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Research Article

A Hybrid Projection Algorithm for Finding Solutions of Mixed Equilibrium Problem and Variational Inequality Problem

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We propose a modified hybrid projection algorithm to approximate a common fixed point of a k-strict pseudocontraction and of two sequences of nonexpansive mappings. We prove a strong convergence theorem of the proposed method and we obtain, as a particular case, approximation of solutions of systems of two equilibrium problems.

1. Introduction

In this paper, we define an iterative method to approximate a common fixed point of a k-strict pseudocontraction and of two sequences of nonexpansive mappings generated by two sequences of firmly nonexpansive mappings and two nonlinear mappings. Let us recall from [1] that the k-strict pseudocontractions in Hilbert spaces were introduced by Browder and Petryshyn in [2].

Definition 1.1. $S : C \to C$ is said to be *k-strict pseudocontractive* if there exists $k \in [0,1[$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

The iterative approximation problems for nonexpansive mappings, asymptotically nonexpansive mappings, and asymptotically pseudocontractive mappings were studied extensively by Browder [3], Goebel and Kirk [4], Kirk [5], Liu [6], Schu [7], and Xu [8, 9] in the setting of Hilbert spaces or uniformly convex Banach spaces. Although nonexpansive mappings are 0-strict pseudocontractions, iterative methods for k-strict pseudocontractions are far less developed than those for nonexpansive mappings. The reason, probably, is that the second term appearing in the previous definition impedes the convergence analysis for iterative algorithms used to find a fixed point of the k-strict pseudocontraction S. However, k-strict pseudocontractions have more powerful applications than nonexpansive mappings do in solving inverse problems. In the recent years the study of iterative methods like Mann's like methods and CQ-methods has been extensively studied by many authors [1, 10–13] and the references therein.

If C is a closed and convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ is a bi-function we call equilibrium problem

$$\text{Find } \overline{x} \in C \quad \text{s.t. } F(\overline{x}, y) \geq 0, \quad \forall y \in C, \tag{1.2}$$

and we will indicate the set of solutions with $EP(F)$.

If $A : C \to H$ is a nonlinear mapping, we can choose $F(x, y) = \langle Ax, y - x \rangle$, so an equilibrium point (i.e., a point of the set $EP(F)$) is a solution of variational inequality problem (VIP)

$$\text{Find } \overline{x} \in C \quad \text{s.t. } \langle A\overline{x}, y - \overline{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

We will indicate with $VI(C, A)$ the set of solutions of VIP.

The equilibrium problems, in its various forms, found application in optimization problems, fixed point problems, convex minimization problems; in other words, equilibrium problems are a unified model for problems arising in physics, engineering, economics, and so on (see [10]).

As in the case of nonexpansive mappings, also in the case of k-strict pseudocontraction mappings, in the recent years many papers concern the convergence of iterative methods to a solutions of variational inequality problems or equilibrium problems; see example for, [10, 14–18].

Here we prove a strong convergence theorem of the proposed method and we obtain, as a particular case, approximation of solutions of systems of two equilibrium problems.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H.
We denote by P_C the metric projection of H onto C. It is well known [19] that

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0, \quad \forall x \in H \text{ and } y \in C. \tag{2.1}$$

Lemma 2.1. *(see [20]) Let X be a Banach space with weakly sequentially continuous duality mapping J, and suppose that $(x_n)_{n\in\mathbb{N}}$ converges weakly to $x_0 \in X$, then for any $x \in X$,*

$$\liminf_{n\to\infty} \|x_n - x_0\| \leq \liminf_{n\to\infty} \|x_n - x\|. \tag{2.2}$$

Moreover if X is uniformly convex, equality holds in (2.2) *if and only if $x_0 = x$.*

Recall that a point $u \in C$ is a solution of a VIP if and only if

$$u = P_C(I - \lambda A)u \quad \forall \lambda > 0, \quad \text{that is, } u \in VI(C,A) \Longleftrightarrow u \in \operatorname{Fix}(P_C(I - \lambda A)), \quad \forall \lambda > 0. \tag{2.3}$$

Definition 2.2. An operator $A : C \to H$ is said to be *α-inverse strongly monotone operator* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \quad \forall x, y \in C. \tag{2.4}$$

If $\alpha = 1$ we say that A is firmly nonexpansive. Note that every α-inverse strongly monotone operator is also $1/\alpha$ Lipschitz continuous (see [21]).

Lemma 2.3. *(see [2]). Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a k-strict pseudocontractive mapping. Then $S_t := tI + (1 - t)S$ with $t \in [k, 1[$ is a nonexpansive mapping with* $\operatorname{Fix}(S_t) = \operatorname{Fix}(S)$.

3. Main Theorem

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let*

(i) *A be an α-inverse strongly monotone mapping of C into H,*

(ii) *B a β-inverse strongly monotone mapping of C into H,*

(iii) *$(T_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ two sequences of firlmy nonexpansive mappings from C to H.*

Let $S : C \to C$ be a k-strict pseudocontraction $\operatorname{Fix}(S) \neq \emptyset$.
Set $S_k = kI + (1 - k)S$ and let us define the sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{gathered} x_1 \in C, \\ C_1 = C, \\ u_n = T_n(I - r_n A)x_n \\ z_n = V_n(I - \lambda_n B)u_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)S_k z_n, \\ C_{n+1} = \{ w \in C_n : \ \|y_n - w\| \leq \|x_n - w\| \}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in N, \end{gathered} \tag{3.1}$$

where

(i) $(\alpha_n)_{n\in\mathbb{N}} \subset [0, a]$ *with* $a < 1$;

(ii) $(\lambda_n)_{n\in\mathbb{N}} \subset [b, c] \subset (0, 2\beta)$;

(iii) $(r_n)_{n\in\mathbb{N}} \subset [d, e] \subset (0, 2\alpha)$.

