - 2\beta_n\langle u, Ju_n\rangle - 2(1-\beta_n)\langle u, JJ_{r_n}u_n\rangle \\
&\quad + \beta_n\|u_n\|^2 + (1-\beta_n)\|J_{r_n}u_n\|^2 - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&\leq \beta_n\phi(u, u_n) + (1-\beta_n)\phi(u, J_{r_n}u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&\leq \beta_n\phi(u, u_n) + (1-\beta_n)\phi(u, u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&= \phi(u, u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|),
\end{aligned}$$

$$
\begin{aligned}
\phi(u,\widetilde{y}_n) &= \phi\Big(u, J^{-1}\Big(\widetilde{\beta}_n J\widetilde{u}_n + \Big(1-\widetilde{\beta}_n\Big)J\widetilde{J}_{r_n}\widetilde{u}_n\Big)\Big) \\
&= \|u\|^2 - 2\Big\langle u, \widetilde{\beta}_n J\widetilde{u}_n + \Big(1-\widetilde{\beta}_n\Big)J\widetilde{J}_{r_n}\widetilde{u}_n \Big\rangle + \Big\|\widetilde{\beta}_n J\widetilde{u}_n + (1-\widetilde{\beta}_n)J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|^2 \\
&\le \|u\|^2 - 2\widetilde{\beta}_n\langle u, J\widetilde{u}_n\rangle - 2\Big(1-\widetilde{\beta}_n\Big)\Big\langle u, J\widetilde{J}_{r_n}\widetilde{u}_n \Big\rangle \\
&\quad + \widetilde{\beta}_n\|\widetilde{u}_n\|^2 + \Big(1-\widetilde{\beta}_n\Big)\Big\|\widetilde{J}_{r_n}\widetilde{u}_n\Big\|^2 - \widetilde{\beta}_n\Big(1-\widetilde{\beta}_n\Big)g\Big(\Big\|J\widetilde{u}_n - J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|\Big) \\
&\le \widetilde{\beta}_n\phi(u,\widetilde{u}_n) + \Big(1-\widetilde{\beta}_n\Big)\phi\Big(u,\widetilde{J}_{r_n}\widetilde{u}_n\Big) - \widetilde{\beta}_n\Big(1-\widetilde{\beta}_n\Big)g\Big(\Big\|J\widetilde{u}_n - J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|\Big) \\
&\le \widetilde{\beta}_n\phi(u,\widetilde{u}_n) + \Big(1-\widetilde{\beta}_n\Big)\phi(u,\widetilde{u}_n) - \widetilde{\beta}_n\Big(1-\widetilde{\beta}_n\Big)g\Big(\Big\|J\widetilde{u}_n - J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|\Big) \\
&= \phi(u,\widetilde{u}_n) - \widetilde{\beta}_n\Big(1-\widetilde{\beta}_n\Big)g\Big(\Big\|J\widetilde{u}_n - J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|\Big).
\end{aligned}
\tag{4.19}
$$

Hence,

$$
\begin{aligned}
\phi(u,\widetilde{x}_n) &= \phi\Big(u, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jy_n)\Big) \\
&= \|u\|^2 - 2\langle u, \alpha_n Jx_0 + (1-\alpha_n)Jy_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)Jy_n\|^2 \\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_0\rangle - 2(1-\alpha_n)\langle u, Jy_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|y_n\|^2 \\
&= \alpha_n\phi(u,x_0) + (1-\alpha_n)\phi(u,y_n) \\
&\le \alpha_n\phi(u,x_0) + \phi(u,y_n) \\
&\le \alpha_n\phi(u,x_0) + \phi(u,u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&= \alpha_n\phi(u,x_0) + \phi(u,K_{r_n}x_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&\le \alpha_n\phi(u,x_0) + \phi(u,x_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|), \\
\phi(u,x_{n+1}) &= \phi\Big(u, J^{-1}(\widetilde{\alpha}_n Jx_0 + (1-\widetilde{\alpha}_n)J\widetilde{y}_n)\Big) \\
&= \|u\|^2 - 2\langle u, \widetilde{\alpha}_n Jx_0 + (1-\widetilde{\alpha}_n)J\widetilde{y}_n\rangle + \|\widetilde{\alpha}_n Jx_0 + (1-\widetilde{\alpha}_n)J\widetilde{y}_n\|^2 \\
&\le \|u\|^2 - 2\widetilde{\alpha}_n\langle u, Jx_0\rangle - 2(1-\widetilde{\alpha}_n)\langle u, J\widetilde{y}_n\rangle + \widetilde{\alpha}_n\|x_0\|^2 + (1-\widetilde{\alpha}_n)\|\widetilde{y}_n\|^2 \\
&= \widetilde{\alpha}_n\phi(u,x_0) + (1-\widetilde{\alpha}_n)\phi(u,\widetilde{y}_n) \\
&\le \widetilde{\alpha}_n\phi(u,x_0) + \phi(u,\widetilde{y}_n) \\
&\le \widetilde{\alpha}_n\phi(u,x_0) + \phi(u,\widetilde{u}_n) - \widetilde{\beta}_n\Big(1-\widetilde{\beta}_n\Big)g\Big(\Big\|J\widetilde{u}_n - J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|\Big) \\
&= \widetilde{\alpha}_{\mathrm{n}}\phi(u,x_0) + \phi(u,K_{r_n}\widetilde{x}_n) - \widetilde{\beta}_n\Big(1-\widetilde{\beta}_n\Big)g\Big(\Big\|J\widetilde{u}_n - J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|\Big) \\
&\le \widetilde{\alpha}_n\phi(u,x_0) + \phi(u,\widetilde{x}_n) - \widetilde{\beta}_n\Big(1-\widetilde{\beta}_n\Big)g\Big(\Big\|J\widetilde{u}_n - J\widetilde{J}_{r_n}\widetilde{u}_n\Big\|\Big).)
\end{aligned}
\tag{4.20}
$$

