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Review Article

Normal Structure and Common Fixed Point Properties for Semigroups of Nonexpansive Mappings in Banach Spaces

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In 1965, Kirk proved that if C is a nonempty weakly compact convex subset of a Banach space with normal structure, then every nonexpansive mapping $T : C \to C$ has a fixed point. The purpose of this paper is to outline various generalizations of Kirk's fixed point theorem to semigroup of nonexpansive mappings and for Banach spaces associated to a locally compact group.

1. Introduction

A closed convex subset C of a Banach space E has *normal structure* if for each bounded closed convex subset D of C which contains more than one point, there is a point $x \in D$ which is not a diametral point of D, that is, $\sup\{\|x - y\| : y \in D\} < \delta(D)$, where $\delta(D)$ = the diameter of D.

The set C is said to have *fixed point property* (FPP) if every nonexpansive mapping $T : C \to C$ has a fixed point. In [1], Kirk proved the following important celebrated result.

Theorem 1.1 (Kirk [1]). *Let E be a Banach space, and C a nonempty closed convex subset of E. If C is weakly compact and has normal structure, then C has the FPP.*

As well known, compact convex subset of a Banach space E always has normal structure (see [2]). It was an open problem for over 15 years whether every weakly compact convex subset of E has normal structure. This problem was answered negatively by Alspach [3] when he showed that there is a weakly compact convex subset C of $L^1[0,1]$ which does not have the fixed point property. In particular, C cannot have normal structure.

It is the purpose of this paper to outline the relation of normal structure and fixed point property for semigroup of nonexpansive mappings. This paper is organized as follows. In Section 3, we will focus on generalizations of Kirk's fixed point theorem to semigroups of nonexpansive mappings. In Section 4, we will discuss about fixed point properties and normal structure on Banach spaces associated to a locally compact group.

2. Some Preliminaries

All topologies in this paper are assumed to be Hausdorff. If E is a Banach space and $A \subseteq E$, then $\overline{A}$ and $\overline{\text{co}}\ A$ will denote the closure of A and the closed convex hull of A in E, respectively.

Let E be a Banach space and let C a subset of E. A mapping T from C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. A Banach space E is said to be *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$ for each $x, y \in E$ satisfying $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$.

Let S be a semigroup, $\ell^\infty(S)$ the Banach space of bounded real valued functions on S with the supremum norm. Then a subspace X of $\ell^\infty(S)$ is *left (resp., right) translation invariant* if $\ell_a(\mathrm{X}) \subseteq \mathrm{X}$ (resp., $r_a(\mathrm{X}) \subseteq \mathrm{X}$) for all $a \in S$, where $(\ell_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$, $s \in S$.

A *semitoplogical semigroup* S is a semigroup with Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S into S are continuous. Examples of semitopological semigroups include all topological groups, the set $M(n, \mathbb{C})$ of all $n \times n$ matrices with complex entries, matrix multiplication, and the usual topology, the unit ball of ℓ^∞ with weak*-topology and pointwise multiplication, or $\mathcal{B}(H)$(= the space of bounded linear operators on a Hilbert space H) with the weak*-topology and composition.

If S is a semitopological semigroup, we denote $\mathrm{CB}(S)$ the closed subalgebra of $\ell^\infty(S)$ consisting of continuous functions. Let $\mathrm{LUC}(S)$ (resp., $\mathrm{RUC}(S)$) be the space of *left* (resp., *right*) *uniformly continuous functions* on S; that is, all $f \in \mathrm{CB}(S)$ such that the mapping from S into $\mathrm{CB}(S)$ defined by $s \to \ell_s f$ (resp., $s \to r_s f$) is continuous when $\mathrm{CB}(S)$ has the sup norm topology. Then as is known (see [4]), $\mathrm{LUC}(S)$ and $\mathrm{RUC}(S)$ are left and right translation invariant closed subalgebras of $\mathrm{CB}(S)$ containing constants. Also let $\mathrm{AP}(S)$ (resp., $\mathrm{WAP}(S)$) denote the space of almost periodic (resp., weakly almost periodic) functions f in $\mathrm{CB}(S)$; that is, all $f \in \mathrm{CB}(S)$ such that $\{\ell_a f; a \in S\}$ is relatively compact in the norm (resp., weak) topology of $\mathrm{CB}(S)$, or equivalently $\{r_a f; a \in S\}$ is relatively compact in the norm (resp., weak) topology of $\mathrm{CB}(S)$. Then as is known [4, page 164], $\mathrm{AP}(S) \subseteq \mathrm{LUC}(S) \cap \mathrm{RUC}(S)$, and $\mathrm{AP}(S) \subseteq \mathrm{WAP}(S)$. When S is a locally compact group, then $\mathrm{WAP}(S) \subseteq \mathrm{LUC}(S) \cap \mathrm{RUC}(S)$ (see [4, page 167]).