Moreover suppose that

(i) $F := \operatorname{Fix}(S)\bigcap \cap_n \operatorname{Fix}(V_n(I-\lambda_n B))\bigcap \cap_n \operatorname{Fix}(T_n(I-r_n A)) \neq \emptyset$;

(ii) $(T_n(I-r_nA))_{n\in\mathbb{N}}$ *pointwise converges in* C *to an operator* R *and* $(V_n(I-\lambda_nB))_{n\in\mathbb{N}}$ *pointwise converges in* C *to an operator* W;

(iii) $\operatorname{Fix}(W) = \cap_n \operatorname{Fix}(V_n(I-\lambda_nB))$ *and* $\operatorname{Fix}(R) = \cap_n \operatorname{Fix}(T_n(I-r_nA))$.

Then $(x_n)_{n\in\mathbb{N}}$ *strongly converges to* $x^* = P_F x_1$.

Proof. We begin to observe that the mappings $T_n(I-r_nA)$ and $V_n(I-\lambda_nB)$ are nonexpansive for all $n \in \mathbb{N}$ since they are compositions of nonexpansive mappings (see [22, page 419]). As a rule, if $p \in F$

$$\begin{aligned} \|u_n - p\|^2 &\le \|x_n - p\|^2, \\ \|z_n - p\|^2 &\le \|u_n - p\|^2 \le \|x_n - p\|^2. \end{aligned} \tag{3.2}$$

Now we divide the proof in more steps.

Step 1. C_n is closed and convex for each $n \in \mathbb{N}$.

Indeed C_{n+1} is the intersection of C_n with the half space

$$\{w \in H : \langle w, x_n - y_n\rangle \le L\}, \tag{3.3}$$

where $L = (\|x_n\|^2 - \|y_n\|^2)/2$.

Step 2. $F \subseteq C_n$ for each $n \in \mathbb{N}$.

For each $w \in F$ we have

$$\begin{aligned} \|y_n - w\| &= \|\alpha_n x_n + (1-\alpha_n)S_k z_n - w\| \\ &\le \alpha_n\|x_n - w\| + (1-\alpha_n)\|z_n - w\| \\ &= \alpha_n\|x_n - w\| + (1-\alpha_n)\|V_n(I-\lambda_nB)u_n - w\| \\ &\le \alpha_n\|x_n - w\| + (1-\alpha_n)\|u_n - w\| \\ &= \alpha_n\|x_n - w\| + (1-\alpha_n)\|T_n(I-r_nA)x_n - w\| \\ &\le \alpha_n\|x_n - w\| + (1-\alpha_n)\|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{3.4}$$

So the claim immediately follows by induction.

Step 3. $\lim_{n\to+\infty}\|x_n - x_1\|$ exists and $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular, that is, $\lim_{n\to+\infty}\|x_{n+1} - x_n\| = 0$.

Since $x_n = P_{C_n}x_1, x_{n+1} = P_{C_{n+1}}x_1$, and $C_{n+1} \subseteq C_n$, by (2.1) choosing $y = x_{n+1}, x = x_1$ and $C = C_n$, we have

$$\begin{aligned} 0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &\le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|, \end{aligned} \tag{3.5}$$

that is, $\|x_n - x_1\| \le \|x_{n+1} - x_1\|$.

By $x_n = P_{C_n}x_1$ and $F \subseteq C_n$, we have

$$\|x_1 - x_n\| \le \|x_1 - P_F x_1\|. \tag{3.6}$$

Then $\lim_{n\to+\infty}\|x_n - x_1\|$ exists and $(x_n)_{n\in\mathbb{N}}$ is bounded. Moreover

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 + x_1 - x_n\|^2 \\ &= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 + 2\langle x_{n+1} - x_1, x_1 - x_n\rangle \\ &= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 + 2\langle x_{n+1} - x_n, x_1 - x_n\rangle - 2\|x_n - x_1\|^2 \\ &\le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 \ \text{by (3.5)}, \end{aligned} \tag{3.7}$$

and consequently $\lim_{n\to+\infty}\|x_{n+1} - x_n\| = 0$.

Step 4. $\lim_{n\to+\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to+\infty}\|x_n - S_k z_n\| = 0$.

By $x_{n+1} \in C_{n+1}$, it follows

$$\begin{gathered} \|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|, \\ \|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\| \longrightarrow 0. \end{gathered} \tag{3.8}$$

Moreover

$$\|y_n - x_n\| = (1 - \alpha_n)\|x_n - S_k z_n\|, \tag{3.9}$$

and by boundedness of $(\alpha_n)_{n\in\mathbb{N}}$, it follows that $\lim_{n\to+\infty}\|x_n - S_k z_n\| = 0$.

Step 5. $\lim_{n\to+\infty}\|Bu_n - Bw\| = 0$, for each $w \in F$.

For $w \in F$, we have

$$\begin{aligned}\|y_n - w\|^2 &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|S_k z_n - w\|^2\\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|z_n - w\|^2\\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|V_n(I-\lambda_n B)u_n - V_n(I-\lambda_n B)w\|^2\\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|(I-\lambda_n B)u_n - (I-\lambda_n B)w\|^2\\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\Big(\|u_n - w\|^2 + \lambda_n^2\|Bu_n - Bw\|^2 - 2\lambda_n\langle Bu_n - Bw, u_n - w\rangle\Big)\\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\Big(\|u_n - w\|^2 - \lambda_n(2\beta-\lambda_n)\|Bu_n - Bw\|^2\Big)\\ &\le \|x_n - w\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2.\end{aligned} \tag{3.10}$$

Consequently

$$\begin{aligned}(1-\alpha_n)\lambda_n(2\beta-\lambda_n)\|Bu_n - Bw\|^2 &\le \|x_n - w\|^2 - \|y_n - w\|^2\\ &= \big(\|x_n - w\| - \|y_n - w\|\big)\big(\|x_n - w\| + \|y_n - w\|\big)\\ &\le \big(\|x_n - y_n\|\big)\big(\|x_n - w\| + \|y_n - w\|\big),\end{aligned} \tag{3.11}$$

and by Step 4, the assumptions on $(\alpha_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$, we obtain the claim of Step 5.