Consequently, the last two inequalities yield that

$$\begin{aligned}
\phi(u,x_{n+1}) &\le \tilde{\alpha}_n\phi(u,x_0)+\phi(u,\tilde{x}_n)-\tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{u}_n-J\tilde{J}_{r_n}\tilde{u}_n\right\|\right)\\
&\le \tilde{\alpha}_n\phi(u,x_0)+\alpha_n\phi(u,x_0)+\phi(u,x_n)-\beta_n(1-\beta_n)g(\|Ju_n-JJ_{r_n}u_n\|)\\
&\quad-\tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{u}_n-J\tilde{J}_{r_n}\tilde{u}_n\right\|\right)\\
&=\phi(u,x_n)+(\alpha_n+\tilde{\alpha}_n)\phi(u,x_0)-\beta_n(1-\beta_n)g(\|Ju_n-JJ_{r_n}u_n\|)\\
&\quad-\tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{u}_n-J\tilde{J}_{r_n}\tilde{u}_n\right\|\right).
\end{aligned}\tag{4.21}$$

Thus, we have

$$\begin{aligned}
&\beta_n(1-\beta_n)g(\|Ju_n-JJ_{r_n}u_n\|)+\tilde{\beta}_n\left(1-\tilde{\beta}_n\right)g\left(\left\|J\tilde{u}_n-J\tilde{J}_{r_n}\tilde{u}_n\right\|\right)\\
&\quad\le\phi(u,x_n)-\phi(u,x_{n+1})+(\alpha_n+\tilde{\alpha}_n)\phi(u,x_0).
\end{aligned}\tag{4.22}$$

By the proof of Proposition 4.1, it is known that $\{\phi(u,x_n)\}$ is convergent; since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\tilde{\alpha}_n=0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, and $\liminf_{n\to\infty}\tilde{\beta}_n(1-\tilde{\beta}_n)>0$, then we have

$$\lim_{n\to\infty}g(\|Ju_n-JJ_{r_n}u_n\|)=\lim_{n\to\infty}g\left(\left\|J\tilde{u}_n-J\tilde{J}_{r_n}\tilde{u}_n\right\|\right)=0.\tag{4.23}$$

Taking into account the properties of g, as in the proof of Theorem 3.1, we have

$$\begin{aligned}
&\lim_{n\to\infty}\|Ju_n-JJ_{r_n}u_n\|=\lim_{n\to\infty}\|u_n-J_{r_n}u_n\|=0,\\
&\lim_{n\to\infty}\left\|J\tilde{u}_n-J\tilde{J}_{r_n}\tilde{u}_n\right\|=\lim_{n\to\infty}\left\|\tilde{u}_n-\tilde{J}_{r_n}\tilde{u}_n\right\|=0,
\end{aligned}\tag{4.24}$$

since J^{-1} is uniformly norm-to-norm continuous on bounded subsets of X^*.

