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Research Article

Halpern's Iteration in CAT(0) Spaces

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Motivated by Halpern's result, we prove strong convergence theorem of an iterative sequence in CAT(0) spaces. We apply our result to find a common fixed point of a family of nonexpansive mappings. A convergence theorem for nonself mappings is also discussed.

1. Introduction

Let (X, d) be a metric space and $x, y \in X$ with $l = d(x, y)$. A *geodesic path* from x to y is an isometry $c : [0, l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a *geodesic segment*. A metric space X is a *(uniquely) geodesic space* if every two points of X are joined by only one geodesic segment. A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\triangle(x_1, x_2, x_3)$ is the triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean space $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j})$ for all $i, j = 1, 2, 3$.

A geodesic space X is a *CAT(0) space* if for each geodesic triangle $\triangle := \triangle(x_1, x_2, x_3)$ in X and its comparison triangle $\overline{\triangle} := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$, the *CAT(0) inequality*

$$d(x, y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y}) \tag{1.1}$$

is satisfied by all $x, y \in \triangle$ and $\overline{x}, \overline{y} \in \overline{\triangle}$. The meaning of the CAT(0) inequality is that a geodesic triangle in X is at least thin as its comparison triangle in the Euclidean plane. A thorough discussion of these spaces and their important role in various branches of mathematics are given in [1, 2]. The complex Hilbert ball with the hyperbolic metric is an example of a CAT(0) space (see [3]).

The concept of Δ-convergence introduced by Lim in 1976 was shown by Kirk and Panyanak [4] in CAT(0) spaces to be very similar to the weak convergence in Banach space setting. Several convergence theorems for finding a fixed point of a nonexpansive mapping have been established with respect to this type of convergence (e.g., see [5–7]). The purpose of this paper is to prove strong convergence of iterative schemes introduced by Halpern [8] in CAT(0) spaces. Our results are proved under weaker assumptions as were the case in previous papers and we do not use Δ-convergence. We apply our result to find a common fixed point of a countable family of nonexpansive mappings. A convergence theorem for nonself mappings is also discussed.

In this paper, we write $(1-t)x \oplus ty$ for the the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y), \qquad d(z,y) = (1-t)d(x,y). \tag{1.2}$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$. A subset C of a CAT(0) space is *convex* if $[x,y] \subset C$ for all $x,y \in C$. For elementary facts about CAT(0) spaces, we refer the readers to [1] (or, briefly in [5]).

The following lemma plays an important role in our paper.

Lemma 1.1. *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \tag{1.3}$$

is satisfied by all $x,y,z \in X$ and all $t \in [0,1]$. In particular, if x,y,z are points in a CAT(0) space and $t \in [0,1]$, then

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z). \tag{1.4}$$

Recall that a continuous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a *Banach limit* if $\|\mu\| = \mu(1,1,\ldots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

Lemma 1.2 (see [9, Proposition 2]). *Let $(a_1, a_2, \ldots) \in l^\infty$ be such that $\mu_n(a_n) \le 0$ for all Banach limits μ and $\limsup_n (a_{n+1} - a_n) \le 0$. Then $\limsup_n a_n \le 0$.*

Lemma 1.3 (see [10, Lemma 2.3]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, and $\{t_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} t_n \le 0$. Suppose that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n t_n + u_n \quad \forall n \in \mathbb{N}. \tag{1.5}$$

Then $\lim_{n\to\infty} s_n = 0$.

2. Halpern's Iteration for a Single Mapping

Lemma 2.1. *Let C be a closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0,1)$, the mapping $S_t : C \to C$ defined by*

$$S_t x = tu \oplus (1-t)Tx \quad \text{for } x \in C \tag{2.1}$$

has a unique fixed point $x_t \in C$, that is,

$$x_t = S_t x_t = tu \oplus (1-t)Tx_t. \tag{2.2}$$

Proof. For $x, y \in C$, we consider the triangle $\triangle(u, Tx, Ty)$ and its comparison triangle and we have the following:

$$\begin{aligned} d\big(tu \oplus (1-t)Tx, tu \oplus (1-t)Ty\big) &\le d_{\mathbb{R}^2}\Big(\overline{tu \oplus (1-t)Tx}, \overline{tu \oplus (1-t)Ty}\Big) \\ &= (1-t)d_{\mathbb{R}^2}\Big(\overline{Tx}, \overline{Ty}\Big) \\ &= (1-t)d\big(Tx, Ty\big) \\ &\le (1-t)d\big(x, y\big). \end{aligned} \tag{2.3}$$