Step 6. $\lim_{n\to+\infty}\|u_n - z_n\| = 0$.

Since V_n is firmly nonexpansive, for any $w \in F$, we have

$$\begin{aligned}\|z_n - w\|^2 &\le \langle (I-\lambda_n B)u_n - (I-\lambda_n B)w, z_n - w\rangle\\ &= \frac{1}{4}\Big\{\|(I-\lambda_n B)u_n - (I-\lambda_n B)w + (z_n - w)\|^2\\ &\qquad - \|(I-\lambda_n B)u_n - (I-\lambda_n B)w - (z_n - w)\|^2\Big\}\\ &\le \frac{1}{4}\Big\{\|u_n - w\|^2 - \lambda_n(2\beta-\lambda_n)\|Bu_n - Bw\|^2 + \|z_n - w\|^2\\ &\qquad - \|u_n - z_n - \lambda_n(Bu_n - Bw)\|^2\Big\}\\ &\le \frac{1}{4}\Big\{\|u_n - w\|^2 + \|z_n - w\|^2 - \|u_n - z_n - \lambda_n(Bu_n - Bw)\|^2\Big\}\\ &= \frac{1}{4}\Big\{\|u_n - w\|^2 + \|z_n - w\|^2 - \|u_n - z_n\|^2\\ &\qquad + 2\lambda_n\langle u_n - z_n, Bu_n - Bw\rangle - \lambda_n^2\|Bu_n - Bw\|^2\Big\}\end{aligned} \tag{3.12}$$

which implies

$$\begin{aligned} 3\|z_n - w\|^2 &\le \|u_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle u_n - z_n, Bu_n - Bw\rangle \\ &\le \|x_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|u_n - z_n\| \, \|Bu_n - Bw\|. \end{aligned} \tag{3.13}$$

Consequently

$$\begin{aligned} \|y_n - w\|^2 &\le \alpha_n \|x_n - w\|^2 + (1-\alpha_n)\|z_n - w\|^2 \\ &\le \|x_n - w\|^2 - (1-\alpha_n)\|u_n - z_n\|^2 + 2(1-\alpha_n)\lambda_n \|u_n - z_n\| \|Bu_n - Bw\| \end{aligned} \tag{3.14}$$

which implies

$$\begin{aligned} (1-\alpha_n)\|u_n - z_n\|^2 &\le \|x_n - w\|^2 - \|y_n - w\|^2 + 2(1-\alpha_n)\lambda_n \|u_n - z_n\| \, \|Bu_n - Bw\| \\ &\le \left(\|x_n - w\| - \|y_n - w\|\right)\left(\|x_n - w\| + \|y_n - w\|\right) \\ &\quad + 2(1-\alpha_n)\lambda_n \|u_n - z_n\| \, \|Bu_n - Bw\| \\ &\le \left(\|x_n - y_n\|\right)\left(\|x_n - w\| + \|y_n - w\|\right) + 2(1-\alpha_n)\lambda_n \|u_n - z_n\| \, \|Bu_n - Bw\|. \end{aligned} \tag{3.15}$$

By the assumptions on $(\alpha_n)_{n\in\mathbb{N}}$, Steps 4 and 6, and the boundedness of $(x_n)_{n\in\mathbb{N}}$ $(y_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ the claim follows.

Step 7. $\lim_{n\to+\infty}\|x_n - u_n\| = 0$ and $\lim_{n\to+\infty}\|x_n - S_k x_n\| = 0$.

Since T_n is firmly nonexpansive, for each $p \in \cap_n \operatorname{Fix}(T_n(I - r_n)A)$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_n(I - r_nA)x_n - T_n(I - r_nA)p\|^2 \\ &\le \langle u_n - p, (I - r_nA)x_n - (I - r_nA)p\rangle \\ &= \frac{1}{2}\Big(\|(I - r_nA)x_n - (I - r_nA)p\|^2 + \|u_n - p\|^2 \\ &\qquad - \|(I - r_nA)x_n - (I - r_nA)p - (u_n - p)\|^2\Big) \\ &= \frac{1}{2}\Big(\|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 + \|u_n - p\|^2 \\ &\qquad - \|x_n - u_n - r_n(Ax_n - Ap)\|^2\Big) \\ &\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\ &\qquad - r_n^2\|Ax_n - Ap\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ap\rangle\Big), \end{aligned} \tag{3.16}$$

and consequently

$$\|u_n - p\|^2 \leq \left(\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|\right). \tag{3.17}$$

Then, for each $w \in F$, we have

$$\begin{aligned}\|y_n - w\|^2 &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|u_n - w\|^2 \\ &\leq \|x_n - w\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 \\ &\quad + 2(1-\alpha_n)r_n\|x_n - u_n\| \ \|Ax_n - Aw\| \text{ by (3.17)},\end{aligned} \tag{3.18}$$

consequently

$$\begin{aligned}(1-\alpha_n)\|x_n - u_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 + 2(1-\alpha_n)r_n\|x_n - u_n\|\|Ax_n - Aw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) + 2(1-\alpha_n)r_n\|x_n - u_n\|\|Ax_n - Aw\|,\end{aligned} \tag{3.19}$$

and by the assumptions on $(\alpha_n)_{n\in\mathbb{N}}$, Step 4 and the boundedness of $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ it follows that $\|x_n - u_n\| \to 0$ as $n \to +\infty$. By Step 6 we note that also $\|x_n - z_n\| \to 0$.