A semitopological semigroup S is *left reversible* if any two closed right ideals of S have nonvoid intersection.

The class $\mathbb{S}$ of all left reversible semitopological semigroups includes trivially all semitopological semigroups which are algebraically groups, and all commuting semigroups.

The class $\mathbb{S}$ is closed under the following operations.

(a) If $S \in \mathbb{S}$ and S' is a continuous homomorphic image of S, then $S' \in \mathbb{S}$.

(b) Let $S_\alpha \in \mathbb{S}$, $\alpha \in I$ and S be the semitopological semigroup consisting of the set of all functions f on I such that $f(\alpha) \in S_\alpha$, $\alpha \in I$, the binary operation defined by $fg(\alpha) = f(\alpha)g(\alpha)$ for all $\alpha \in I$ and $f, g \in S$, and the product topology. Then $S \in \mathbb{S}$.

(c) Let S be a semitopological semigroup and S_α, $\alpha \in I$, semitopological subsemigroups of S with the property that $S = \cup S_\alpha$ and, if $\alpha_1, \alpha_2 \in I$, then there exists $\alpha_3 \in I$ such that $S_{\alpha_3} \supseteq S_{\alpha_1} \cup S_{\alpha_2}$. If $S_\alpha \in \mathbb{S}$ for each $\alpha \in I$, then $S \in \mathbb{S}$.

Let S be a nonempty set and X a translation invariant subspace of $\ell^\infty(S)$ containing constants. Then $\mu \in X^*$ is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. As well known, μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \tag{2.1}$$

for each $f \in X$.

Also μ is called a *left (resp., right) invariant mean* if $\mu(\ell_a f) = \mu(f)$ (resp., $\mu(r_a f) = \mu(f)$) for all $a \in S$, $f \in X$.

Lemma 2.1. *Let S be a semitopological semigroup and X a left translation invariant subspace of* CB(S) *containing constants and which separates closed subsets of S. If X has a left invariant mean, then S is left reversible.*

Proof. Let μ be a left invariant mean of X, I_1 and I_2 disjoint nonempty closed right ideals of S. By assumption, there exists $f \in X$ such that $f \equiv 1$ on I_1 and $f \equiv 0$ on I_2. Now if $a_1 \in I_1$, then $\ell_{a_1} f = 1$. So,

$$\mu(f) = \mu(\ell_{a_1} f) = 1. \tag{2.2}$$

But if $a_2 \in I_2$, then $\ell_{a_2} f \equiv 0$. So $\mu(f) = \mu(\ell_{a_2} f) = 0$, which is impossible. □

Corollary 2.2. *If S is normal and CB(S) has a left invariant mean, then S is left reversible.*

See [5] for details.

A discrete semigroup S is called *left amenable* [6] if $\ell^\infty(S)$ has a left invariant mean. In particular every left amenable discrete semigroup is left reversible by Corollary 2.2. The semigroup S is *amenable* if it is both left and right amenable. In this case, there is always an invariant mean on $\ell^\infty(S)$.

Remark 2.3. Lemma 2.1 is not true without normality. Let S be a topological space which is regular and Hausdorff and CB(S) consists of constant functions only [7]. Define on S the multiplication $st = s$ for all $s, t \in S$. Let $a \in S$ be fixed. Define $\mu(f) = f(a)$ for all $a \in S$. Then μ is a left invariant mean on $C(S)$, but S is not left reversible.

3. Generalizations of Kirk's Fixed Point Theorem

By a (nonlinear) *submean* on X, we will mean a real-valued function μ on X satisfying the following properties:

(1) $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;

(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$;

(3) for $f, g \in X$, $f \leq q$ implies $\mu(f) \leq \mu(g)$;

(4) $\mu(c) = c$ for every constant function c.

Clearly every mean is a submean. See [8] for details.