This implies that S_t is a contraction mapping and hence the conclusion follows. □

The following result is proved by Kirk in [11, Theorem 26] under the boundedness assumption on C. We present here a new proof which is modified from Kirk's proof.

Lemma 2.2. *Let C, T be as the preceding lemma. Then $F(T) \neq \emptyset$ if and only if $\{x_t\}$ given by the formula* (2.2) *remains bounded as $t \to 0$. In this case, the following statements hold:*

(1) *$\{x_t\}$ converges to the unique fixed point z_0 of T which is nearest u;*

(2) *$d^2(u, z_0) \le \mu_n d^2(u, x_n)$ for all Banach limits μ and all bounded sequences $\{x_n\}$ with $x_n - Tx_n \to 0$.*

Proof. If $F(T) \neq \emptyset$, then it is clear that $\{x_t\}$ is bounded. Conversely, suppose that $\{x_t\}$ is bounded. Let $\{t_n\}$ be any sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and define $g : C \to \mathbb{R}$ by

$$g(z) = \limsup_{n\to\infty} d^2(x_{t_n}, z) \tag{2.4}$$

for all $z \in C$. By the boundedness of $\{x_{t_n}\}$, we have $\delta := \inf\{g(z) : z \in C\} < \infty$. We choose a sequence $\{z_m\}$ in C such that $\lim_{m\to\infty} g(z_m) = \delta$. It follows from Lemma 1.1 that

$$d^2\left(x_{t_n}, \frac{1}{2}z_m \oplus \frac{1}{2}z_k\right) \le \frac{1}{2}d^2(x_{t_n}, z_m) + \frac{1}{2}d^2(x_{t_n}, z_k) - \frac{1}{4}d^2(z_m, z_k). \tag{2.5}$$

Then, by the convexity of C,

$$\delta \leq \limsup_{n\to\infty} d^2\left(x_{t_n}, \frac{1}{2}z_m \oplus \frac{1}{2}z_k\right) \leq \frac{1}{2}g(z_m) + \frac{1}{2}g(z_k) - \frac{1}{4}d^2(z_m, z_k). \tag{2.6}$$

This implies that $\{z_m\}$ is a Cauchy sequence in C and hence it converges to a point $z_0 \in C$. Suppose that $\widehat{z}$ is a point in C satisfying $g(\widehat{z}) = \delta$. It follows then that

$$\delta \leq \limsup_{n\to\infty} d^2\left(x_{t_n}, \frac{1}{2}z_0 \oplus \frac{1}{2}\widehat{z}\right) \leq \frac{1}{2}g(z_0) + \frac{1}{2}g(\widehat{z}) - \frac{1}{4}d^2(z_0, \widehat{z}), \tag{2.7}$$

and hence $\widehat{z} = z_0$. Moreover, z_0 is a fixed point of T. To see this, we consider

$$d(x_{t_n}, Tx_{t_n}) = \frac{t_n}{1-t_n} d(u, x_{t_n}) \longrightarrow 0, \tag{2.8}$$

and

$$\begin{aligned}
\limsup_{n\to\infty} d^2(x_{t_n}, Tz_0) &\leq \limsup_{n\to\infty} \left(d(x_{t_n}, Tx_{t_n}) + d(Tx_{t_n}, Tz_0)\right)^2 \\
&\leq \limsup_{n\to\infty} \left(d(x_{t_n}, Tx_{t_n}) + d(x_{t_n}, z_0)\right)^2 \\
&= \limsup_{n\to\infty} d^2(x_{t_n}, z_0) = \delta.
\end{aligned} \tag{2.9}$$

This implies that $z_0 = Tz_0$ and hence $F(T) \neq \varnothing$.