Finally

$$\begin{aligned}\|x_n - S_k x_n\| &\leq \|x_n - S_k z_n\| + \|S_k z_n - S_k x_n\| \\ &\leq \|x_n - S_k z_n\| + \|z_n - x_n\| \\ &\leq \|x_n - S_k z_n\| + \|z_n - u_n\| + \|u_n - x_n\|,\end{aligned} \tag{3.20}$$

and by previous steps, it follows that $\|x_n - S_k x_n\| \to 0$ as $n \to +\infty$.

Step 8. The set of weak cluster points of $(x_n)_{n\in\mathbb{N}}$ is contained in F.

We will use three times the Opial's Lemma 2.1.

Let p be a weak cluster point of $(x_n)_{n\in\mathbb{N}}$ and let $(x_{n_j})_{j\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_j} \rightharpoonup p$.

We prove that $p \in \mathrm{Fix}(S) = \mathrm{Fix}(S_k)$. We suppose for absurd that $p \neq S_k p$. By Opial's Lemma 2.1 and $\|x_n - S_k x_n\| \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}\liminf_{j\to+\infty}\|x_{n_j} - p\| &< \liminf_{j\to+\infty}\|x_{n_j} - S_k p\| \\ = \liminf_{j\to+\infty}\|x_{n_j} - S_k x_{n_j} - S_k x_{n_j} - S_k p\| &\leq \liminf_{j\to+\infty}\left[\|x_{n_j} - S_k x_{n_j}\| + \|S_k x_{n_j} - S_k p\|\right] \\ &= \liminf_{j\to+\infty}\|x_{n_j} - p\|\end{aligned} \tag{3.21}$$

which is a contradiction.

Since $\mathrm{Fix}(R) = \cap_n \mathrm{Fix}(T_n(I - r_n A))$ it is enough to prove that $p \in \mathrm{Fix}(R)$. Now if $p \neq Rp$ we note that

$$\begin{aligned}
\liminf_{j\to+\infty} \left\|x_{n_j} - p\right\| &< \liminf_{j\to+\infty} \left\|x_{n_j} - Rp\right\| \\
&\leq \liminf_{j\to+\infty} \Big[\left\|x_{n_j} - T_{n_j}\left(I - r_{n_j}A\right)x_{n_j}\right\| \\
&\quad + \left\|T_{n_j}\left(I - r_{n_j}A\right)x_{n_j} - T_{n_j}\left(I - r_{n_j}A\right)p\right\| + \left\|T_{n_j}\left(I - r_{n_j}A\right)p - Rp\right\|\Big] \\
&\leq \liminf_{j\to+\infty} \Big[\left\|x_{n_j} - u_{n_j}\right\| + \left\|x_{n_j} - p\right\| + \left\|T_{n_j}\left(I - r_{n_j}A\right)p - Rp\right\|\Big] \\
&= \liminf_{j\to+\infty} \left\|x_{n_j} - p\right\|.
\end{aligned} \tag{3.22}$$

This leads to a contraddiction again. By the hypotheses and Step 7 the claim follows. By the same idea and using Step 6, we prove that $p \in \mathrm{Fix}(W) = \cap_n \mathrm{Fix}(V_n(I - \lambda_n B))$.

Step 9. $x_n \to x^* = P_F x_1$.

Since $x^* = P_F x_1 \in C_n$ and $x_n = P_{C_n} x_1$, we have

$$\|x_1 - x_n\| \leq \|x_1 - x^*\|. \tag{3.23}$$

Let $(x_{n_j})_{j\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_j} \rightharpoonup p$. By Step 8, $p \in F$. Thus

$$\begin{aligned}
\|x_1 - x^*\| \leq \left\|x_1 - p\right\| &\leq \liminf_{j\to+\infty} \left\|x_1 - x_{n_j}\right\| \\
&\leq \limsup_{j\to+\infty} \left\|x_1 - x_{n_j}\right\| \leq \|x_1 - x^*\|.
\end{aligned} \tag{3.24}$$

Therefore we have

$$\|x_1 - x^*\| = \left\|x_1 - p\right\| = \lim_{j\to+\infty} \left\|x_1 - x_{n_j}\right\|. \tag{3.25}$$

Since H has the Kadec-Klee property, then $x_{n_j} \to p$ as $j \to +\infty$.

Moreover, by $\|x_1 - x^*\| = \|x_1 - p\|$ and by the uniqueness of the projection $P_F x_1$, it follows that $p = x^* = P_F x_1$.

Thence every subsequence $(x_{n_j})_{j\in\mathbb{N}}$ converges to x^* as $j \to +\infty$ and consequently $x_n \to x^*$, as $n \to +\infty$. □

Remark 3.2. Let us observe that one can choose $(T_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ as sequences of γ_n-inverse strongly monotone operators and η_n-inverse strongly monotone operators provided $\gamma_n \geq 1, \eta_n \geq 1$ for all $n \in \mathbb{N}$.

The hypotheses (ii) and (iii) in the main Theorem 3.1 seem very strong but, in the sequel, we furnish two cases in which (ii) and (iii) are satisfied.

Let us remember that the metric projection on a convex closed set P_C is a firmly nonexpansive mapping (see [19]) so we claim that have the following proposition.

Proposition 3.3. *If $(r_n)_{n\in\mathbb{N}} \subset (0,\infty)$ is such that $\lim_n r_n = r > 0$ and A an α-inverse strongly monotone, then $P_C(I - r_n A)$ realizes conditions (ii) and (iii) with $R = P_C(I - rA)$.*

Proof. To prove (ii) we note that for each $x \in C$,

$$\|P_C(I - r_n A)x - P_C(I - rA)x\| \leq \|(I - r_n A)x - (I - rA)x\| \leq |r_n - r|\|Ax\|. \tag{3.26}$$

Moreover, (iii) follows directly by (2.2). □

Now we consider the mixed equilibrium problem

$$\text{Find } x \in C : f(x,y) + h(x,y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{3.27}$$

In the sequel we will indicate with $MEP(f,h,A)$ the set of solution of our mixed equilibrium problem. If $A = 0$ we denote $MEP(f,h,0)$ with $MEP(f,h)$.

