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Research Article

Strong Convergence Theorems of a New General Iterative Process with Meir-Keeler Contractions for a Countable Family of λ_i-Strict Pseudocontractions in q-Uniformly Smooth Banach Spaces

Yanlai Song and Changsong Hu

Department of Mathematics, Hubei Normal University, Huangshi 435002, China

Correspondence should be addressed to Yanlai Song, songyanlai2009@163.com

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We introduce a new iterative scheme with Meir-Keeler contractions for strict pseudocontractions in q-uniformly smooth Banach spaces. We also discuss the strong convergence theorems for the new iterative scheme in q-uniformly smooth Banach space. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E, respectively. Let C be a subset of E, and lrt T be a non-self-mapping of C. We use $F(T)$ to denote the set of fixed points of T.

The norm of a Banach space E is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for all x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. If, for each $y \in S(E)$, the limit (1.1) is uniformly attained for $x \in S(E)$, then the norm of E is said to be uniformly Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if, for each $x \in S(E)$, the limit (1.1) is attained uniformly for $y \in S(E)$. The norm of E is said to be uniformly Fréchet differentiable (or uniformly smooth) if the limit (1.1) is attained uniformly for $x, y \in S(E) \times S(E)$.

Let $\rho_E : [0,1) \to [0,1)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E),\ \|y\| \le t \right\}. \tag{1.2}$$

A Banach space E is said to be uniformly smooth if $\rho_E(t)/t \to 0$ as $t \to 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \le 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$.

By a gauge we mean a continuous strictly increasing function φ defined $\mathbb{R}^+ := [0,\infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty}\varphi(r) = \infty$. We associate with a gauge φ a (generally multivalued) duality map $J_\varphi : E \to E^*$ defined by

$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\}. \tag{1.3}$$

In particular, the duality mapping with gauge function $\varphi(t) = t^{q-1}$ denoted by J_q, is referred to the (generalized) duality mapping. The duality mapping with gauge function $\varphi(t) = t$ denoted by J, is referred to the normalized duality mapping. Browder [1] initiated the study J_φ. Set for $t \ge 0$

$$\Phi(t) = \int_0^t \varphi(r)dr. \tag{1.4}$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|\cdot\|)$ at x. It is well known that if E is smooth, then J_q is single valued, which is denoted by j_q.

The duality mapping J_q is said to be weakly sequentially continuous if the duality mapping J_q is single valued and for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_q(x_n) \overset{*}{\rightharpoonup} J_q(x)$. Every l^p $(1 < p < \infty)$ space has a weakly sequentially continuous duality map with the gauge $\varphi(t) = t^{p-1}$. Gossez and Lami Dozo [2] proved that a space with a weakly continuous duality mapping satisfies Opial's condition. Conversely, if a space satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then it has a weakly continuous duality mapping. We already know that in q-uniformly smooth Banach space, there exists a constant $C_q > 0$ such that

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q, \tag{1.5}$$

for all $x, y \in E$.

Recall that a mapping T is said to be nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C. \tag{1.6}$$

T is said to be a λ-strict pseudocontraction in the terminology of Browder and Petryshyn [3], if there exists a constant $\lambda > 0$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^q - \lambda\|(I-T)x-(I-T)y\|^q, \tag{1.7}$$

for every x, y, and C for some $j_q(x-y) \in J_q(x-y)$. It is clear that (1.7) is equivalent to the following:

$$\langle (I-T)x-(I-T)y, j_q(x-y)\rangle \geq \lambda\|(I-T)x-(I-T)y\|^q. \tag{1.8}$$

The following famous theorem is referred to as the Banach contraction principle.

Theorem 1.1 (Banach [4]). *Let (X,d) be a complete metric space and let f be a contraction on X, that is, there exists $r \in (0,1)$ such that $d(f(x),f(y)) \leq \mathrm{rd}(x,y)$ for all $x,y \in X$. Then f has a unique fixed point.*

Theorem 1.2 (Meir and Keeler [5]). *Let (X,d) be a complete metric space and let ϕ be a Meir-Keeler contraction (MKC, for short) on X, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x,y) < \varepsilon+\delta$ implies $d(\phi(x),\phi(y)) < \varepsilon$ for all $x,y \in X$. Then ϕ has a unique fixed point.*

This theorem is one of generalizations of Theorem 1.1, because contractions are Meir-Keeler contractions.