(1) is proved in [12, Theorem 26]. In fact, it is shown that z_0 is the nearest point of $F(T)$ to u. Finally, we prove (2). Suppose that $\{z_{t_m}\}$ is a sequence given by the formula (2.2), where $\{t_m\}$ is a sequence in $(0,1)$ such that $\lim_{m\to\infty} t_m = 0$. We also assume that $z_0 = \lim_{m\to\infty} z_{t_m}$ is the nearest point of $F(T)$ to u. By the first inequality in Lemma 1.1, we have

$$\begin{aligned}
d^2(x_n, z_{t_m}) &= d^2(x_n, t_m u \oplus (1-t_m)Tz_{t_m}) \\
&\leq t_m d^2(x_n, u) + (1-t_m)d^2(x_n, Tz_{t_m}) - t_m(1-t_m)d^2(u, Tz_{t_m}) \\
&\leq t_m d^2(x_n, u) + (1-t_m)\left(d(x_n, Tx_n) + d(Tx_n, Tz_{t_m})\right)^2 - t_m(1-t_m)d^2(u, Tz_{t_m}) \\
&\leq t_m d^2(x_n, u) + (1-t_m)\left(d(x_n, Tx_n) + d(x_n, z_{t_m})\right)^2 - t_m(1-t_m)d^2(u, Tz_{t_m}).
\end{aligned} \tag{2.10}$$

Let μ be a Banach limit. Then

$$\mu_n d^2(x_n, z_{t_m}) \leq t_m \mu_n d^2(x_n, u) + (1-t_m)\mu_n d^2(x_n, z_{t_m}) - t_m(1-t_m)d^2(u, Tz_{t_m}). \tag{2.11}$$

This implies that

$$\mu_n d^2(x_n, z_{t_m}) \leq \mu_n d^2(x_n, u) - (1-t_m)d^2(u, Tz_{t_m}). \tag{2.12}$$

Letting $m \to \infty$ gives

$$\mu_n d^2(x_n, z) \le \mu_n d^2(x_n, u) - d^2(u, z). \tag{2.13}$$

In particular,

$$d^2(u, z) \le \mu_n d^2(x_n, u) \quad \text{for all Banach limits } \mu. \tag{2.14}$$

□

Inspired by the results of Wittmann [13] and of Shioji and Takahashi [9], we use the iterative scheme introduced by Halpern to obtain a strong convergence theorem for a nonexpansive mapping in CAT(0) space setting. A part of the following theorem is proved in [14].

Theorem 2.3. *Let C be a closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set $F(T)$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is iteratively generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T x_n \quad \forall n \ge 1, \tag{2.15}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

- (C1) $\lim_{n\to\infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ converges to $z \in F(T)$ which is the nearest point of $F(T)$ to u.

Proof. We first show that the sequence $\{x_n\}$ is bounded. Let $p \in F(T)$. Then

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n u \oplus (1 - \alpha_n) T x_n, p) \\ &\le \alpha_n d(u, p) + (1 - \alpha_n) d(T x_n, p) \\ &\le \alpha_n d(u, p) + (1 - \alpha_n) d(x_n, p) \\ &\le \max\{d(u, p), d(x_n, p)\}. \end{aligned} \tag{2.16}$$

By induction, we have

$$d(x_{n+1}, p) \le \max\{d(u, p), d(x_1, p)\} \tag{2.17}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so is the sequence $\{T x_n\}$.

Next, we show that $d(x_{n+1}, x_n) \to 0$. To see this, we consider the following:

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n u \oplus (1-\alpha_n)Tx_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1})Tx_{n-1}) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n)Tx_n, \alpha_n u \oplus (1-\alpha_n)Tx_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1-\alpha_n)Tx_{n-1}, \alpha_{n-1} u \oplus (1-\alpha_{n-1})Tx_{n-1}) \\ &\leq (1-\alpha_n)d(Tx_n, Tx_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, Tx_{n-1}) \\ &\leq (1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, Tx_{n-1}). \end{aligned} \tag{2.18}$$