We notice that for $h = 0$ and $A = 0$ the problem is the well-known equilibrium problem [23–25]. If $h = 0$ and A is an α-inverse strongly monotone operator we have the equilibrium problems studied firstly in [26] and then in [18, 22, 27]. If $h(x,y) = \varphi(y) - \varphi(x)$ and $A = 0$ we refound the mixed equilibrium problem studied in [16, 28, 29].

Definition 3.4. A bi-function $g : C \times C \to \mathbb{R}$ is monotone if $g(x,y) + g(y,x) \leq 0$ for all $x, y \in C$.

A function $G : C \to \mathbb{R}$ is upper hemicontinuous if

$$\limsup_{t\to 0} G(tx + (1-t)y) \leq G(y). \tag{3.28}$$

Next lemma examines the case in which $A = 0$.

Lemma 3.5. *Let C be a convex closed subset of a Hilbert space H.*

Let $f : C \times C \to \mathbb{R}$ be a bi-function such that

(f1) *$f(x,x) = 0$ for all $x \in C$;*

(f2) *f is monotone and upper hemicontinuous in the first variable;*

(f3) *f is lower semicontinuous and convex in the second variable.*

Let $h : C \times C \to \mathbb{R}$ be a bi-function such that

(h1) *$h(x,x) = 0$ for all $x \in C$;*

(h2) *h is monotone and weakly upper semicontinuous in the first variable;*

(h3) *h is convex in the second variable.*

Moreover let us suppose that

(H) *for fixed* $r > 0$ *and* $x \in C$*, there exists a bounded set* $K \subset C$ *and* $a \in K$ *such that for all* $z \in C \setminus K$, $-f(a,z) + h(z,a) + (1/r)\langle a-z, z-x\rangle < 0$,

for $r > 0$ *and* $x \in H$ *let* $T_r : H \to C$ *be a mapping defined by*

$$T_r x = \left\{ z \in C : f(z,y) + h(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}, \tag{3.29}$$

called resolvent of f *and* h.

Then

(1) $T_r x \neq \emptyset$;

(2) $T_r x$ *is a single value;*

(3) T_r *is firmly nonexpansive;*

(4) $MEP(f,h) = \operatorname{Fix}(T_r)$ *and it is closed and convex.*

Proof. Let $x_0 \in H$. For any $y \in C$ define

$$G_{r,x_0} y = \left\{ z \in C : -f(y,z) + h(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \right\}. \tag{3.30}$$

We will prove that, by KKM's lemma, $\cap_{y \in C} G_{r,x_0} y$ is nonempty.

First of all we claim that G_{r,x_0} is a KKM's map. In fact if there exists $\{y_1, \ldots, y_N\} \subset C$ such that $\overline{y} = \sum_i \alpha_i y_i$ (with $\sum_i \alpha_i = 1$) does not appartiene to $G_{r,x_0} y_i$ for any $i = 1, \ldots, N$ then

$$-f(y_i, \overline{y}) + h(\overline{y}, y_i) + \frac{1}{r}\langle y_i - \overline{y}, \overline{y} - x_0\rangle < 0, \quad \forall i. \tag{3.31}$$

By the convexity of f and h and the monotonicity of f, we obtain that

$$\begin{aligned}
0 &= f(\overline{y}, \overline{y}) + h(\overline{y}, \overline{y}) + \frac{1}{r}\langle \overline{y} - \overline{y}, \overline{y} - x_0\rangle \\
&\leq \sum_i \alpha_i f(\overline{y}, y_i) + \sum_i \alpha_i h(\overline{y}, y_i) + \frac{1}{r}\sum_i \alpha_i \langle y_i - \overline{y}, \overline{y} - x_0\rangle \\
&\leq -\sum_i \alpha_i f(y_i, \overline{y}) + \sum_i \alpha_i h(\overline{y}, y_i) + \frac{1}{r}\sum_i \alpha_i \langle y_i - \overline{y}, \overline{y} - x_0\rangle \\
&= \sum_i \alpha_i \left[-f(y_i, \overline{y}) + h(\overline{y}, y_i) + \frac{1}{r}\langle y_i - \overline{y}, \overline{y} - x_0\rangle \right] < 0,
\end{aligned} \tag{3.32}$$

that is absurd.

Now we prove that $\overline{G_{r,x_0}}^w = G_{r,x_0}$. We recall that, by the weak lower semicontinuity of $\|\cdot\|^2$, the relation

$$\limsup_m \langle y - z_m, z_m - x_0\rangle \leq \langle y - z, z - x_0\rangle \tag{3.33}$$

holds. Let $z \in \overline{G_{r,x_0}y}^w$ and let $(z_m)_m$ be a sequence in $G_{r,x_0}y$ such that $z_m \rightharpoonup z$.