In a smooth Banach space, we define an operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \geq \overline{\gamma}\|x\|^2, \quad \|aI - bA\| = \sup_{\|x\|\leq 1}\{|\langle (aI-bA)x, J(x)\rangle| : a \in [0,1],\ b \in [0,1]\}, \tag{1.9}$$

where I is the identity mapping and J is the normalized duality mapping.

Attempts to modify the normal Mann's iteration method for nonexpansive mappings and λ-strictly pseudocontractions so that strong convergence is guaranteed have recently been made; see, for example, [6–11] and the references therein.

Kim and Xu [6] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \quad n \geq 0, \end{aligned} \tag{1.10}$$

where T is a nonexpansive mapping of C into itself $u \in C$ is a given point. They proved the sequence $\{x_n\}$ defined by (1.10) converges strongly to a fixed point of T, provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Hu and Cai [12] introduced the following iteration process:

$$
\begin{aligned}
x_1 &= x \in C,\\
y_n &= P_C\left[\beta_n x_n + (1-\beta_n)\sum_{i=1}^{N}\eta_i^{(n)} T_i x_n\right], \qquad (1.11)\\
x_{n+1} &= \alpha_n \gamma f(x_n) + \gamma_n x_n + \left[(1-\gamma_n)I - \alpha_n A\right] y_n, \quad n \geq 1.
\end{aligned}
$$

where T_i is non-self-λ_i-strictly pseudocontraction, f is a contraction and A is a strong positive linear bounded operator in Banach space. They have proved, under certain appropriate assumptions on the sequences $\{\alpha_n\}, \{\gamma_n\}$, and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.11) converges strongly to a common fixed point of a finite family of λ_i-strictly pseudocontractions, which solves some variational inequality.

Question 1. Can Theorem 3.1 of Zhou [8], Theorem 2.2 of Hu and Cai [12] and so on be extended from finite λ_i-strictly pseudocontraction to infinite λ_i-strictly pseudocontraction?

Question 2. We know that the Meir-Keeler contraction (MKC, for short) is more general than the contraction. What happens if the contraction is replaced by the Meir-Keeler contraction?

The purpose of this paper is to give the affirmative answers to these questions mentioned above. In this paper we study a general iterative scheme as follows:

$$
\begin{aligned}
x_1 &= x \in C,\\
y_n &= P_C\left[\beta_n x_n + (1-\beta_n)\sum_{i=1}^{\infty}\eta_i^{(n)} T_i x_n\right], \qquad (1.12)\\
x_{n+1} &= \alpha_n \gamma \phi(x_n) + \gamma_n x_n + \left[(1-\gamma_n)I - \alpha_n A\right] y_n, \quad n \geq 1,
\end{aligned}
$$

where T_n is non-self λ_n-strictly pseudocontraction, ϕ is a MKC contraction and A is a strong positive linear bounded operator in Banach space. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\mu_i^n\}$, that $\{x_n\}$ defined by (1.12) converges strongly to a common fixed point of an infinite family of λ_i-strictly pseudocontractions, which solves some variational inequality.

2. Preliminaries

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (see [13]). *Let $\{x_n\}, \{z_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ which satisfies the following condition: $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)x_n + \beta_n z_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.2 (see Xu [14]). *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that* $\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n$, *where* γ_n *is a sequence in (0, 1) and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.3 (see [15] demiclosedness principle). *Let* C *be a nonempty closed convex subset of a reflexive Banach space* E *which satisfies Opial's condition, and suppose* $T : C \to E$ *is nonexpansive. Then the mapping* $I - T$ *is demiclosed at zero, that is,* $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ *implies* $x = Tx$.

Lemma 2.4 (see [16, Lemmas 3.1, 3.3]). *Let* E *be real smooth and strictly convex Banach space, and* C *be a nonempty closed convex subset of* E *which is also a sunny nonexpansive retraction of* E. *Assume that* $T : C \to E$ *is a nonexpansive mapping and* P *is a sunny nonexpansive retraction of* E *onto* C, *then* $F(T) = F(PT)$.