Now let us show that

$$\lim_{n\to\infty}\phi(u,x_n)=\lim_{n\to\infty}\phi(u,\tilde{x}_n)=\lim_{n\to\infty}\phi(u,u_n)=\lim_{n\to\infty}\phi(u,\tilde{u}_n).\tag{4.25}$$

Indeed, from (4.6e) we get

$$\phi(u,x_{n+1})-\tilde{\alpha}_n\phi(u,x_0)\le\phi(u,\tilde{x}_n)\le\phi(u,x_n)+\alpha_n\phi(u,x_0),\tag{4.26}$$

which, together with $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\tilde{\alpha}_n=0$, yields that

$$\lim_{n\to\infty}\phi(u,\tilde{x}_n)=\lim_{n\to\infty}\phi(u,x_n).\tag{4.27}$$

From (4.6d) it follows that

$$\phi(u, x_{n+1}) - \tilde{\alpha}_n \phi(u, x_0) \leq \phi(u, \tilde{y}_n) \leq \phi(u, \tilde{x}_n), \tag{4.28}$$

which, together with $\lim_{n\to\infty}\phi(u, \tilde{x}_n) = \lim_{n\to\infty}\phi(u, x_n)$, yields that

$$\lim_{n\to\infty} \phi(u, \tilde{y}_n) = \lim_{n\to\infty} \phi(u, x_n). \tag{4.29}$$

From (4.6c) it follows that

$$\phi(u, \tilde{x}_n) - \alpha_n \phi(u, x_0) \leq \phi(u, y_n) \leq \phi(u, x_n), \tag{4.30}$$

which, together with $\lim_{n\to\infty}\phi(u, \tilde{x}_n) = \lim_{n\to\infty}\phi(u, x_n)$, yields that

$$\lim_{n\to\infty} \phi(u, y_n) = \lim_{n\to\infty} \phi(u, x_n). \tag{4.31}$$

From (4.6c) it follows that

$$\phi(u, \tilde{y}_n) \leq \phi(u, \tilde{u}_n) \leq \phi(u, \tilde{x}_n), \tag{4.32}$$

which together with

$$\lim_{n\to\infty} \phi(u, \tilde{x}_n) = \lim_{n\to\infty} \phi(u, \tilde{y}_n) = \lim_{n\to\infty} \phi(u, x_n), \tag{4.33}$$

yields that

$$\lim_{n\to\infty} \phi(u, \tilde{u}_n) = \lim_{n\to\infty} \phi(u, x_n). \tag{4.34}$$

From (4.6a) it follows that

$$\phi(u, y_n) \leq \phi(u, u_n) \leq \phi(u, x_n) \tag{4.35}$$

which, together with $\lim_{n\to\infty}\phi(u, y_n) = \lim_{n\to\infty}\phi(u, x_n)$, yields that

$$\lim_{n\to\infty} \phi(u, u_n) = \lim_{n\to\infty} \phi(u, x_n). \tag{4.36}$$

On the other hand, let us show that

$$\lim_{n\to\infty} \|x_n - \tilde{x}_n\| = 0. \tag{4.37}$$

Indeed, let $s = \sup\{\|x_n\|, \|u_n\|, \|\tilde{x}_n\|, \|\tilde{u}_n\| : n \geq 0\}$. From Lemma 2.7, there exists a continuous, strictly increasing, and convex function g_1 with $g_1(0) = 0$ such that

$$g_1(\|x - y\|) \leq \phi(x, y), \quad \forall x, y \in B_s. \tag{4.38}$$

Since $u_n = K_{r_n} x_n$ and $\tilde{u}_n = K_{r_n} \tilde{x}_n$, we deduce from Proposition 2.11 that for $u \in \Omega$,

$$\begin{aligned} g_1(\|u_n - x_n\|) &\leq \phi(u_n, x_n) \leq \phi(u, x_n) - \phi(u, u_n), \\ g_1(\|\tilde{u}_n - \tilde{x}_n\|) &\leq \phi(\tilde{u}_n, \tilde{x}_n) \leq \phi(u, \tilde{x}_n) - \phi(u, \tilde{u}_n). \end{aligned} \tag{4.39}$$

This implies that

$$\lim_{n\to\infty} g_1(\|u_n - x_n\|) = \lim_{n\to\infty} g_1(\|\tilde{u}_n - \tilde{x}_n\|) = 0. \tag{4.40}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of X, from the properties of g_1, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|u_n - x_n\| &= \lim_{n\to\infty} \|Ju_n - Jx_n\| = 0, \\ \lim_{n\to\infty} \|\tilde{u}_n - \tilde{x}_n\| &= \lim_{n\to\infty} \|J\tilde{u}_n - J\tilde{x}_n\| = 0. \end{aligned} \tag{4.41}$$