By the conditions (C2) and (C3), we have

$$d(x_{n+1}, x_n) \longrightarrow 0. \tag{2.19}$$

Consequently, by the condition (C1),

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x_n, x_{n+1}) + d(\alpha_n u \oplus (1-\alpha_n)Tx_n, Tx_n) \\ &= d(x_n, x_{n+1}) + \alpha_n d(u, Tx_n) \longrightarrow 0. \end{aligned} \tag{2.20}$$

From Lemma 2.2, let $z = \lim_{t\to 0} x_t$ where x_t is given by the formula (2.2). Then z is the nearest point of $F(T)$ to u. We next consider the following:

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\alpha_n u \oplus (1-\alpha_n)Tx_n, z) \\ &\leq \alpha_n d^2(u, z) + (1-\alpha_n)d^2(Tx_n, z) - \alpha_n(1-\alpha_n)d^2(u, Tx_n) \\ &\leq (1-\alpha_n)d^2(x_n, z) + \alpha_n\Big(d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\Big). \end{aligned} \tag{2.21}$$

By Lemma 2.2, we have $\mu_n(d^2(u, z) - d^2(u, x_n)) \leq 0$ for all Banach limits μ. Moreover, since $x_{n+1} - x_n \to 0$,

$$\limsup_{n\to\infty}\Big(d^2(u, z) - d^2(u, x_n)\Big) - \Big(d^2(u, z) - d^2(u, x_{n+1})\Big) = 0. \tag{2.22}$$

It follows from $x_n - Tx_n \to 0$ and Lemma 1.2 that

$$\limsup_{n\to\infty}\Big(d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\Big) = \limsup_{n\to\infty}\Big(d^2(u, z) - d^2(u, x_n)\Big) \leq 0. \tag{2.23}$$

Hence the conclusion follows by Lemma 1.3. □

3. Halpern's Iteration for a Family of Mappings

3.1. Finitely Many Mappings

We use the "cyclic method" [15] and Bauschke's condition [16] to obtain the following strong convergence theorem for a finite family of nonexpansive mappings.

Theorem 3.1. *Let X be a complete CAT(0) space and C a closed convex subset of X. Let $T_1, T_2, \ldots, T_N : C \to C$ be nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \varnothing$ and let $u, x_1 \in C$ be arbitrarily chosen. Define an iterative sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T_{n \bmod N} x_n \quad \forall n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+N}| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+N}) = 1$.

Suppose, in addition, that

$$\bigcap_{i=1}^{N} F(T_i) = F(T_N \circ T_{N-1} \circ \cdots \circ T_1). \tag{3.2}$$

Then $\{x_n\}$ converges to $z \in \bigcap_{i=1}^{N} F(T_i)$ which is nearest u.

Here the $\bmod N$ function takes values in $\{1, 2, \ldots, N\}$.

Proof. By [16, Theorem 2], we have

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1 \circ T_N \circ T_{N-1} \circ \cdots \circ T_2) = \cdots = F(T_{N-1} \circ T_N \circ T_1 \circ \cdots \circ T_{N-2}). \tag{3.3}$$

The proof line now follows from the proofs of Theorem 2.3 and [15, Theorem 3.1]. □

3.2. Countable Mappings

The following concept is introduced by Aoyama et al. [10]. Let X be a complete CAT(0) space and C a subset of X. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of mappings from C into itself. We say that a family $\{T_n\}$ satisfies *AKTT-condition* if

$$\sum_{n=1}^{\infty} \sup\{d(T_{n+1}z, T_n z) : z \in B\} < \infty \tag{3.4}$$

for each bounded subset of B of C.

If C is a closed subset and $\{T_n\}$ satisfies AKTT-condition, then we can define $T : C \to C$ such that

$$Tx = \lim_{n\to\infty} T_n x \quad (x \in C). \tag{3.5}$$

In this case, we also say that $(\{T_n\}, T)$ satisfies AKTT-condition.