If S is a semigroup and X is left translation invariant, a submean μ on X is left *subinvariant* if $\mu(\ell_a f) \geq \mu(f)$ for each $f \in X$ and $a \in S$.

Let S be a semitopological semigroup, C a nonempty subset of a Banach space E, then a representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings from C into C is *continuous* if $S \times C \to C$ defined by $(s, x) \to T_s x$, $s \in S$, $x \in C$ is continuous when $S \times C$ has the product topology. It is called *separately continuous* if for each $x \in C$ and $s \in S$, the maps $s \to T_s x$ from S into C and the map $x \to T_s x$ from C into C are continuous.

Theorem 3.1. *Let S be a semitopological semigroup, let C a nonempty weakly compact convex subset of a Banach space E which has normal structure and let $\mathcal{S} = \{T_s; s \in S\}$ a continuous representation of S as nonexpansive self-mappings on C. Suppose that $RUC(S)$ has a left subinvariant submean. Then $\mathcal{S}$ has a common fixed point in C.*

Corollary 3.2. *Let S be a left reversible semitopological semigroup. Let C be a nonempty weakly compact convex subset of a Banach space E which has normal structure and let $\mathcal{S} = \{T_s; s \in S\}$ a continuous representation of S as nonexpansive self-mappings on C. Then $\mathcal{S}$ has a fixed point in C.*

Proof. If S is left reversible, define $\mu(f) = \inf_s \sup_{t \in sS} f(t)$. Then the proof of Lemma 3.6 in [9] shows that μ is a submean on $CB(S)$ such that $\mu(\ell_a f) \geq \mu(f)$ for all $f \in CB(S)$ and $a \in S$, that is, μ is left subinvariant. □

Note that since every compact convex set has normal structure, Corollary 3.2 implies the following.

Corollary 3.3 (DeMarr [10]). *Let E be a Banach space and C a nonempty compact convex subset of E. If $\mathcal{F}$ is a commuting family of nonexpansive mappings of C into C, then the family $\mathcal{F}$ has a common fixed point in C.*

Remark 3.4. Theorem 3.1 is proved by Lau and Takahashi in [11]. Mitchell [12] generalized the theorems of DeMarr [10, page 1139] and Takahaski [13, page 384] by showing that if C is a nonempty compact convex subset of a Banach space and S is a left-reversible discrete semigroup of nonexpansive mappings from C into C, then C contains a common fixed point for S. Belluce and Kirk [14] also improved DeMarr's result in [10] and proved that if C is a nonempty weakly compact convex subset of a Banach space and if C has complete normal structure, then every family of commuting nonexpansive self-maps on C has a common fixed point.

This result was extended to the class of left reversible semitopological semigroup by Holmes and Lau in [15]. Corollary 3.2 is due to Lim [16] who showed that normal structure and complete normal structure are equivalent.

The following related theorem was also established in [15].

Theorem 3.5. *Let S be a left reversible semitopological semigroup, let C a nonempty, bounded, closed convex subset of a Banach space E, and let $\mathcal{S} = \{T_s;\ s \in S\}$ a separately continuous representation of S as nonexpansive self-maps on C. If there is a nonempty compact subset $M \subseteq C$ and $a \in S$ such that*

a commutes with all elements of S and for each $x \in C$, the closure of the set $\{a^n(x) \mid n = 1, 2, \ldots\}$ contains a point of M, then M contains a common fixed point of S.

Let S be a semitopological semigroup and C is a nonempty subset of a Banach space E, and $\mathcal{S} = \{T_s;\ s \in S\}$ a separately continuous representation of S as mappings from C into C. We say that the representation is asymptotically nonexpansive if for each $x, y \in C$, there is a left ideal $J \subseteq S$ such that $\|T_s x - T_s y\| \leq \|x - y\|$ for all $s \in J$.

We also say that the representation has *property (B)* if for each $x \in C$, whenever a net $\{s_\alpha(x);\ \alpha \in I\}, s_\alpha \in S$, converges to x, then the net $\{(s_\alpha a)x;\ \alpha \in I\}$ also converges to $a(x)$ for each $a \in S$.

Clearly condition (B) is automatically satisfied when S is commutative.

The semitopological semigroup S is right reversible if

$$\overline{s}_a \cap \overline{s}_b \neq \emptyset \quad \text{for each } a, b \in S. \tag{3.1}$$

The following theorem is proved in [17].