Note that

$$\begin{aligned} \phi(x_n, u_n) - \phi(u_n, x_n) &= \|x_n\|^2 - 2\langle x_n, Ju_n\rangle + \|u_n\|^2 - \left[\|x_n\|^2 - 2\langle u_n, Jx_n\rangle + \|u_n\|^2\right] \\ &= -2\langle x_n, Ju_n\rangle + 2\langle u_n, Jx_n\rangle \\ &= 2\langle x_n, Jx_n - Ju_n\rangle + 2\langle u_n - x_n, Jx_n\rangle \\ &\leq 2\|x_n\|\|Jx_n - Ju_n\| + 2\|u_n - x_n\|\|x_n\|, \\ \phi(x_n, J_{r_n}u_n) &= \|x_n\|^2 - 2\langle x_n, JJ_{r_n}u_n\rangle + \|J_{r_n}u_n\|^2 \\ &= \|x_n\|^2 - \|x_n\|^2 + \|J_{r_n}u_n\|^2 - \|x_n\|^2 + 2\langle x_n, Jx_n - JJ_{r_n}u_n\rangle \\ &= (\|J_{r_n}u_n\| - \|x_n\|)(\|J_{r_n}u_n\| + \|x_n\|) + 2\langle x_n, Jx_n - JJ_{r_n}u_n\rangle \\ &\leq \|J_{r_n}u_n - x_n\|(\|J_{r_n}u_n\| + \|x_n\|) + 2\|x_n\|\|Jx_n - JJ_{r_n}u_n\| \\ &= \|J_{r_n}u_n - u_n + u_n - x_n\|(\|J_{r_n}u_n\| + \|x_n\|) \\ &\quad + 2\|x_n\|\|Jx_n - Ju_n + Ju_n - JJ_{r_n}u_n\| \\ &\leq (\|J_{r_n}u_n - u_n\| + \|u_n - x_n\|)(\|J_{r_n}u_n\| + \|x_n\|) \\ &\quad + 2\|x_n\|(\|Jx_n - Ju_n\| + \|Ju_n - JJ_{r_n}u_n\|). \end{aligned} \tag{4.42}$$

Since $\phi(u_n, x_n) \to 0$, it follows from (4.24) and (4.41) that $\phi(x_n, u_n) \to 0$ and $\phi(x_n, J_{r_n}u_n) \to 0$.

Also, observe that

$$\begin{aligned}
\phi(x_n, y_n) &= \phi\Big(x_n, J^{-1}(\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n)\Big) \\
&= \|x_n\|^2 - 2\big\langle x_n, \beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\big\rangle + \big\|\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\big\|^2 \\
&\le \|x_n\|^2 - 2\beta_n\langle x_n, J u_n\rangle - 2(1-\beta_n)\langle x_n, J J_{r_n} u_n\rangle + \beta_n\|u_n\|^2 + (1-\beta_n)\|J_{r_n} u_n\|^2 \\
&= \beta_n\phi(x_n, u_n) + (1-\beta_n)\phi(x_n, J_{r_n} u_n) \\
&\le \phi(x_n, u_n) + \phi(x_n, J_{r_n} u_n),
\end{aligned} \tag{4.43}$$

and hence

$$\begin{aligned}
\phi(x_n, \tilde{x}_n) &= \phi\Big(x_n, J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J y_n)\Big) \\
&= \|x_n\|^2 - 2\big\langle x_n, \alpha_n J x_0 + (1-\alpha_n) J y_n\big\rangle + \big\|\alpha_n J x_0 + (1-\alpha_n) J y_n\big\|^2 \\
&\le \|x_n\|^2 - 2\alpha_n\langle x_n, J x_0\rangle - 2(1-\alpha_n)\langle x_n, J y_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\big\|y_n\big\|^2 \\
&= \alpha_n\phi(x_n, x_0) + (1-\alpha_n)\phi(x_n, y_n) \\
&\le \alpha_n\phi(x_n, x_0) + \phi(x_n, y_n) \\
&\le \alpha_n\phi(x_n, x_0) + \phi(x_n, u_n) + \phi(x_n, J_{r_n} u_n).
\end{aligned} \tag{4.44}$$

Thus, from $\alpha_n \to 0$, $\phi(x_n, u_n) \to 0$, and $\phi(x_n, J_{r_n} u_n) \to 0$, it follows that $\phi(x_n, \tilde{x}_n) \to 0$. In terms of Lemma 2.2, we derive $\|x_n - \tilde{x}_n\| \to 0$.

