Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 547828, 9 pages
doi:10.1155/2010/547828

Research Article

Convergence of Paths for Perturbed Maximal Monotone Mappings in Hilbert Spaces

Yuan Qing,[1] Xiaolong Qin,[1] Haiyun Zhou,[2] and Shin Min Kang[3]

[1] *Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China*
[2] *Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, China*
[3] *Department of Mathematics, Gyeongsang National University, Jinju 660-701, Republic of Korea*

Correspondence should be addressed to Shin Min Kang, smkang@gnu.ac.kr

Received 16 July 2010; Revised 30 November 2010; Accepted 20 December 2010

Academic Editor: Ljubomir B. Ciric

Let H be a Hilbert space and C a nonempty closed convex subset of H. Let $A : C \to H$ be a maximal monotone mapping and $f : C \to C$ a bounded demicontinuous strong pseudocontraction. Let $\{x_t\}$ be the unique solution to the equation $f(x) = x + tAx$. Then$\{x_t\}$ is bounded if and only if $\{x_t\}$ converges strongly to a zero point of A as $t \to \infty$ which is the unique solution in $A^{-1}(0)$, where $A^{-1}(0)$ denotes the zero set of A, to the following variational inequality $\langle f(p) - p, y - p\rangle \leq 0$, for all $y \in A^{-1}(0)$.

1. Introduction and Preliminaries

Throughout this work, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and A a nonlinear mapping. We use $D(A)$ and $R(A)$ to denote the domain and the range of the mapping A. $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Recall the following well-known definitions.

(1) A mapping $A : C \to H$ is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{1.1}$$

(2) The single-valued mapping $A : C \to H$ is *maximal* if the graph $G(A)$ of A is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping A is *maximal* if and only if for $(x, Ax) \in H\times H$, $\langle x-y, Ax-g\rangle \geq 0$ for every $(y, g) \in G(A)$ implies $g = Ay$.

(3) $A : C \to H$ is said to be *pseudomonotone* if for any sequence $\{x_n\}$ in C which converges weakly to an element x in C with $\limsup_{n\to\infty}\langle Ax_n, x_n - x\rangle \le 0$ we have

$$\liminf_{n\to\infty}\langle Ax_n, x_n - y\rangle \ge \langle Ax, x - y\rangle, \quad \forall y \in C. \tag{1.2}$$

(4) $A : C \to H$ is said to be *bounded* if it carries bounded sets into bounded sets; it is coercive if $\langle Ax, x\rangle / \|x\| \to \infty$ as $\|x\| \to \infty$.

(5) Let X, Y be linear normed spaces. $T : D(T) \subset X \to Y$ is said to be *demicontinuous* if, for any $\{x_n\} \subset D(T)$ we have $Tx_n \rightharpoonup Tx_0$ as $x_n \to x_0$.

(6) Let T be a mapping of a linear normed space X into its dual space X^*. T is said to be *hemicontinuous* if it is continuous from each line segment in X to the weak topology in X^*.

(7) The mapping f with the domain $D(f)$ and the range $R(f)$ in H is said to be *pseudocontractive* if

$$\langle f(x) - f(y), x - y\rangle \le \|x - y\|^2, \quad \forall x, y \in D(f). \tag{1.3}$$

(8) The mapping f with the domain $D(f)$ and the range $R(f)$ in H is said to be *strongly pseudocontractive* if there exists a constant $\beta \in (0,1)$ such that

$$\langle f(x) - f(y), x - y\rangle \le \beta\|x - y\|^2, \quad \forall x, y \in D(f). \tag{1.4}$$

Remark 1.1. For the maximal monotone operator A, we can defined the resolvent of A by $J_t = (I + tA)^{-1}$, $t > 0$. It is well know that $J_t : H \to D(A)$ is nonexpansive.

Remark 1.2. It is well-known that if T is demicontinuous, then T is hemicontinuous, however, the converse, in general, may not be true. In reflexive Banach spaces, for monotone mappings defined on the whole Banach space, demicontinuity is equivalent to hemicontinuity.

To find zeroes of maximal monotone operators is the central and important topics in nonlinear functional analysis. We observe that p is a zero of the monotone mapping A if and only if it is a fixed point of the pseudocontractive mapping $T := I - A$. Consequently, considerable research works, especially, for the past 40 years or more, have been devoted to the existence and convergence of zero points for monotone mappings or fixed points of pseudocontractions, see, for instance, [1–23].