We want to prove that

$$-f(y,z) + h(z,y) + \frac{1}{r}\langle y - z, z - x_0\rangle \geq 0. \tag{3.34}$$

Since f is lower semicontinuous and convex in the second variable and h is weakly upper semicontinuous in the first variable, then

$$\begin{aligned}
0 &\leq \limsup_m \left[-f(y,z_m) + h(z_m,y) + \frac{1}{r}\langle y - z, z - x_0\rangle\right] \\
&\leq \limsup_m(-f(y,z_m)) + \limsup_m h(z_m,y) + \frac{1}{r}\ \limsup_m \langle y - z, z - x_0\rangle \\
&\leq -\liminf_m f(y,z_m) + \limsup_m h(z_m,y) + \frac{1}{r}\ \limsup_m \langle y - z, z - x_0\rangle \\
&\leq -f(y,z) + h(z,y) + \frac{1}{r}\langle y - z, z - x_0\rangle.
\end{aligned} \tag{3.35}$$

Now we observe that $\overline{G_{r,x_0}y}^w = G_{r,x_0}y$ is weakly compact for at least a point $y \in C$. In fact by hypothesis (H) there exist a bounded $K \subset C$ and $a \in K$, such that for all $z \in C \setminus K$ it results $z \notin G_{r,x_0}a$. Then $G_{r,x_0}\mathrm{a} \subset K$, that is, it is bounded. It follows that $G_{r,x_0}a$ is weakly compact. Then by KKM's lemma $\cap_{y\in C}G_{r,x_0}y$ is nonempty. However if $z \in \cap_{y\in C}G_{r,x_0}$ then

$$-f(y,z) + h(z,y) + \frac{1}{r}\langle y - z, z - x_0\rangle \geq 0, \quad \forall y \in C. \tag{3.36}$$

As in [24, Lemma 3], since f is upper hemicontinuous and convex in the first variable and monotone, we obtain that (3.36) is equivalent to claim that z is such that

$$f(z,y) + h(z,y) + \frac{1}{r}\langle y - z, z - x_0\rangle \geq 0, \quad \forall y \in C, \tag{3.37}$$

that is, $z \in T_r(x_0)$. This prove (1). To prove (2) and (3) we consider $z_1 \in T_r x_1$ and $z_2 \in T_r x_2$. They satisfy the relations

$$\begin{aligned}
&f(z_1,z_2) + h(z_1,z_2) + \frac{1}{r}\langle z_2 - z_1, z_1 - x_1\rangle \geq 0, \\
&f(z_2,z_1) + h(z_2,z_1) + \frac{1}{r}\langle z_1 - z_2, z_2 - x_2\rangle \geq 0.
\end{aligned} \tag{3.38}$$

By the monotonicity of f and h, summing up both the terms,

$$\begin{aligned} 0 &\leq \frac{1}{r}[\langle z_2 - z_1, z_1 - x_1\rangle - \langle z_2 - z_1, z_2 - x_2\rangle] \\ &= \frac{1}{r}[\langle z_2 - z_1, z_1 - x_1 - z_2 + x_2\rangle] \\ &= \frac{1}{r}\left[-\|z_2 - z_1\|^2 + \langle z_2 - z_1, x_2 - x_1\rangle\right] \end{aligned} \tag{3.39}$$

so we conclude

$$\|z_2 - z_1\|^2 \leq \langle z_2 - z_1, x_2 - x_1\rangle \tag{3.40}$$

that means simultaneously that $z_1 = z_2$ if $x_1 = x_2$ and T_r is firmly nonexpansive.

To prove (4), it is enough to follow (iii) and (iv) in [25, Lemma 2.12]. □

Remark 3.6. We note that if $h = 0$, our lemma reduces to [25, Lemma 2.12]. The coercivity condition (H) is fulfilled.

Moreover our lemma is more general than [16, Lemma 2.2]. In fact

(i) our hypotheses on f are weaker (f weak upper semicontinuous implies f upper hemicontinuous);

(ii) if φ satisfies the condition in Lemma 2.2 , choosing $h(x,y) = \varphi(y) - \varphi(x)$ one has that h is concave and upper semicontinuous in the first variable and convex and lower semicontinous in the second variable;

(iii) the coercivity condition (H) by the equivalence of (3.36) and (3.37) is the same.

Lemma 3.7. *Let us suppose that (f1)–(f3), (h1)–(h3) and (H) hold. Let $x, y \in H$, $r_1, r_2 > 0$. Then*

$$\|T_{r_2}y - T_{r_1}x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}y - y\|. \tag{3.41}$$

Proof. By Lemma 3.5, defining $u_1 = T_{r_1}x$ and $u_2 := T_{r_2}y$, we know that

$$\begin{aligned} f(u_2, z) + h(u_2, z) + \frac{1}{r_2}\langle z - u_2, u_2 - y\rangle \geq 0, \quad \forall z \in C, \\ f(u_1, z) + h(u_1, z) + \frac{1}{r_1}\langle z - u_1, u_1 - x\rangle \geq 0, \quad \forall z \in C. \end{aligned} \tag{3.42}$$

In particular,

$$\begin{aligned} f(u_2, u_1) + h(u_2, u_1) + \frac{1}{r_2}\langle u_1 - u_2, u_2 - y\rangle \geq 0, \\ f(u_1, u_2) + h(u_1, u_2) + \frac{1}{r_1}\langle u_2 - u_1, u_1 - x\rangle \geq 0. \end{aligned} \tag{3.43}$$

Hence, summing up this two inequalities and using the monotonicity of f and h,

$$\left\langle u_2 - u_1, \frac{u_1 - x}{r_1} - \frac{u_2 - y}{r_2} \right\rangle \geq 0. \tag{3.44}$$

We derive from (3.44) that

$$\left\langle u_2 - u_1, u_1 - u_2 - x + u_2 - \frac{r_1}{r_2}(u_2 - y) \right\rangle \geq 0, \tag{3.45}$$

and so

$$-\|u_2 - u_1\|^2 + \left\langle u_2 - u_1, (u_2 - y)\left(1 - \frac{r_1}{r_2}\right) + (y - x) \right\rangle \geq 0. \tag{3.46}$$

Then,

$$\|u_2 - u_1\|^2 \leq \|u_2 - u_1\| \left(\|y - x\| + \left|1 - \frac{r_1}{r_2}\right| \|u_2 - y\| \right), \tag{3.47}$$

and thus the claim holds. □

Proposition 3.8. *Let us suppose that f and h are two bi-functions satisfying the hypotheses of Lemma 3.5. Let T_r be the resolvent of f and h. Let A be an α-inverse strongly monotone operator. Let us suppose that $(r_n)_{n\in\mathbb{N}} \subset (0, \infty)$ is such that $\lim_n r_n = r > 0$. Then $T_{r_n}(I - r_nA)$ realize (ii) and (iii) in Theorem 3.1.*

Proof. Let x be in a bounded closed convex subset K of C. To prove (i) it is enough to observe that by Lemma 3.7

$$\|T_{r_n}(I - r_nA)x - T_r(I - rA)x\| \leq |r_n - r|\|Ax\| + \frac{|r_n - r|}{r}\|T_r(I - rA)x - (I - rA)x\|. \tag{3.48}$$

When $n \to \infty$, by boundedness of the terms that do not depend on n, we obtain (ii).