Theorem 3.6. *Let C be a nonempty compact convex subset of a Banach space E and S a right reversible semitopological semigroup. If $\mathcal{S} = \{T_s;\ s \in S\}$ is a separately continuous asymptotically nonexpansive representation of S as mappings from C into C with property (B), then C contains a common fixed point for $\mathcal{S}$.*

The following example from [17] shows a simple situation where our fixed point theorem applies, but DeMarr's fixed point theorem does not.

Let $K = \{(r, \theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$ be the closed unit disc in R^2 with polar coordinates and the usual Euclidean norm. Define continuous mappings f, g from K into K by

$$\begin{aligned} f(r, \theta) &= \left(\frac{r}{2}, \theta\right), \\ g(r, \theta) &= (r, 2\theta (\text{mod } 2\pi)). \end{aligned} \tag{3.2}$$

Then the semigroup of continuous mappings from K to K generated by f and g under usual composition is commutative and asymptotically nonexpansive. However, the action of S (or any ideal of S) on K is *not* nonexpansive.

Open Problem 1. Can right reversibility of S and property (B) in Theorem 3.6 be replaced by amenability of S?

Let C be a nonempty closed convex subset of a Banach space E. Then C has the *fixed point property* for nonexpansive mappings if every nonexpansive mapping $T : C \to C$ has a fixed point; C has the only*conditional fixed point property for nonexpansive mappings* if every nonexpansive mapping $T : C \to C$ satisfies either T has no fixed point in C, or T has a fixed point in every nonempty bounded closed convex T-invariant subset of C. For commuting family of nonexpansive mappings, the following is a remarkable common fixed point property due to Bruck [18].

Theorem 3.7. *Let E be a Banach space and C a nonempty closed convex subset of E. If C has both the fixed point property and the conditional fixed point property for nonexpansive mappings, then for any commuting family S of nonexpansive mappings of C into C, there is a common fixed point for S.*

Theorem 3.7 was proved by Belluce and Kirk [19] when S is finite and C is weakly compact and has normal structure, by Belluce and Kirk [14] when C is weakly compact and has complete normal structure, Browder [20] when E is uniformly convex and C is bounded, Lau and Holmes [15] when S is left reversible and C is compact, and finally by Lim [16] when S is left reversible and C is weakly compact and has normal structure.

Open Problem 2 (Bruck [18]). Can commutativity of $\mathcal{S}$ be replaced by left reversibility?

The answer to Problem 2 is not known even when the semigroup is left amenable.

Let $(\Sigma, \circ)$ be a compact right topological semigroup, that is, a semigroup and a compact Hausdorff topological space such that for each $\tau \in \Sigma$ the mapping $\gamma \to \gamma \circ \tau$ from Σ into Σ is continuous. In this case, Σ must contain minimal left ideals. Any minimal left ideal in Σ is closed and any two minimal left ideals of Σ are homeomorphic and algebraically isomorphic.

Let X be a nonempty weakly compact convex subset of a Banach space E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup S as nonexpansive and weak-weak continuous mappings from X into X. Let Σ be the closure of $\mathcal{S}$ in the product space $(X,\text{weak})^X$. Then Σ is a compact right topological semigroup consisting of nonexpansive mappings from X into X. Further, for any $T \in \Sigma$, there exists a sequence $\{T_n\}$ of convex combination of operators from $\mathcal{S}$ such that $\|T_n x - Tx\| \to 0$ for every $x \in X$. See [21] for details.

Σ is called the enveloping semigroup of S.

Theorem 3.8. *Let X be a nonempty weakly compact convex subset of a Banach space, E and X has normal structure. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup as norm nonexpansive and weakly continuous mappings from X into X and let Σ be the enveloping semigroup of $\mathcal{S}$. Let I be a minimal left ideal of Σ and let Y a minimal $\mathcal{S}$-invariant closed convex subset of X. Then there exists a nonempty weakly closed subset C of Y such that I is constant on C.*

Corollary 3.9. *Let Σ and X as in Theorem 3.8. Then there exist $T_0 \in \Sigma$ and $x \in X$ such that $T_0 T x = T_0 x$ for every $T \in \Sigma$.*

Proof. Pick $x \in C$ and $T_0 \in I$ of the above theorem. □

Remark 3.10. If S is commutative, then for any $T \in \Sigma$ and $s \in S, T_s \circ T = T \circ T_s$, that is, $z = T_0 x$ is in fact a common fixed point for Σ (and, hence, for $\mathcal{S}$). Note that if X is norm compact, the weak and norm topology agree on X. Hence every nonexpansive mapping from X into X must be weakly continuous. Therefore, Corollary 3.9 improves the fixed point theorem of DeMarr [10] for commuting semigroups of nonexpansive mappings on compact convex sets.