In 1965, Browder [1] proved the existence result of fixed point for demicontinuous pseudocontractions in Hilbert spaces. To be more precise, he proved the following theorem.

Theorem B1. *Let H be a Hilbert space, C a nonempty bounded and closed convex subset of H and $T : C \to C$ a demicontinuous pseduo-contraction. Then T has a fixed point in C.*

In 1968, Browder [4] proved the existence results of zero points for maximal monotone mappings in reflexive Banach spaces. To be more precise, he proved the following theorem.

Theorem B2. *Let X be a reflexive Banach space, $T_1 : D(T_1) \subseteq X \to 2^{X^*}$ a maximal monotone mapping and T_2 a bounded, pseudomonotone and coercive mapping. Then, for any $h \in X^*$, there exists $u \in X$ such that $h \in (T_1 + T_2)u$, or $R(T_1 + T_2)$ is all of X^*.*

For the existence of continuous paths for continuous pseudocontractions in Banach spaces, Morales and Jung [15] proved the following theorem.

Theorem MJ. *Let E be a Banach space. Suppose that C is a nonempty closed convex subset of E and $T : C \to E$ is a continuous pseudocontraction satisfying the weakly inward condition. Then for each $z \in C$, there exists a unique continuous path $t \mapsto y_t \in C$, $t \in [0,1)$, which satisfies the following equation $y_t = (1-t)z + tTy_t$.*

In 2002, Lan and Wu [14] partially improved the result of Morales and Jung [15] from continuous pseudocontractions to demicontinuous pseudocontractions in the framework of Hilbert spaces. To be more precise, they proved the following theorem.

Theorem LW. *Let K be a bounded closed convex set in H. Assume that $T : K \to H$ is a demicontinuous weakly inward pseudocontractive map. Then T has a fixed point in K. Moreover; for every $x_0 \in K$, $\{x_t\}$ defined by $x_t = (1-t)Tx_t + tx_0$ converges to a fixed point of T.*

In this work, motivated by Browder [3], Lan and Wu [14], Morales and Jung [15], Song and Chen [19], and Zhou [22, 23], we consider the existence of convergence of paths for maximal monotone mappings in the framework of real Hilbert spaces.

2. Main Results

Lemma 2.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ a demicontinuous monotone mapping. Then T is pseudomonotone.*

Proof. For any sequence $\{x_n\} \subset C$ which converges weakly to an element x in C such that

$$\limsup_{n\to\infty}\langle Tx_n, x_n - x\rangle \leq 0, \tag{2.1}$$

we see from the monotonicity of T that

$$\langle Tx, x_n - x\rangle \leq \langle Tx_n, x_n - x\rangle. \tag{2.2}$$

Combining (2.1) with (2.2), we obtain that

$$\limsup_{n\to\infty}\langle Tx, x_n - x\rangle = 0. \tag{2.3}$$

By taking $[z, g] \in \mathrm{Graph}(T)$, we arrive at

$$\langle Tx_n, x_n - z\rangle = \langle Tx_n, x_n - x\rangle + \langle Tx_n, x - z\rangle, \tag{2.4}$$

which yields that

$$\liminf_{n\to\infty}\langle Tx_n, x_n - z\rangle = \liminf_{n\to\infty}\langle Tx_n, x - z\rangle. \tag{2.5}$$

Noticing that

$$\langle g, x_n - z\rangle \le \langle Tx_n, x_n - z\rangle, \tag{2.6}$$

we have

$$\langle g, x - z\rangle \le \liminf_{n\to\infty}\langle Tx_n, x - z\rangle. \tag{2.7}$$

Let $z_t = (1-t)x + ty$, for all $y \in C$ and $t \in (0,1)$. By taking $z_t = z$ and $g_t = g$ in (2.7), we see that

$$\langle g_t, x - y\rangle \le \liminf_{n\to\infty}\langle Tx_n, x - y\rangle. \tag{2.8}$$

Noting that $z_t \to x$, $t \to 0$, $g_t = Tz_t$, and $T : C \to H$ is demicontinuous, we have $g_t = Tz_t \rightharpoonup Tx$ as $t \to 0$, and hence

$$\liminf_{n\to\infty}\langle Tx_n, x_n - y\rangle = \liminf_{n\to\infty}\langle Tx_n, x - y\rangle \ge \langle Tx, x - y\rangle. \tag{2.9}$$

This completes the proof. □

Lemma 2.2. *Let C be a nonempty closed convex subset of a Hilbert space H, $A : C \to H$ a maximal monotone mapping, and $T : C \to H$ a bounded, demicontinuous, and strongly monotone mapping. Then $A + T$ has a unique zero in C.*

Proof. By using Lemma 2.1 and Theorem B2, we can obtain the desired conclusion easily. □

Lemma 2.3. *Let C be a nonempty closed convex subset of a Hilbert space H, $A : C \to H$ a maximal monotone mapping, and $f : C \to H$ a bounded, demicontinuous strong pseudocontraction with the coefficient $\beta \in (0,1)$. For $t > 0$, consider the equation*

$$0 = Tx + tAx, \tag{2.10}$$

where $T = I - f$. Then, One has the following.

