Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 470149, 9 pages
doi:10.1155/2010/470149

Research Article

Biorthogonal Systems Approximating the Solution of the Nonlinear Volterra Integro-Differential Equation

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Received 22 March 2010; Accepted 14 June 2010

Academic Editor: Juan J. Nieto

This paper deals with obtaining a numerical method in order to approximate the solution of the nonlinear Volterra integro-differential equation. We define, following a fixed-point approach, a sequence of functions which approximate the solution of this type of equation, due to some properties of certain biorthogonal systems for the Banach spaces $C[0,1]$ and $C[0,1]^2$.

1. Introduction

The aim of this paper is to introduce a numerical method to approximate the solution of the nonlinear Volterra integro-differential equation, which generalizes that developed in [1]. Let us consider the nonlinear Volterra integro-differential equation

$$\begin{gathered} y'(t) = f(t, y(t)) + \int_0^t K(t, s, y(s))\, ds \quad (t \in [0,1]), \\ y(0) = y_0, \end{gathered} \tag{1.1}$$

where $y_0 \in \mathbb{R}$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying a Lipschitz condition with respect to the last variables: there exist $L_f, L_K \geq 0$ such that

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &\leq L_f |y_1 - y_2|, \\ |K(t,s,y_1) - K(t,s,y_2)| &\leq L_K |y_1 - y_2|, \end{aligned} \tag{1.2}$$

for $t, s \in [0,1]$ and for $y_1, y_2 \in \mathbb{R}$. In the sequel, these conditions will be assumed. It is a simple matter to check that a function $z : [0,1] \to \mathbb{R}$ is a solution of (1.1) if, and only if, it is a fixed point of the self-operator of the Banach space $C[0,1]$ (usual supnorm) $T : C[0,1] \to C[0,1]$ given by the formula

$$Ty(t) = y_0 + \int_0^t f(v, y(v))dv + \int_0^t \int_0^v K(v,s,y(s))ds\ dv. \tag{1.3}$$

Section 2 shows that operator T satisfies the hypothesis of the Banach fixed point theorem and thus the sequence $\{T^m(z_0)\}_{m\in\mathbb{N}}$ converges to the solution z of (1.1) for any $z_0 \in C[0,1]$. However, such a sequence cannot be determined in an explicit way. The method we present consists of replacing the first element of the convergent sequence, Tz_0, by the new easy to calculate function $z_1 \in C[0,1]$, and in such a way that the error $\|Tz_0 - z_1\|$ is small enough. By repeating the same process for the function Tz_1 and so on, we obtain a sequence $\{z_m\}_{m\geq 0}$ that approximates the solution z of (1.1) in the uniform sense. To obtain such sequence, we will make use of some biorthogonal systems, the usual Schauder bases for the spaces $C[0,1]$ and $C[0,1]^2$, as well as their properties. These questions are also reviewed in Section 2. In Section 3 we define the sequence $\{z_m\}_{m\in\mathbb{N}}$ described above and we study the error $\|z - z_k\|$. Finally, in Section 4 we apply the method to two examples.

Volterra integro-differential equations are usually difficult to solve in an analytical way. Many authors have paid attention to their study and numerical treatment (see for instance [2–15] for the classical methods and recent results). Among the main advantages of our numerical method as opposed to the classical ones, such as collocation or quadrature, we can point out that it is not necessary to solve algebraic equation systems; furthermore, the integrals involved are immediate and therefore we do not have to require any quadrature method to calculate them. Let us point out that our method clearly applies to the case where the involved functions are defined in $[t_0, \mathrm{T}]$, although we have chosen the unit interval for the sake of simplicity. Schauder bases have been used in order to solve numerically some differential and integral problems (see [1, 16–20]).

2. Preliminaries

We first show that operator T^n also satisfies a suitable Lipschitz condition. This result is proven by using an inductive argument. The proof is similar to that of the linear case (see [1, Lemma 2]).

Lemma 2.1. *For any $p, q \in C[0,1]$ and $n \in \mathbb{N}$, we have*

$$\|T^n p - T^n q\| \le \frac{L^n}{n!}\|p - q\|, \tag{2.1}$$

where $L := L_f + L_K$.