To prove (iii) let $W = T_r(I - rA)$ the pointwise limit of $T_{r_n}(I - r_nA)$. It is necessary to prove only that $\text{Fix}(W) \subset \cap_n \text{Fix}(T_{r_n}(I - r_nA))$. Let $x \in \text{Fix}(W)$. We want to prove that $x \in MEP(f, h, A)$. Let $w_n = T_{r_n}(I - r_nA)x$. Thus, by definition of T_{r_n}, w_n is the unique point such that

$$f(w_n, y) + h(w_n, y) + \frac{1}{r_n}\langle y - w_n, w_n - (I - r_nA)x\rangle \geq 0, \quad \forall y. \tag{3.49}$$

By monotonicity of f and h this implies

$$h(w_n, y) + \frac{1}{r_n}\langle y - w_n, w_n - (I - r_nA)x\rangle \geq f(y, w_n). \tag{3.50}$$

Passing to the limit on n, by (f3) and (h2) we obtain

$$h(x,y) + \langle y - x, Ax \rangle \geq f(y,x), \quad \forall y. \tag{3.51}$$

Let now $u = ty + (1-t)x$ with $t \in [0,1]$. Then by the convexity of f and h

$$\begin{aligned} 0 = f(u,u) + h(u,u) &\leq t\left[f(u,y) + h(u,y)\right] + (1-t)\left[f(u,x) + h(u,x)\right] \\ &\leq t\left[f(u,y) + h(u,y)\right] + \langle u - x, Ax \rangle \\ &= t\left[f(u,y) + h(u,y) + \langle y - x, Ax \rangle\right]. \end{aligned} \tag{3.52}$$

Passing $t \to 0^+$ we obtain by (f1) and (h1)

$$f(x,y) + h(x,y) + \langle Ax, y - x \rangle \geq 0. \tag{3.53}$$

That is, $x \in MEP(f,h,A)$. At this point we observe that from the definitions of $MEP(f,h,A)$ and T_{r_n}, one has $MEP(f,h,A) = \mathrm{Fix}(T_{r_n}(I - r_n A))$. □

By Propositions 3.3 and 3.8 we can exhibit iterative methods to approximate fixed points of the k-strict pseudo contraction that are also

(1) solution of a system of two variational inequalities VI(C,A) and VI(C,B) ($V_n = T_n = P_C$);

(2) solution of a system of two mixed equilibrium problems ($T_n = T_{r_n}$ and $V_n = T_{\lambda_n}$);

(3) solution of a mixed equilibrium problem and a variational inequality ($T_n = T_{r_n}$ and $V_n = P_C$).

However when the properties of the mapping T_n and V_n are well known, one can prove convergence theorems like Theorem 3.1 without use of Opial's lemma.

In next theorem our purpose is to prove a strong convergence theorem to approximate a fixed point of S that is also a solution of a mixed equilibrium problem and a solution of a variational inequality $VI(C,B)$. One can note that we relax the hypotheses on the convergence of the sequences $(r_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$.

Theorem 3.9. *Let C be a closed convex subset of a real Hilbert space H, let $f,h : C \times C \to \mathbb{R}$ be two bi-functions satisfying (f1)–(f3),(h1)–(h3), and (H). Let $S : C \to C$ be a k-strict pseudocontraction.*

Let A be an α-inverse strongly monotone mapping of C into H and let B be a β-inverse strongly monotone mapping of C into H.

Let us suppose that $F = \mathrm{Fix}(S) \cap MEP(f,h,A) \cap VI(C,B) \neq \emptyset$.

Set $S_k = kI + (1-k)S$, one defines the sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{gathered}
x_1 \in C, \\
C_1 = C, \\
f(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle + \langle Ax_n, y - u_n\rangle \geq 0, \\
z_n = P_C(I - \lambda_n B)u_n, \\
y_n = \alpha_n x_n + (1-\alpha_n)S_k z_n, \\
C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\
x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in N,
\end{gathered} \tag{3.54}$$

where

(i) $(\alpha_n)_{n\in\mathbb{N}} \subset [0, a]$ *with* $a < 1$;

(ii) $(\lambda_n)_{n\in\mathbb{N}} \subset [b, c] \subset (0, 2\beta)$;

(iii) $(r_n)_{n\in\mathbb{N}} \subset [d, e] \subset (0, 2\alpha)$.

Then $(x_n)_{n\in\mathbb{N}}$ *strongly converges to* $x^* = P_F x_1$.

Proof. First of all we observe that by Lemma 3.5 we have that $u_n = T_{r_n}(I - r_n A)x_n$. We can follow the proof of Theorem 3.1 from Steps 1–7. We prove only the following.

Step 10. The set of weak cluster points of $(x_n)_{n\in\mathbb{N}}$ is contained in F.