(i) *Equation* (2.10) *has a unique solution $x_t \in C$ for every $t > 0$.*

(ii) *If $\{x_t\}$ is bounded, then $\|Ax_t\| \to 0$ as $t \to \infty$.*

(iii) *If $A^{-1}(0) \neq \emptyset$, then $\{x_t\}$ is bounded and satisfies*

$$\langle x_t - f(x_t), x_t - p\rangle \le 0, \quad \forall p \in A^{-1}(0), \tag{2.11}$$

where $A^{-1}(0)$ denotes the zero set of A.

Proof. (i) From Lemma 2.2, one can obtain the desired conclusion easily.

(ii) We use $x_t \in C$ to denote the unique solution of (2.10). That is, $0 = Tx_t + tAx_t$. It follows that $0 = (I - f)x_t + tAx_t$. Notice that

$$Ax_t = \frac{f(x_t) - x_t}{t}. \tag{2.12}$$

From the boundedness of f and $\{x_t\}$, one has $\lim_{t\to\infty}\|Ax_t\| = 0$.

(iii) For $p \in A^{-1}(0)$, one obtains that

$$\begin{aligned} \|x_t - p\|^2 &= \langle x_t - p, x_t - p\rangle \\ &= \langle f(x_t) - p, x_t - p\rangle - \langle Ax_t, x_t - p\rangle \\ &\le \langle f(x_t) - f(p), x_t - p\rangle + \langle f(p) - p, x_t - p\rangle - \langle Ax_t, x_t - p\rangle \\ &\le \beta\|x_t - p\|^2 + \langle f(p) - p, x_t - p\rangle. \end{aligned} \tag{2.13}$$

It follows that

$$\|x_t - p\|^2 \le \frac{1}{1-\beta}\langle f(p) - p, x_t - p\rangle. \tag{2.14}$$

That is, $\|x_t - p\| \le (1/(1-\beta))\|f(p) - p\|$, for all $t > 0$. This shows that $\{x_t\}$ is bounded. Noticing that $x_t - f(x_t) = -tAx_t$, one arrives at

$$\langle x_t - f(x_t), x_t - p\rangle = -t\langle Ax_t, x_t - p\rangle \le 0, \quad \forall t > 0. \tag{2.15}$$

This completes the proof. □

Lemma 2.4. *Let C be a nonempty closed convex subset of a Hilbert space H and A a maximal monotone mapping. Then $C \subsetneq (I + A)C$. If one defines $g : C \to C$ by $g(x) = (I + A)^{-1}x$, for all $x \in C$, then $g : C \to C$ is a nonexpansive mapping with $F(g) = A^{-1}(0)$ and $\|x - g(x)\| \le \|Ax\|$, where $F(g)$ denotes the set of fixed points of g.*

Proof. Noticing that A is maximal monotone, one has $R(I + A) = H$. It follows that $C \subsetneq (I + A)C$. For any $x, y \in C$, one sees that

$$\|g(x) - g(y)\| = \left\|(I + A)^{-1}x - (I + A)^{-1}y\right\| \le \|x - y\|, \tag{2.16}$$

which yields that g is nonexpansive mapping. Notice that

$$x = g(x) \Longleftrightarrow (I + A)x = x \Longleftrightarrow Ax = 0. \tag{2.17}$$

That is, $F(g) = A^{-1}(0)$. On the other hand, for any $x \in C$, we have

$$\begin{aligned}\|x - g(x)\| &= \left\|gg^{-1}(x) - g(x)\right\| \\ &\leq \left\|g^{-1}(x) - x\right\| \\ &= \|(I + A)x - x\| \\ &= \|Ax\|.\end{aligned} \tag{2.18}$$

This completes the proof. □

Set $S = (0, 1)$. Let $B(S)$ denote the Banach space of all bounded real value functions on S with the supremum norm, X a subspace of $B(S)$, and μ an element in X^*, where X^* denotes the dual space of X. Denote by $\mu(f)$ the value of μ at $f \in X$. If $e(s) = 1$, for all $x \in S$, sometimes $\mu(e)$ will be denoted by $\mu(1)$. When X contains constants, a linear functional μ on X is called a mean on X if $\mu(1) = \|\mu\| = 1$. We also know that if X contains constants, then the following are equivalent.