The above theorem proved in [21] provides a new approach using enveloping semigroups in the study of common fixed point of a semigroup of nonexpansive mappings on a weakly compact convex subset of a Banach space.

Open Problem 3. Can the above technique applied to give a proof of Lim's fixed point theorem for left reversible semigroup in [16].

The following generalization of DeMarr's fixed point theorem was proved in [22].

Theorem 3.11. *Let S a be semitopological semigroup.*

If $\mathrm{AP}(S)$ *has a left invariant mean, then S has the following fixed point property. Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a separately continuous representation of S as nonexpansive self-mappings on a compact convex subset C of a Banach space, then C contains a common fixed point for $\mathcal{S}$.*

Quite recently the Lau and Zhang [23] are able to establish the following related fixed point property.

Theorem 3.12. *Let S be a separable semitopological semigroup. If $WAP(S)$ has a left invariant mean, then S has the following fixed point property:*

Whenever $\mathcal{S} = \{T_s; s \in S\}$ is a continuous representation of S as nonexpansive self-mappings on a weakly compact convex subset C of a Banach space E such that the closure of $\mathcal{S}$ in C^C with the product of weak topology consists entirely of continuous functions, then C contains a common fixed point of C.

Remark 3.13. (a) The converse of Theorem 3.12 also holds when S has an identity by considering $\mathcal{S} = \{r_s;\ r \in S\}$, the semigroup of right translations, on the weakly compact convex sets $C_f = \overline{\mathrm{co}}\{r_s f;\ s \in S\}$ for each $f \in \mathrm{WAP}(S)$ (see [24]).

(b) When S is a discrete semigroup, the following implication diagram is known:

$$\begin{tikzcd}
S \text{ left amenable} \arrow[d, Rightarrow, shift right] & \\
S \text{ left reversible} \arrow[u, Rightarrow, shift right, "/" marking] \arrow[dr, shift right] \arrow[dd, Rightarrow, shift right] & \\
& \mathrm{WAP(S)\ has\ LIM} \arrow[ul, shift right, "/" marking] \arrow[dl, shift right] \\
\mathrm{AP(S)\ has\ LIM} \arrow[uu, Rightarrow, shift right, "/" marking] \arrow[ur, shift right, "/" marking] &
\end{tikzcd}$$

The implication "S is left reversible $\Longrightarrow \mathrm{AP}(S)$ has a LIM" for any semitopological semigroup was established in [22]. During the 1984 Richmond, Virginia conference on analysis on semigroups, T. Mitchell [12] gave two examples to show that for discrete semigroups "$\mathrm{AP}(S)$ has LIM" $\not\Rightarrow$ "S is left reversible" (see [25] or [23]). The implication "S is left reversible $\Longrightarrow \mathrm{WAP}(S)$ has LIM" for discrete semigroups was proved by Hsu [26]. Recently, it is shown in [23] that if S_1 is the bicyclic semigroup generated by $\{e, a, b, c\}$ such that e is the unit of S_1 and $ab = e$ and $ac = e$, then $\mathrm{WAP}(S)$ has a LIM, but S_1 is not left reversible. Also if S_2 is the bicyclic semigroup generated by $\{e, a, b, c, d\}$, where e is the unit element and $ac = bd = e$, then $\mathrm{AP}(S_2)$ has a LIM, but $\mathrm{WAP}(S_2)$ does not have a LIM.

The following is proved in [5] (see also [27]).

Theorem 3.14. *Let S be a left reversible discrete semigroup. Then S has the following fixed point property.*

Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as norm nonexpansive weak-weak* continuous mappings of a norm-separable weak*-compact convex subset C of a dual Banach space E into C, then C contains a common fixed point for S.*

It can be shown that the following fixed point property on a discrete semigroup S implies that S is left amenable.

(G) Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as norm nonexpansive weak*-weak* continuous mappings of a weak*-compact convex subset C of a dual Banach space E into C, then C contains a common fixed point for $\mathcal{S}$.