Let p be a cluster point of x_n; we begin to prove that $p \in MEP(f, h, A)$. We know that

$$f(u_n, y) + h(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.55}$$

and by (f2)

$$h(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n), \quad \forall y \in C. \tag{3.56}$$

Let $(x_{n_j})_{j\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ weakly convergent to p, then by Step 7 $u_{n_j} \rightharpoonup p$ as $j \to +\infty$. Let $\rho_t := ty + (1-t)p$, $t \in]0, 1]$. Then by (3.56)

$$\begin{aligned}
\left\langle \rho_t - u_{n_j}, A\rho_t \right\rangle &= \left\langle \rho_t - u_{n_j}, A\rho_t - Ax_{n_j} \right\rangle + \left\langle Ax_{n_j}, \rho_t - u_{n_j} \right\rangle \\
&\geq \left\langle \rho_t - u_{n_j}, A\rho_t - Ax_{n_j} \right\rangle + f\left(y, u_{n_j}\right) - h\left(u_{n_j}, y\right) - \frac{1}{r_{n_j}}\left\langle y - u_{n_j}, u_{n_j} - x_{n_j} \right\rangle \\
&= \left\langle \rho_t - u_{n_j}, A\rho_t - Au_{n_j} \right\rangle + \left\langle \rho_t - u_{n_j}, Au_{n_j} - Ax_{n_j} \right\rangle \\
&\quad + f\left(y, u_{n_j}\right) - h\left(u_{n_j}, y\right) - \frac{1}{r_{n_j}}\left\langle y - u_{n_j}, u_{n_j} - x_{n_j} \right\rangle \\
&\geq \left\langle \rho_t - u_{n_j}, Au_{n_j} - Ax_{n_j} \right\rangle + f\left(y, u_{n_j}\right) - h\left(u_{n_j}, y\right) - \frac{1}{r_{n_j}}\left\langle y - u_{n_j}, u_{n_j} - x_{n_j} \right\rangle.
\end{aligned} \tag{3.57}$$

Since A is Lipschitz continuous and $\|u_{n_j} - x_{n_j}\| \to 0$ as $j \to +\infty$, we have $\|Au_{n_j} - Ax_{n_j}\| \to 0$ as $j \to +\infty$.

By condition $(f3)$, for $x \in H$ fixed, the function $f(x,\cdot)$ is lower semicontinuos and convex, and thus weakly lower semicontinuous [30].

Since $\|x_n - u_n\| \to 0$, as $n \to \infty$ and by the assumption on r_n we obtain $(u_{n_j} - x_{n_j})/r_{n_j} \to 0$. Then we obtain by (h2)

$$\langle \rho_t - p, A\rho_t \rangle \geq f(y,p) - h(p,y). \tag{3.58}$$

Using (f1), (f3), (h1), (h3) we obtain

$$\begin{aligned} 0 &= f(\rho_t,\rho_t) + h(\rho_t,\rho_t) \leq t f(\rho_t,y) + (1-t) f(\rho_t,p) + t h(\rho_t,y) + (1-t) h(\rho_t,p) \\ &\leq t f(\rho_t,y) + t h(\rho_t,y) + (1-t)\left(f(\rho_t,p) - h(p,\rho_t)\right) \\ &\leq t f(\rho_t,y) + t h(\rho_t,y) + (1-t)\langle \rho_t - p, A\rho_t \rangle \\ &= t\left(f(\rho_t,y) + h(\rho_t,y) + (1-t)\langle y - p, A\rho_t \rangle\right). \end{aligned} \tag{3.59}$$

Consequently

$$f(\rho_t,y) + h(\rho_t,y) + (1-t)\langle y - p, A\rho_t \rangle \geq 0 \tag{3.60}$$

by (f2) and (h2), as $t \to 0$, we obtain $p \in MEP(f,h,A)$.

Now we prove that $p \in VI(C,B)$.

We define the maximal monotone operator

$$Tx = \begin{cases} Bx + N_C x, & \text{if } x \in C, \\ \emptyset, & \text{se } \ x \notin C, \end{cases} \tag{3.61}$$

where $N_C x$ is the normal cone to C at x, that is,

$$N_C x = \{ w \in H : \langle x - u, w \rangle \geq 0, \forall u \in C \}. \tag{3.62}$$

Since $z_n \in C$, by the definition of N_C we have

$$\langle x - z_n, y - Bx \rangle \geq 0. \tag{3.63}$$

But $z_n = P_C(I - \lambda_n B)u_n$, then

$$\langle x - z_n, z_n - (I - \lambda_n B)u_n \rangle \geq 0, \tag{3.64}$$

and hence

$$\left\langle x - z_n, \frac{z_n - u_n}{\lambda_n} + Bu_n \right\rangle \geq 0. \tag{3.65}$$

By (3.63), (3.65), and by the β-inverse monotonicity of B, we obtain

$$\begin{aligned}\left\langle x - z_{n_j}, y\right\rangle &\geq \left\langle x - z_{n_j}, Bx\right\rangle \\ &\geq \left\langle x - z_{n_j}, Bx\right\rangle - \left\langle x - z_{n_j}, \frac{z_{n_j} - u_{n_j}}{\lambda_{n_j}} + Bu_{n_j}\right\rangle \\ &= \left\langle x - z_{n_j}, Bx - Bz_{n_j}\right\rangle + \left\langle x - z_{n_j}, Bz_{n_j} - Bu_{n_j}\right\rangle \\ &\quad - \left\langle x - z_{n_j}, \frac{z_{n_j} - u_{n_j}}{\lambda_{n_j}}\right\rangle.\end{aligned} \tag{3.66}$$

By $\|x_n - z_n\| \to 0$ as $n \to +\infty$ (immediately consequence of Steps 6 and 7), it follows that $z_{n_j} \rightharpoonup p$ as $j \to +\infty$. Then

$$\langle x - p, y\rangle \geq 0, \tag{3.67}$$

moreover, since T is a maximal operator, $0 \in Tp$, that is, $p \in VI(C, B)$.

Finally, to prove that $p \in \mathrm{Fix}(S) = \mathrm{Fix}(S_k)$ we follow Step 8 as in Theorem 3.1.

Since also Step 9 can be followed as in Theorem 3.1, we obtain the claim. □

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