(1) $\mu(1) = \|\mu\| = 1$.

(2) $\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$, for all $f \in X$.

To prove our main results, we also need the following lemma.

Lemma 2.5 (see [20, Lemma 4.5.4]). *Let C be a nonempty and closed convex subset of a Banach space E. Suppose that norm of E is uniformly Gâteaux differentiable. Let $\{x_t\}$ be a bounded set in X and $z \in C$. Let μ_t be a mean on X. Then*

$$\mu_t \|x_t - z\|^2 = \min_{y \in C} \|x_t - y\| \tag{2.19}$$

if and only if

$$\mu_t \langle y - z, x_t - z \rangle \leq 0, \quad y \in C. \tag{2.20}$$

Now, we are in a position to prove the main results of this work.

Theorem 2.6. *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let $A : C \to H$ be a maximal monotone mapping and $f : C \to C$ a bounded demicontinuous strong pseudocontraction. Let $\{x_t\}$ be as in Lemma 2.3. Then $\{x_t\}$ is bounded if and only if $\{x_t\}$ converges strongly to a zero point p of A as $t \to \infty$ which is the unique solution in $A^{-1}(0)$ to the following variational inequality:*

$$\langle f(p) - p, y - p \rangle \leq 0, \quad \forall y \in A^{-1}(0). \tag{2.21}$$

Proof. The part ($\Leftarrow$) is obvious and we only prove ($\Rightarrow$). From Lemma 2.3, one sees that $\|Ax_t\| \to 0$ as $t \to \infty$. It follows from Lemma 2.4 that $\|x_t - g(x_t)\| \to 0$ as $t \to \infty$.

Define $h(x) = \mu_t\|x_t - x\|$, $x \in C$, where μ_t is a Banach limit. Then h is a convex and continuous function with $h(x) \to \infty$ as $\|x\| \to \infty$. Put

$$K = \left\{ x \in C : h(x) = \min_{y \in C} h(y) \right\}. \tag{2.22}$$

From the convexity and continuity of h, we can get the convexity and continuity of the set K. Since h is continuous and H is a Hilbert space, we see that h attains its infimum over K; see [20] for more details. Then K is nonempty bounded and closed convex subset of C. Indeed, K contains one point only. Set $g(x) = (I + A)^{-1}x$, where $g : K \to K$. Notice that g is nonexpansive. Since every nonempty bounded and closed convex subset has the fixed point property for nonexpansive self-mapping in the framework of Hilbert spaces, then g has a fixed point p in K, that is, $g(p) = p$. It follows from Lemma 2.4 that $A(p) = 0$. On the other hand, one has $\mu_t\|x_t - p\| = \min_{y \in C} h(y)$. In view of Lemma 2.5, we obtain that

$$\mu_t\langle y - p, x_t - p\rangle \le 0, \quad \forall y \in C. \tag{2.23}$$

By taking $y = f(p)$ in (2.23), we arrive at

$$\mu_t\langle f(p) - p, x_t - p\rangle \le 0, \quad \forall y \in C. \tag{2.24}$$

Combining (2.14) with (2.23) yields that $\mu_t\|x_t - p\|^2 = 0$. Hence, there exists a subnet $\{x_{t_\alpha}\}$ of $\{x_t\}$ such that $\{x_{t_\alpha}\} \to p$. From (iii) of Lemma 2.3, one has

$$\langle x_{t_\alpha} - f(x_{t_\alpha}), x_{t_\alpha} - y\rangle \le 0, \quad \forall y \in A^{-1}(0). \tag{2.25}$$

Taking limit in (2.25), one gets that

$$\langle p - f(p), p - y\rangle \le 0, \quad \forall y \in A^{-1}(0). \tag{2.26}$$

If there exists another subset $\{x_{t_\beta}\}$ of $\{x_t\}$ such that $\{x_{t_\beta}\} \to q$, then q is also a zero of A. It follows from (2.26) that

$$\langle p - f(p), p - q\rangle \le 0. \tag{2.27}$$

By using (iii) of Lemma 2.3 again, one arrives at

$$\left\langle x_{t_\beta} - f\left(x_{t_\beta}\right), x_{t_\beta} - p \right\rangle \le 0. \tag{2.28}$$