In view of the Banach fixed point theorem and Lemma 2.1, T has a unique fixed point z and

$$\text{for all } z_0 \in C([0,1]) \text{ and } m \ge 1, \quad \|z - T^m z_0\| \le \sum_{k=m}^{\infty} \frac{L^k}{k!}\|Tz_0 - z_0\|. \tag{2.2}$$

Now let us consider a special kind of biorthogonal system for a Banach space. Let us recall that a sequence $\{b_n\}_{n\ge1}$ in a Banach space E is said to be a *Schauder basis* if for every $x \in E$ there exists a unique sequence of scalars $\{\beta_n\}_{n\ge1}$ such that $x = \sum_{n\ge1} \beta_n b_n$. The associated sequence of (continuous and linear) *projections* $\{P_n\}_{n\ge1}$ is defined by the partial sums $P_n(\sum_{k\ge1} \beta_k b_k) = \sum_{k=1}^{n} \beta_k b_k$. We now consider the usual Schauder basis for the space $C[0,1]$ (supnorm), also known as the *Faber-Schauder* basis: for a dense sequence of distinct points $\{t_i\}_{i\ge1}$, with $t_1 = 0$ and $t_2 = 1$, we define $b_1(t) := 1$, $(t \in [0,1])$ and for all $n \ge 2$ we use b_n to stand for the piecewise linear function with nodes at the points $\{t_1, \dots, t_n\}$ with $b_n(t_k) = 0$ for all $k < n$ and $b_n(t_n) = 1$. It is straightforward to show (see [21]) that the sequence of projections $\{P_n\}_{n\ge1}$ satisfies the following interpolation property:

$$x \in C[0,1], \; n \ge 1, \quad i \le n \Longrightarrow P_n(x)(t_i) = x(t_i). \tag{2.3}$$

In order to define an analogous basis for the Banach space $C[0,1]^2$ (supnorm), let us consider the mapping $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ given by (for a real number x, $[x]$ denotes its integer part)

$$\sigma(n) := \begin{cases} (\sqrt{n}, \sqrt{n}) & \text{if } [\sqrt{n}] = \sqrt{n}, \\ \left(n - [\sqrt{n}]^2, [\sqrt{n}] + 1\right) & \text{if } 0 < n - [\sqrt{n}]^2 \le [\sqrt{n}], \\ \left([\sqrt{n}] + 1, n - [\sqrt{n}]^2 - [\sqrt{n}]\right) & \text{if } [\sqrt{n}] < n - [\sqrt{n}]^2. \end{cases} \tag{2.4}$$

If $\{b_i\}_{i\ge1}$ is a Schauder base for the space $C[0,1]$, then the sequence

$$B_n(s,t) := b_i(s) b_j(t) \quad (s, t \in [0,1]), \tag{2.5}$$

with $\sigma(n) = (i,j)$, is a Schauder basis for $C[0,1]^2$ (see [21]). Therefore, from now on, if $\{t_i : i \ge 1\}$ is a dense subset of distinct points in $[0,1]$, with $t_1 = 0$ and $t_2 = 1$, and $\{b_i\}_{i\ge1}$ is the associated usual Schauder basis, then we will write $\{B_n\}_{n\ge1}$ to denote the Schauder basis for $C[0,1]^2$ obtained in this "natural" way. It is not difficult to check that this basis satisfies similar properties to the ones for the one-dimensional case: for instance, the sequence of projections $\{Q_n\}_{n\geq 1}$ satisfies, for all $x \in C[0,1]^2$ and for all $n, i, j \in \mathbb{N}$ with $\sigma^{-1}(i,j) \leq n$,

$$Q_n(x)(t_i, t_j) = x(t_i, t_j). \tag{2.6}$$

Under certain weak conditions, we can estimate the rate of convergence of the sequence of projections. For this purpose, consider the dense subset $\{t_i\}_{i\geq 1}$ of distinct points in $[0,1]$ and let T_n be the set $\{t_1, \ldots, t_n\}$ ordered in an increasing way for $n \geq 2$. Clearly, T_n is a partition of $[0,1]$. Let ΔT_n denote the norm of the partition T_n. The following remarks follow easily from the interpolating properties (2.3) and (2.6) and the mean-value theorems for one and two variables:

$$x \in C^1[0,1], \quad n \geq 2 \Longrightarrow \|x - P_n(x)\| \leq 2\|x'\|\Delta T_n, \tag{2.7}$$

$$x \in C^1[0,1]^2, \quad n \geq 2 \Longrightarrow \|x - Q_{n^2}(x)\| \leq 4\max\left\{\left\|\frac{\partial x}{\partial s}\right\|, \left\|\frac{\partial x}{\partial t}\right\|\right\}\Delta T_n. \tag{2.8}$$

3. A Method for Approximating the Solution

We now turn to the main purpose of this paper, that is, to approximate the unique fixed point of the nonlinear operator $T : C[0,1] \to C[0,1]$ given by (1.3), with the adequate conditions. We then define the approximating sequence described in the Introduction.