Open Problem 4. Does left amenability of S imply (G)?

Other related results for this section can also be found in [9, 28–38].

4. Normal Structure in Banach Spaces Associated to Locally Compact Groups

A Banach space has *weak-normal structure* if every nontrivial weakly compact convex subset has normal structure. If the Banach space is also a dual space then it has weak*-normal structure if every nontrivial weak* compact convex subset has normal structure. It is clear that a dual Banach space has weak-normal structure whenever it has weak∗-normal structure.

A [dual] Banach space E is said to have the weak-fixed point property (weak-FPP) [(FPP*)] if for every weakly [weak*] compact convex subset C of E and for every nonexpansive $T : C \to C$, T has a fixed point in C. Kirk proved that if E has weak-normal structure then E has property FPP [1]. Subsequently, Lim [39] proved that a dual Banach space has property FPP* whenever it has weak*-normal structure.

A Banach space E is said to have the *Kadec-Klee property* (KK) if whenever (x_n) is a sequence in the unit ball of E that converges weakly to x, and $\operatorname{sep}((x_n)) > 0$, where

$$\operatorname{sep}((x_n)) \equiv \inf\{\|x_n - x_m\| : n \neq m\}, \tag{4.1}$$

then $\|x\| < 1$ (see [40]).

For dual Banach spaces, we have the similar properties replacing weak converges by weak* converges.

A Banach space E is said to have the *uniformly Kadec-Klee property* (UKK) if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever (x_n) is a sequence in the unit ball of E converging weakly to x and $\operatorname{sep}((x_n)) > \varepsilon$ then $\|x\| \leq \delta$. This property was introduced by Huff [40] who showed that property UKK is strictly stronger than property KK. van Dulst and Sims showed that a Banach space with property UKK has property weak FPP [41].

It is natural to define a property similar to UKK by replacing the weak convergence by weak* convergence in UKK and calling it UKK*. However, van Dulst and Sims found that the following definition is more useful.

A dual Banach space E has property UKK* if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever A is a subset of the closed unit ball of E containing a sequence (x_n) with $\operatorname{sep}((x_n)) > \varepsilon$, then there is an x in weak*-closure (A) such that $\|x\| \leq \delta$.

They proved that a dual Banach space with property UKK* has property FPP* [41]. Moreover, they observed that if the dual unit ball is weak* sequentially compact then property UKK*, as defined above, is equivalent to the condition obtained from UKK by replacing weak convergence by weak* convergence.

We now summarize the various properties defined above by

$$\begin{tikzcd}
\text{KK} & \text{UKK} \arrow[l, Rightarrow] \arrow[dd, Rightarrow] & \text{UKK}^* \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{KK}^* \\
 & & \text{weak}^* \text{ n.s.} \arrow[d, Rightarrow] & \\
 & \text{weak-FPP} & \text{FPP}^* &
\end{tikzcd}$$

(where n.s. = normal structure).

Let $X = (\ell_2 \oplus \ell_3 \oplus \cdots \oplus \ell_n \oplus \cdots)_2$. Then, as noted by Huff [40], X is reflexive and has property KK but not UKK.

Let X be a locally compact Hausdorff space, and $C(X)$ the space of bounded continuous complex-valued functions defined on X with the supremum norm. Let $C_0(X)$ be the subspace of $C(X)$ consisting of functions "vanishing at infinity," and $M(X)$ be the space of bounded regular Borel measure on X, with the variation norm. Let $M_d(X)$ be the subspace of $M(X)$ consisting of the discrete measures on X. It is well known that the dual of $C_0(X)$ can be identified with $M(X)$, and that $M_d(X)$ is isometrically isomorphic to $\ell_1(X)$.

Lennard [42] proved the following theorem.