Taking limit in (2.28), we obtain that

$$\langle q - f(q), q - p\rangle \le 0. \tag{2.29}$$

Adding (2.27) and (2.29), we have

$$\langle p - q + f(q) - f(p), p - q \rangle \leq 0, \tag{2.30}$$

which yields that

$$\|p - q\|^2 \leq \langle f(p) - f(q), p - q \rangle \leq \beta \|p - q\|^2. \tag{2.31}$$

It follows that $p = q$. That is, $\{x_t\}$ converges strongly to $p \in A^{-1}(0)$, which is the unique solution to the following variational inequality:

$$\langle f(p) - p, y - p \rangle \leq 0, \quad \forall y \in A^{-1}(0). \tag{2.32}$$

□

Remark 2.7. From Theorem 2.6, we can obtain the following interesting fixed point theorem. The composition of bounded, demicontinuous, and strong pseudocontractions with the metric projection has a unique fixed point. That is, $p = Pf(p)$.

Acknowledgment

The third author was supported by the National Natural Science Foundation of China (Grant no. 10771050).

References

[1] F. E. Browder, "Fixed-point theorems for noncompact mappings in Hilbert space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1272–1276, 1965.

[2] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82–90, 1967.

[3] F. E. Browder, "Nonlinear operators and nonlinear equations of evolution in Banach spaces," in *Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968)*, pp. 1–308, American Mathematical Society, Providence, RI, USA, 1976.

[4] F. E. Browder, "Nonlinear maximal monotone operators in Banach space," *Mathematische Annalen*, vol. 175, pp. 89–113, 1968.

[5] F. E. Browder, "Nonlinear monotone and accretive operators in Banach spaces," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 61, pp. 388–393, 1968.

[6] R. E. Bruck, Jr., "A strongly convergent iterative solution of $0 \in Ux$ for a maximal monotone operator U in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 48, pp. 114–126, 1974.

[7] R. Chen, P.-K. Lin, and Y. Song, "An approximation method for strictly pseudocontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 11, pp. 2527–2535, 2006.

[8] C. E. Chidume and C. Moore, "Fixed point iteration for pseudocontractive maps," *Proceedings of the American Mathematical Society*, vol. 127, no. 4, pp. 1163–1170, 1999.

[9] C. E. Chidume and M. O. Osilike, "Nonlinear accretive and pseudo-contractive operator equations in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 31, no. 7, pp. 779–789, 1998.

[10] C. E. Chidume and H. Zegeye, "Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps," *Proceedings of the American Mathematical Society*, vol. 132, no. 3, pp. 831–840, 2004.

[11] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[12] K. Deimling, "Zeros of accretive operators," *Manuscripta Mathematica*, vol. 13, pp. 365–374, 1974.

[13] T. Kato, "Demicontinuity, hemicontinuity and monotonicity," *Bulletin of the American Mathematical Society*, vol. 70, pp. 548–550, 1964.

[14] K. Q. Lan and J. H. Wu, "Convergence of approximants for demicontinuous pseudo-contractive maps in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 49, no. 6, pp. 737–746, 2002.

[15] C. H. Morales and J. S. Jung, "Convergence of paths for pseudocontractive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3411–3419, 2000.

[16] C. H. Morales and C. E. Chidume, "Convergence of the steepest descent method for accretive operators," *Proceedings of the American Mathematical Society*, vol. 127, no. 12, pp. 3677–3683, 1999.

[17] X. Qin and Y. Su, "Approximation of a zero point of accretive operator in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 415–424, 2007.

[18] X. Qin, S. M. Kang, and Y. J. Cho, "Approximating zeros of monotone operators by proximal point algorithms," *Journal of Global Optimization*, vol. 46, no. 1, pp. 75–87, 2010.

[19] Y. Song and R. Chen, "Convergence theorems of iterative algorithms for continuous pseudocontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 2, pp. 486–497, 2007.

[20] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, Japan, 2000.

[21] H. Zhou, "Iterative solutions of nonlinear equations involving strongly accretive operators without the Lipschitz assumption," *Journal of Mathematical Analysis and Applications*, vol. 213, no. 1, pp. 296–307, 1997.

[22] H. Zhou, "Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 2977–2983, 2008.

[23] H. Zhou, "Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 546–556, 2008.