Theorem 3.2. *Let X be a complete CAT(0) space and C a closed convex subset of X. Let $\{T_n\} : C \to C$ be a countable family of nonexpansive mappings with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) T_n x_n \quad \forall n \geq 1, \tag{3.6}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Suppose, in addition, that

(M1) $(\{T_n\}, T)$ *satisfies AKTT-condition;*

(M2) $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Then $\{x_n\}$ converges to $z \in \bigcap_{n=1}^{\infty} F(T_n)$ which is nearest u.

Proof. Since the proof of this theorem is very similar to that of Theorem 2.3, we present here only the sketch proof. First, we notice that both sequences $\{x_n\}$ and $\{T_n x_n\}$ are bounded and

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(\alpha_n u \oplus (1-\alpha_n) T_n x_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1}) T_{n-1} x_{n-1}) \\
&\leq d(\alpha_n u \oplus (1-\alpha_n) T_n x_n, \alpha_n u \oplus (1-\alpha_n) T_n x_{n-1}) \\
&\quad + d(\alpha_n u \oplus (1-\alpha_n) T_n x_{n-1}, \alpha_n u \oplus (1-\alpha_n) T_{n-1} x_{n-1}) \\
&\quad + d(\alpha_n u \oplus (1-\alpha_n) T_{n-1} x_{n-1}, \alpha_{n-1} u \oplus (1-\alpha_{n-1}) T_{n-1} x_{n-1}) \\
&\leq (1-\alpha_n) d(T_n x_n, T_n x_{n-1}) + (1-\alpha_n) d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} x_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} x_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} x_{n-1}) \\
&\quad + \sup\{ d(T_n y, T_{n-1} y) : y \in \{x_n\} \}.
\end{aligned} \tag{3.7}$$

By conditions (C2), (C3), AKTT-condition, and Lemma 1.3, we have

$$d(x_{n+1}, x_n) \longrightarrow 0. \tag{3.8}$$

Consequently, $d(x_n, T_n x_n) \to 0$ and hence

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, T_n x_n) + d(T_n x_n, Tx_n) \\ &\le d(x_n, T_n x_n) + \sup\{d(T_n z, Tz) : z \in \{x_n\}\} \\ &\le d(x_n, T_n x_n) + \sum_{k=n}^{\infty} \sup\{d(T_k z, T_{k+1} z) : z \in \{x_n\}\} \longrightarrow 0. \end{aligned} \tag{3.9}$$

Let $z \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ be the nearest point of $F(T)$ to u. As in the proof of Theorem 2.3, we have $d^2(u, z) \le \mu_n d^2(u, x_n)$ for all Banach limits μ and $\limsup_{n\to\infty}(d^2(u, z) - d^2(u, x_n)) - (d^2(u, z) - d^2(u, x_{n+1})) = 0$. We observe that

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\alpha_n u \oplus (1-\alpha_n) T_n x_n, z) \\ &\le \alpha_n d^2(u, z) + (1-\alpha_n) d^2(T_n x_n, z) - \alpha_n (1-\alpha_n) d^2(u, T_n x_n) \\ &\le (1-\alpha_n) d^2(x_n, z) + \alpha_n \Big(d^2(u, z) - (1-\alpha_n) d^2(u, T_n x_n) \Big), \end{aligned} \tag{3.10}$$

and this implies that

$$\limsup_{n\to\infty} \Big(d^2(u, z) - (1-\alpha_n) d^2(u, T_n x_n) \Big) = \limsup_{n\to\infty} \Big(d^2(u, z) - d^2(u, x_n) \Big) \le 0. \tag{3.11}$$

Therefore, $\lim_{n\to\infty} d^2(x_n, z) = 0$ and hence $\{x_n\}$ converges to z. □

We next show how to generate a family of mappings from a given family of mappings to satisfy conditions (M1) and (M2) of the preceding theorem. The following is an analogue of Bruck's result [17] in CAT(0) space setting. The idea using here is from [10].