Theorem 4.1. *Let H be a Hilbert space. Then $\mathfrak{C}(H)$, the trace class operators on H, has the property UKK* and has FPP* when regarded as the dual space of $\mathcal{C}(H)$, the C^*-algebra of compact operator on H.*

Theorem 4.2. *Let G be a locally compact group. Then the following statements are equivalent.*

(1) *G is discrete.*

(2) *$M(G)$ is isometrically isomorphic to $\ell_1(G)$.*

(3) *$M(G)$ has property UKK*.*

(4) *$M(G)$ has property KK*.*

(5) *Weak* convergence and weak convergence of sequences agree on the unit sphere of $M(G)$.*

(6) *$M(G)$ has weak* normal structure.*

(7) *$M(G)$ has property FPP*.*

Theorem 4.3. *Let G be a locally compact group. Then the group algebra $L^1(G)$ has the weak fixed point property for left reversible semigroups if and only if G is discrete.*

Theorem 4.4. *Let G be a locally compact group. Let N be a C^*-subalgebra of* WAP(*G*) *containing $C_0(G)$ and the constants. Then the following statements are equivalent.*

(1) *G is finite.*

(2) *N^* has property UKK*.*

(3) *N^* has property KK*.*

(4) *Weak* convergence and weak convergence for sequences agree on the unit sphere of N^*.*

(5) *N^* has weak*-normal structure.*

Theorem 4.5. *Let G be a locally compact group. Then*

(1) *Weak* convergence and weak convergence for sequences agree on the unit sphere of* $\mathrm{LUC}(G)^*$ *if and only if G is discrete.*

(2) $\mathrm{LUC}(G^*)$ *has weak*-normal structure if and only if G is finite.*

Let G be a locally compact group. We define $C^*(G)$, the group C^*-algebra of G, to be the completion of $L_1(G)$ with respect to the norm

$$\|f\|_* = \sup\|\pi_f\|, \tag{4.2}$$

where the supremum is taken over all nondegenerate representations π of $L_1(G)$ as an algebra of bounded operator on a Hilbert space. Let $C(G)$ be the Banach space of bounded continuous complex-valued function on G with the supremum norm. Denote the set of continuous positive definite functions on G by $P(G)$, and the set of continuous functions on G with compact support by $C_{00}(G)$. Define the Fourier-Stieltjes algebra of G, denoted by $B(G)$, to be the linear span of $P(G)$. The Fourier algebra of G, denoted by $A(G)$, is defined to be the closed linear span of $P(G) \cap C_{00}(G)$. Finally, let λ be the left regular representation of G, that is, for each $f \in L_1(G), \lambda(f)$ is the bounded operator in $\mathfrak{B}(L_2(G))$ defined on $L_2(G)$ by $\lambda(f)(h) = f * h$ (the convolution of f and h). Then denote by $\mathrm{VN}(G)$ to be the closure of $\{\lambda(f) : f \in L_1(G)\}$ in the weak operator topology in $\mathfrak{B}(L_2(G))$. It is known that $C^*(G)^* = B(G)$ and $A(G)^* = \mathrm{VN}(G)$. Furthermore, if G is amenable (e.g., when G is compact), then

$$C^*(G) \cong \text{norm closure of } \{\lambda(f) : f \in L_1(G)\} \subseteq \mathrm{VN}(G). \tag{4.3}$$

We refer the reader to [43] for more details on these spaces.

Notice that when G is an abelian locally compact group, then $B(G) \cong M(\widehat{G})$ and $C^*(G) \cong C_0(\widehat{G})$, where $\widehat{G}$ is the dual group of G. It follows from Theorem 4.2 that $B(G)$ was the weak*-normal structure if and only if $\widehat{G}$ is discrete, or equivalently, G is compact.

Theorem 4.6. *If G is compact, then $B(G)$ has weak*-normal structure and hence the FPP*.*

For a Banach space (resp., dual Banach space) E, we say that E has the weak-FPP (weak*-FPP) for left reversible semigroup if whenever S is a left reversible semitopological semigroup and C is a weak (resp., weak*) compact convex subset of E, and $\mathcal{S} = \{T_s : s \in S\}$ is a separately continuous representation of S as nonexpansive mappings from C into C, then there is a common fixed point in C for $\mathcal{S}$.

Theorem 4.7. *If G is a separable compact group, then $B(G)$ has the weak*- FPP for left reversible semigroups.*

Open Problem 5. Can separability be dropped from Theorem 4.7?

A locally compact group G is called an [IN]-group if there is a compact neighbourhood of the identity e in G which is invariant under the inner automorphisms. The class of [IN]-group contains all discrete groups, abelian groups and compact groups. Every [IN]-group is unimodular.