Next, let us show that $x_n \rightharpoonup z$, where $z = \lim_{n\to\infty}\Pi_{T^{-1}0\cap\tilde{T}^{-1}0\cap EP} x_n$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$. Hence it follows from (4.24), (4.41), and $\|x_n - \tilde{x}_n\| \to 0$ that $\{u_{n_k}\}$, $\{\tilde{u}_{n_k}\}$, $\{J_{r_{n_k}} u_{n_k}\}$ and $\tilde{J}_{r_{n_k}}\tilde{u}_{n_k}$ converge weakly to the same point $\hat{x}$. Furthermore, from $\liminf_{n\to\infty} r_n > 0$ and (4.24), we have that

$$\begin{aligned}
\lim_{n\to\infty}\|A_{r_n} u_n\| &= \lim_{n\to\infty}\frac{1}{r_n}\|J u_n - J J_{r_n} u_n\| = 0, \\
\lim_{n\to\infty}\Big\|\tilde{A}_{r_n}\tilde{u}_n\Big\| &= \lim_{n\to\infty}\frac{1}{r_n}\Big\|J\tilde{u}_n - J\tilde{J}_{r_n}\tilde{u}_n\Big\| = 0.
\end{aligned} \tag{4.45}$$

If $z^* \in Tz$ and $\tilde{z}^* \in \tilde{T}\tilde{z}$, then it follows from (2.17) and the monotonicity of the operators T, $\tilde{T}$ that for all $k \ge 1$

$$\Big\langle z - J_{r_{n_k}} u_{n_k}, z^* - A_{r_{n_k}} u_{n_k}\Big\rangle \ge 0, \qquad \Big\langle \tilde{z} - \tilde{J}_{r_{n_k}}\tilde{u}_{n_k}, \tilde{z}^* - \tilde{A}_{r_{n_k}}\tilde{u}_{n_k}\Big\rangle \ge 0. \tag{4.46}$$

Letting $k \to \infty$, we obtain that

$$\langle z - \hat{x}, z^*\rangle \ge 0, \qquad \langle \tilde{z} - \hat{x}, \tilde{z}^*\rangle \ge 0. \tag{4.47}$$

Then the maximality of the operators T, $\tilde{T}$ implies that $\hat{x} \in T^{-1}0 \cap \tilde{T}^{-1}0$.

Now, by the definition of $u_n := K_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \tag{4.48}$$

where $F(x, y) = f(x, y) + \langle Ax, y - x\rangle$. Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jx_{n_k} \rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C. \tag{4.49}$$

Since $y \mapsto F(x, y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (4.41) and (A4), we have

$$F(y, \hat{x}) \leq 0, \quad \forall y \in C. \tag{4.50}$$

For t, with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, then $y_t \in C$ and hence $F(y_t, \hat{x}) \leq 0$. So, from (A1), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-\mathrm{t})F(y_t, \hat{x}) \leq tF(y_t, y). \tag{4.51}$$

Dividing by t, we get $F(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, from (A3) it follows that $F(\hat{x}, y) \geq 0$, $\forall y \in C$. So, $\hat{x} \in EP$. Therefore, $\hat{x} \in \Omega$. Let $z_n = \Pi_\Omega x_n$. From Lemma 2.3 and $\hat{x} \in \Omega$, we get

$$\langle z_{n_k} - \hat{x}, Jx_{n_k} - Jz_{n_k} \rangle \geq 0. \tag{4.52}$$

From Proposition 4.1, we also know that $z_n \to z \in \Omega$. Note that $x_{n_k} \rightharpoonup \hat{x}$. Since J is weakly sequentially continuous, then $\langle z - \hat{x}, J\hat{x} - Jz \rangle \geq 0$ as $k \to \infty$. In addition, taking into account the monotonicity of J, we conclude that $\langle z - \hat{x}, J\hat{x} - Jz \rangle \leq 0$. Hence

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle = 0. \tag{4.53}$$

From the strict convexity of X, it follows that $z = \hat{x}$. Therefore, $x_n \rightharpoonup \hat{x}$, where $\hat{x} = \lim_{n\to\infty} \Pi_{T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP} x_n$. This completes the proof. □

Remark 4.3. Compared with the algorithm of Theorem 1.2, the above algorithm (4.1) can be applied to find an element of $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$. But, the algorithm of Theorem 1.2 cannot be applied. Therefore, algorithm (4.1) develops and improves the algorithm of Theorem 1.2.

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