Theorem 3.3. *Let X be a complete CAT(0) space and C a closed convex subset of X. Suppose that $\{T_n\} : C \to X$ is a countable family of nonexpansive mappings with $\bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Then there exist a family of nonexpansive mappings $\{S_n\} : C \to X$ and a nonexpansive mapping $S : C \to X$ such that*

(M1) *$(\{S_n\}, S)$ satisfies AKTT-condition;*

(M2) *$F(S) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 3.4. *Let X and C be as above. Suppose that $S, T : C \to X$ are nonexpansive mappings and $F(S) \cap F(T) \neq \varnothing$. Then, for any $0 < t < 1$, the mapping $U := (1-t)S \oplus tT : C \to X$ is nonexpansive and $F(U) = F(S) \cap F(T)$.*

Proof. To see that U is nonexpansive, we only apply the triangle inequality and two applications of the second inequality in Lemma 1.1. We next prove the latter. It is clear that

$F(S) \cap F(T) \subset F(U)$. To see the reverse inclusion, let $p \in F(U)$ and $q \in F(S) \cap F(T)$. Then, by the first inequality of Lemma 1.1,

$$\begin{aligned} d^2(q,p) &= d^2(q,Up) \\ &= d^2(q,(1-t)Sp \oplus tTp) \\ &\leq (1-t)d^2(q,Sp) + td^2(q,Tp) - t(1-t)d^2(Sp,Tp) \\ &\leq d^2(q,p) - t(1-t)d^2(Sp,Tp). \end{aligned} \tag{3.12}$$

This implies $Sp = Tp$. As $p = Up$, we have $p \in F(S) \cap F(T)$, as desired. □

Proof of Theorem 3.3. We first define a family of mappings $\{S_n\} : C \to X$ by

$$\begin{gathered} S_1x = \frac{1}{2}x \oplus \frac{1}{2}T_1x \\ S_2x = \frac{2^2-1}{2^2}S_1x \oplus \frac{1}{2^2}T_2x \\ \vdots \\ S_nx = \frac{2^n-1}{2^n}S_{n-1}x \oplus \frac{1}{2^n}T_nx \\ \vdots \end{gathered} \tag{3.13}$$

By Lemma 3.4, each S_n is a nonexpansive mapping satisfying $F(S_n) = \bigcap_{k=1}^{n} F(T_k)$. Notice that, for fixed $p \in \bigcap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned} d^2(S_{n+1}x, S_nx) &= d^2\left(\frac{2^{n+1}-1}{2^{n+1}}S_nx \oplus \frac{1}{2^{n+1}}T_{n+1}x, S_nx\right) \\ &= \frac{1}{2^{n+1}}d^2(T_{n+1}x, S_nx) \\ &= \frac{1}{2^{n+1}}\left(d(T_{n+1}x,p) + d(p,S_nx)\right)^2 \\ &\leq \frac{1}{2^{n-1}}d^2(x,p). \end{aligned} \tag{3.14}$$

From the estimation above, we have

$$\sum_{n=1}^{\infty} \sup\{d(S_{n+1}x, S_nx) : x \in B\} < \infty \tag{3.15}$$

for each bounded subset B of C. In particular, $\{S_n x\}$ is a Cauchy sequence for each $x \in C$. We now define the nonexpansive mapping $S : C \to X$ by

$$Sx = \lim_{n \to \infty} S_n x. \tag{3.16}$$

Finally, we prove that

$$F(S) = \bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T_n). \tag{3.17}$$

The latter equality is clearly verified and $\bigcap_{n=1}^{\infty} F(S_n) \subset F(S)$ holds. On the other hand, let $p \in F(S)$ and $q \in \bigcap_{n=1}^{\infty} F(T_n)$. We consider the following:

$$\begin{aligned} d^2(q, S_n p) &= d^2\left(q, \frac{2^n - 1}{2^n} S_{n-1} p \oplus \frac{1}{2^n} T_n p\right) \\ &\leq \frac{2^n - 1}{2^n} d^2(q, S_{n-1} p) + \frac{1}{2^n} d^2(q, T_n p) \\ &\leq \frac{2^n - 1}{2^n} d^2(q, S_{n-1} p) + \frac{1}{2^n} d^2(q, p). \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned} d^2(q, S_n p) &\leq \left(\prod_{k=2}^{n} \frac{2^k - 1}{2^k}\right) d^2(q, S_1 p) + \left(1 - \prod_{k=2}^{n} \frac{2^k - 1}{2^k}\right) d^2(q, p) \\ &\leq \left(\prod_{k=2}^{n} \frac{2^k - 1}{2^k}\right) \left(\frac{1}{2} d^2(q, p) + \frac{1}{2} d^2(q, T_1 p) - \frac{1}{4} d^2(p, T_1 p)\right) \\ &\quad + \left(1 - \prod_{k=2}^{n} \frac{2^k - 1}{2^k}\right) d^2(q, p) \\ &\leq \left(\prod_{k=2}^{n} \frac{2^k - 1}{2^k}\right) \left(d^2(q, p) - \frac{1}{4} d^2(p, T_1 p)\right) + \left(1 - \prod_{k=2}^{n} \frac{2^k - 1}{2^k}\right) d^2(q, p). \end{aligned} \tag{3.19}$$

Letting $n \to \infty$ yields

$$d^2(q, p) \leq \left(\prod_{k=2}^{\infty} \frac{2^k - 1}{2^k}\right) \left(d^2(q, p) - \frac{1}{4} d^2(p, T_1 p)\right) + \left(1 - \prod_{k=2}^{\infty} \frac{2^k - 1}{2^k}\right) d^2(q, p). \tag{3.20}$$

Because $\prod_{k=2}^{\infty}((2^k-1)/2^k) > 0$, we have $p = T_1 p$. Continuing this procedure we obtain that $p \in \bigcap_{n=1}^{\infty} F(T_n)$ and hence $F(S) \subset \bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

4. Nonself Mappings

From Bridson and Haefliger's book (page 176), the following result is proved.

Theorem 4.1. *Let X be a complete CAT(0) space and C a closed convex subset of X. Then the followings hold true.*

(i) *For each $x \in X$, there exists an element $\pi(x) \in C$ such that*

$$d(x, \pi(x)) = \operatorname{dist}(x, C). \tag{4.1}$$

(ii) *$\pi(x) = \pi(x')$ for all $x' \in [x, \pi(x)]$.*

(iii) *The mapping $x \mapsto \pi(x)$ is nonexpansive.*

The mapping π in the preceding theorem is called the *metric projection from X onto C*. From this, we have the following result.

Theorem 4.2. *Let X be a complete CAT(0) space and C a closed convex subset of X. Let $T : C \to X$ be a nonself nonexpansive mapping with $F(T) \neq \emptyset$ and $\pi : X \to C$ the metric projection from X onto C. Then the mapping $\pi \circ T$ is nonexpansive and $F(\pi \circ T) = F(T)$.*

Proof. It follows from Theorem 4.1 that $\pi \circ T$ is nonexpansive. To see the latter, it suffices to show that $F(\pi \circ T) \subset F(T)$. Let $p \in F(\pi \circ T)$ and $q \in F(T)$. Since

$$\begin{aligned} d^2(q,p) &= d^2\left(\pi(q), \pi\left(\frac{1}{2}Tp \oplus \frac{1}{2}p\right)\right) \\ &\leq d^2\left(q, \frac{1}{2}Tp \oplus \frac{1}{2}p\right) \\ &\leq \frac{1}{2}d^2(q,Tp) + \frac{1}{2}d^2(q,p) - \frac{1}{4}d^2(Tp,p) \\ &\leq d^2(q,p) - \frac{1}{4}d^2(Tp,p), \end{aligned} \tag{4.2}$$

we have $p = Tp$ and this finishes the proof. □

By the preceding theorem and Theorem 2.3, we obtain the following result.

Theorem 4.3. *Let $X, C, T : C \to X$, and $\pi : X \to C$ be as the same as Theorem 4.2. Suppose that $u, x_1 \in C$ are arbitrarily chosen and the sequence $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\pi \circ T x_n) \quad \forall n \geq 1, \tag{4.3}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying*

(C1) $\lim_{n\to\infty}\alpha_n = 0$*;*

(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$*;*

(C3) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$*.*

Then $\{x_n\}$ *converges to* $z \in F(T)$ *which is nearest* u*.*

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