Theorem 3.1. *Let $K \in C^1([0,1]^2 \times \mathbb{R})$, $f \in C^1([0,1] \times \mathbb{R})$, $z_0 \in C^1[0,1]$, and $m \in \mathbb{N}$. Let $\{\varepsilon_1, \ldots, \varepsilon_m\}$ be a set of positive numbers and, with the notation above, define inductively, for $k \in \{1, \ldots, m\}$ and $0 \leq t, s \leq 1$, the functions*

$$\begin{aligned} \Psi_{k-1}(t) &:= f(t, z_{k-1}(t)), \\ \Phi_{k-1}(t,s) &:= K(t, s, z_{k-1}(s)), \end{aligned} \tag{3.1}$$

$$z_k(t) := y_0 + \int_0^t P_{m_k}(\Psi_{k-1}(v))dv + \int_0^t \int_0^v Q_{n_k^2}(\Phi_{k-1}(v,s))ds\, dv, \tag{3.2}$$

where

(1) *m_k is a natural number such that $\Delta T_{m_k} \leq \dfrac{\varepsilon_k}{4\|\Psi'_{k-1}\|}$*

(2) *n_k is a natural number such that $\Delta T_{n_k} \leq \dfrac{\varepsilon_k}{4M_{k-1}}$, with*

$$M_{k-1} := \max\left\{\left\|\frac{\partial \Phi_{k-1}}{\partial t}\right\|, \left\|\frac{\partial \Phi_{k-1}}{\partial s}\right\|\right\}. \tag{3.3}$$

Then, for all $k \in \{1, \ldots, m\}$, it is satisfied that

$$\|Tz_{k-1} - z_k\| \leq \varepsilon_k. \tag{3.4}$$

Proof. In view of condition (1) we have, by applying (2.7), that for all $k \in \{1, \ldots, m\}$, the inequality

$$\|\Psi_{k-1} - P_{m_{k-1}}(\Psi_{k-1})\| \leq \frac{\varepsilon_k}{2}. \tag{3.5}$$

is valid. Analogously, it follows from condition (2) and (2.8) that for all $k \in \{1, \ldots, m\}$

$$\|\Phi_{k-1} - Q_{m_{k-1}^2}(\Phi_{k-1})\| \leq \varepsilon_k. \tag{3.6}$$

As a consequence, we derive that for all $t \in [0,1]$ we have

$$\begin{aligned} |Tz_{k-1}(t) - z_k(t)| &\leq \left|\int_0^t (\Psi_{k-1}(v) - P_{m_{k-1}}(\Psi_{k-1}(v)))dv\right| \\ &\quad + \left|\int_0^t \int_0^v \left(\Phi_{k-1}(v,s) - Q_{m_{k-1}^2}(\Phi_{k-1}(v,s))\right) ds\, dv\right| \\ &\leq \frac{\varepsilon_k}{2} + \varepsilon_k \int_0^t \int_0^v ds\, dv \\ &= \varepsilon_k, \end{aligned} \tag{3.7}$$

and therefore,

$$\|Tz_{k-1} - z_k\| \leq \varepsilon_k \tag{3.8}$$

as announced. □

The next result is used in order to establish the fact that the sequence defined in Theorem 3.1 approximates the solution of the nonlinear Volterra integro-differential equation, as well as giving an upper bond of the error committed.

Proposition 3.2. *Let $m \in \mathbb{N}$ and $\{z_0, z_1, \ldots, z_m\}$ be any subset of $C[0,1]$. Then*

$$\|z - z_m\| \leq \sum_{k=m}^{\infty} \frac{L^k}{k!} \|Tz_0 - z_0\| + \sum_{k=1}^{m} \frac{L^{m-k}}{(m-k)!} \|Tz_{k-1} - z_k\|, \tag{3.9}$$

with z being the fixed point of the operator T and $L = L_f + L_K$.