We now investigate the weak fixed point property for a semigroup. A group G is said to an [AU]-group if the von Neumann algebra generated by every continuous unitary representation of G is atomic (i.e., every nonzero projection in the van Neumann algebra majorizes a nonzero minimal projection). It is an [AR]-group if the von Neumann algebra VN(G) is atomic. Since VN(G) is the von Neumann algebra generated by the regular representation, it is clear that every [AU]-group is an [AR]-group. It was shown in [44, Lemma 3.1] that if the predual $\mathfrak{M}_*$ of a von Neumann algebra $\mathfrak{M}$ has the Radon-Nikodym property, then $\mathfrak{M}_*$ has the weak fixed point property. In fact, since the property UKK is hereditary, the proof there actually showed that $\mathfrak{M}_*$ has property UKK and hence has weak normal structure. For the two preduals $A(G)$ and $B(G)$, we know from [45, Theorems 4.1 and 4.2] that the class of groups for which $A(G)$ and $B(G)$ have the Radon-Nikodym property is precisely the [AR]-groups and [AU]-groups, respectively. Thus by Lim's result [16, Theorem 3] we have the following proposition

Proposition 4.8. *Let G be a locally compact group.*

(a) *If G is an [AR]-group, then $A(G)$ has the weak fixed point property for left reversible semigroups.*

(b) *If G is an [AU]-group, then $B(G)$ has the weak fixed point property for left reversible semigroups.*

Proposition 4.9. *Let G be an [IN]-group. Then the following are equivalent.*

(a) *G is compact.*

(b) *$A(G)$ has property UKK.*

(c) *$A(G)$ has weak normal structure.*

(d) *$A(G)$ has the weak fixed point property for left reversible semigroups.*

(e) *$A(G)$ has the weak fixed point property.*

(f) *$A(G)$ has the Radon-Nikodym property.*

(g) *$A(G)$ has the Krein-Milman property.*

A Banach space E is said to have the fixed point property (FPP) if every bounded closed convex subset of E has the fixed point property for nonexpansive mapping. As well known, every uniformly convex space has the FPP.

Theorem 4.10. *Let G be a locally compact group. Then $A(G)$ has the FPP if and only if G is finite.*

Remark 4.11. (a) Theorems 4.1, 4.2, 4.4, 4.5 and 4.6 are proved by Lau and Mah in [46]; Theorems 4.3, 4.7, and Propositions 4.8 and 4.9 are proved by Lau and Mah in [47] and by Lau and Leinert in [48].

(b) Upon the completion of this paper, the author received a preprint from Professor Narcisse Randrianantoanina [49], where he answered an old question in [50] (see also [23]) and showed that for any Hilbert space H (separable or not) the trace class opertors on $H, \mathfrak{T}(H)$ has the weak*-FPP for left reversible semigroups. He is also able to remove the separability condition in our Theorem 4.7, and show that for any locally compact group G:

(i) $A(G)$ has the weak FPP if and only if is an [AR]-group;

(ii) $B(G)$ has the weak-FPP if and only if G is an [AU]-group. In this case, $B(G)$ even has the weak-FPP for left reversible semigroup.

We are grateful to Professor Randrianantoanina for sending us a copy of his work.

(c) An example of an [AU]-group G which is not compact is the Fell group which is the semidirect product of the additive p-adic number field Q_p and the multiplicative compact group of p-adic units for a fixed prime p. So G is solvable and hence amenable. We claim that $B(G)$ cannot have property KK*. Indeed, the Fell group G is separable. Hence (AG) is norm separable (see [29]). So the proof of [51] shows that there is a bounded approximate identity in $A(G)$ consisting of a sequence $\{\phi_n\}$, ϕ_n positive definite with norm 1. The sequence ϕ_n converges to 1 in $B(G)$ in the weak*-topology. Now if $B(G)$ has property KK*, then $\|\phi_n - 1\| \to 0$, and so $1 \in A(G)$. In particular G is compact. See [52] for a more general result.

(d) Theorem 4.10 is proved by Lau and Leinert in [48]. In a preprint of Hernandez Linares and Japon [53] sent to the author just recently, they have shown that if G is compact and separable, then $A(G)$ can be renormed to have the FPP. This generalizes an earlier result of Lin [54] who proves that ℓ_1 can be renormed to have the FPP. Note that if $G = \mathbb{T}$, the circle group, then $A(G)$ is isometric isomorphic to ℓ_1. We are grateful to Professor Japon for providing us with a preprint of their work.

(e) Other related results for this section can also be found in [55].

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