Proof. We know from Lemma 2.1 that

$$\left\|T^{m-k+1} z_{k-1} - T^{m-k} z_k\right\| \leq \frac{L^{m-k}}{(m-k)!} \|Tz_{k-1} - z_k\| \tag{3.10}$$

for $k \in \{1, \dots, m\}$, which implies

$$\sum_{k=1}^{m} \left\| T^{m-k+1} z_{k-1} - T^{m-k} z_k \right\| \leq \sum_{k=1}^{m} \frac{L^{m-k}}{(m-k)!} \| T z_{k-1} - z_k \|. \tag{3.11}$$

The proof is complete by applying (2.2) to $\|z - T^m z_0\|$ and taking into account that

$$\|z - z_m\| \leq \|z - T^m z_0\| + \sum_{k=1}^{m} \left\| T^{m-k+1} z_{k-1} - T^{m-k} z_k \right\|. \tag{3.12}$$

□

As a consequence of Theorem 3.1 and Proposition 3.2, if z is the exact solution of the nonlinear Volterra integro-differential (1.1), then for the sequence of approximating functions $\{z_m\}_{m\geq 0}$ the error $\|z - z_m\|$ is given by

$$\|z - z_m\| \leq \sum_{k=m}^{\infty} \frac{L^k}{k!} \| T z_0 - z_0 \| + \sum_{k=1}^{m} \frac{L^{m-k}}{(m-k)!} \varepsilon_k, \tag{3.13}$$

where $L = L_f + L_K$. In particular, it follows from this inequality that given $\varepsilon > 0$, there exists $m \geq 1$ such that $\|z - z_m\| < \varepsilon$.

In order to choose m_k and n_k (projections P_{m_k} and $Q_{n_k^2}$ in Theorem 3.1), we can observe the fact, which is not difficult to check, that the sequences $\{\Psi'_k\}_{k\geq 0}$ and $\{M_k\}_{k\geq 0}$ are bounded (and hence conditions (1.1) and (1.3)) in Theorem 3.1 are easy to verify), provided that the scalar sequence $\{\varepsilon_k\}_{k\geq 1}$ is bounded, f and K are C^1–functions, and $\partial f/\partial t$, $\partial f/\partial s$, $\partial K/\partial t$, $\partial K/\partial s$, and $\partial K/\partial u$ satisfy a Lipschitz condition at their last variables. Indeed in view of inequality (3.13),

$$\|z - z_k\| \leq e^L \left(\| T z_0 - z_0 \| + \sup_{k\geq 1} \varepsilon_k \right), \tag{3.14}$$

and in particular $\{z_k\}_{k\geq 0}$ is bounded. Therefore, taking into account that the Schauder bases considered are monotone (norm-one projections, see [21]), we arrive at

$$\left\| z'_k \right\| \leq \max_{t\in[0,1]} \left| f(t, z_{k-1}(t)) \right| + \max_{t,s\in[0,1]} |K(t, s, z_{k-1}(s))|. \tag{3.15}$$

Take $M_f := \max_{0\leq t\leq 1} |f(t,0)|$ and $M_K := \max_{0\leq t,s\leq 1} |K(t,s,0)|$ to derive from the triangle inequality and the last inequality that

$$\left\| z'_k \right\| \leq L_f \|z_{k-1}\| + M_f + L_K \|z_{k-1}\| + M_K. \tag{3.16}$$

Finally, since the sequence $\{z_k\}_{k\geq 0}$ is bounded, $\{z'_k\}_{k\geq 0}$ also is. Similarly, one proves that $\{\Psi'_k\}_{k\geq 1}$ is bounded (sequences $\{z_k\}_{k\geq 0}$ and $\{z'_k\}_{k\geq 0}$ are bounded and $\partial f/\partial t$ and $\partial f/\partial s$ are Lipschitz at their second variables) and $\{M_k\}_{k\geq 1}$ is bounded (sequences $\{z_k\}_{k\geq 0}$ and $\{z'_k\}_{k\geq 0}$ are bounded and $\partial K/\partial t$, $\partial K/\partial s$, and $\partial K/\partial s$ are Lipschitz at the third variables).

We have chosen the Schauder bases above for simplicity in the exposition, although our numerical method also works by considering fundamental biorthogonal systems in $C[0,1]$ and $C[0,1]^2$.

4. Numerical Examples

The behaviour of the numerical method introduced above will be illustrated with the following two examples.

Example 4.1. ([22, Problem 2]). The equation

$$y'(t) = 2t - \frac{1}{2}\left(\sin\left(t^4\right)\right) + \int_0^t t^2 s \cos\left(t^2 y(s)\right) ds \quad (t \in [0,1]),$$
$$y(0) = 0 \tag{4.1}$$

has exact solution $z(t) = t^2$.

Example 4.2. Consider the equation

$$y'(t) = 3t^2 + \frac{1}{3}(\cos(y(t)) - 1) + \int_0^t s^2 \sin(y(s)) ds \quad (t \in [0,1]),$$
$$y(0) = 0 \tag{4.2}$$

whose exact solution is $z(t) = t^3$.

The computations associated with the examples were performed using Mathematica 7. In both cases, we choose the dense subset of $[0,1]$

$$\left\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots, \frac{1}{2^k}, \frac{3}{2^k}, \ldots, \frac{2^k - 1}{2^k}, \ldots\right\} \tag{4.3}$$

to construct the Schauder bases in $C[0,1]$ and $C[0,1]^2$. To define the sequence $\{z_m\}_{m\in\mathbb{N}}$ introduced in Theorem 3.1, we take $z_0(t) = y_0$ and $m_k = n_k = j$ (for all $k \in \mathbb{N}$) in the expression (3.2), that is

$$z_k(t) = y_0 + \int_0^t P_j(\Psi_{k-1}(v)) dv + \int_0^t \int_0^v Q_{j^2}(\Phi_{k-1}(v,s)) ds\, dv. \tag{4.4}$$

In Tables 1 and 2 we exhibit, for $j = 9, 17$ and 33, the absolute errors committed in eight points (t_i) of $[0,1]$ when we approximate the exact solution z by the iteration z_m. The results in Table 1 improve those in [22].

Table 1: Absolute errors for Example 4.1.

t_i	$j = 9$ $\lvert z_3(t_i) - z(t_i)\rvert$	$j = 17$ $\lvert z_3(t_i) - z(t_i)\rvert$	$j = 33$ $\lvert z_3(t_i) - z(t_i)\rvert$
0.125	3.81×10^{-6}	1.03×10^{-6}	2.63×10^{-7}
0.250	3.30×10^{-5}	8.42×10^{-6}	2.11×10^{-6}
0.375	1.13×10^{-4}	2.85×10^{-5}	7.14×10^{-6}
0.5	2.69×10^{-4}	6.76×10^{-5}	1.69×10^{-5}
0.625	5.24×10^{-4}	1.31×10^{-4}	3.28×10^{-5}
0.750	8.85×10^{-4}	2.21×10^{-4}	5.53×10^{-5}
0.875	1.30×10^{-3}	3.23×10^{-4}	8.07×10^{-5}
1	1.52×10^{-3}	3.75×10^{-4}	9.36×10^{-5}

Table 2: Absolute errors for Example 4.2.

t_i	$j = 9$ $\lvert z_2(t_i) - z(t_i)\rvert$	$j = 17$ $\lvert z_2(t_i) - z(t_i)\rvert$	$j = 33$ $\lvert z_2(t_i) - z(t_i)\rvert$
0.125	9.76×10^{-4}	2.44×10^{-4}	6.10×10^{-5}
0.250	1.95×10^{-3}	4.88×10^{-4}	1.22×10^{-4}
0.375	2.92×10^{-3}	7.32×10^{-4}	1.83×10^{-4}
0.5	3.90×10^{-3}	9.75×10^{-4}	2.43×10^{-4}
0.625	4.87×10^{-3}	1.21×10^{-3}	3.04×10^{-4}
0.750	5.83×10^{-3}	1.45×10^{-3}	3.64×10^{-4}
0.875	6.77×10^{-3}	1.69×10^{-3}	4.23×10^{-4}
1	7.68×10^{-3}	1.92×10^{-3}	4.80×10^{-4}

Acknowledgment

This research is partially supported by M.E.C. (Spain) and FEDER, project MTM2006-12533, and by Junta de Andaluca Grant FQM359.

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