Lemma 2.5 (see [17, Lemma 2.2]). *Let* C *be a nonempty convex subset of a real* q*-uniformly smooth Banach space* E *and* $T : C \to C$ *be a* λ*-strict pseudocontraction. For* $\alpha \in (0,1)$, *we define* $T_\alpha x = (1-\alpha)x + \alpha Tx$. *Then, as* $\alpha \in (0,\mu]$, $\mu = \min\{1, \{q\lambda/C_q\}^{1/(q-1)}\}$, $T_\alpha : C \to C$ *is nonexpansive such that* $F(T_\alpha) = F(T)$.

Lemma 2.6 (see [12, Remark 2.6]). *When* T *is non-self-mapping, the Lemma 2.5 also holds.*

Lemma 2.7 (see [12, Lemma 2.8]). *Assume that* A *is a strongly positive linear bounded operator on a smooth Banach space* E *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$. *Then,*

$$\|I - \rho A\| \leq 1 - \rho\overline{\gamma}. \tag{2.1}$$

Lemma 2.8 (see [18, Lemma 2.3]). *Let* ϕ *be an MKC on a convex subset* C *of a Banach space* E. *Then for each* $\varepsilon > 0$, *there exists* $r \in (0,1)$ *such that*

$$\|x - y\| \geq \varepsilon \text{ implies } \|\phi x - \phi y\| \leq r\|x - y\| \quad \forall x, y \in C. \tag{2.2}$$

Lemma 2.9. *Let* C *be a closed convex subset of a reflexive Banach space* E *which admits a weakly sequentially continuous duality mapping* J_q *from* E *to* E^*. *Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$ *and* $\phi : C \to C$ *be a MKC,* A *is strongly positive linear bounded operator with coefficient* $\overline{\gamma} > 0$. *Assume that* $0 < \gamma < \overline{\gamma}$. *Then the sequence* $\{x_t\}$ *define by* $x_t = t\gamma\phi(x_t) + (1 - tA)Tx_t$ *converges strongly as* $t \to 0$ *to a fixed point* $\tilde{x}$ *of* T *which solves the variational inequality:*

$$\langle (A - \gamma\phi)\tilde{x}, J_q(\tilde{x} - z)\rangle \leq 0, \quad z \in F(T). \tag{2.3}$$

Proof. The definition of $\{x_t\}$ is well definition. Indeed, from the definition of MKC, we can see MKC is also a nonexpansive mapping. Consider a mapping S_t on C defined by

$$S_t x = t\gamma\phi(x) + (I - tA)Tx, \quad x \in C. \tag{2.4}$$

It is easy to see that S_t is a contraction. Indeed, by Lemma 2.8, we have

$$\begin{aligned}\|S_t x - S_t y\| &\le t\gamma\|\phi(x)-\phi(y)\| + \|(I-tA)(Tx-Ty)\| \\ &\le t\gamma\|\phi(x)-\phi(y)\| + (1-t\overline{\gamma})\|x-y\| \\ &\le t\gamma\|x-y\| + (1-t\overline{\gamma})\|x-y\| \\ &\le \left[1-t(\overline{\gamma}-\gamma)\right]\|x-y\|.\end{aligned} \tag{2.5}$$

Hence, S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma\phi(x_t) + (I-tA)Tx_t. \tag{2.6}$$

We next show the uniqueness of a solution of the variational inequality (2.3). Suppose both $\tilde{x} \in F(T)$ and $\hat{x} \in F(T)$ are solutions to (2.3), not lost generality, we may assume there is a number ε such that $\|\hat{x}-\tilde{x}\| \ge \varepsilon$. Then by Lemma 2.8, there is a number r such that $\|\phi\hat{x}-\phi\tilde{x}\| \le r\|\hat{x}-\tilde{x}\|$. From (2.3), we know

$$\begin{aligned}&\left\langle (A-\gamma\phi)\tilde{x}, J_q(\tilde{x}-\hat{x})\right\rangle \le 0, \\ &\left\langle (A-\gamma\phi)\hat{x}, J_q(\hat{x}-\tilde{x})\right\rangle \le 0.\end{aligned} \tag{2.7}$$

Adding up (2.7) gets

$$\left\langle (A-\gamma\phi)\hat{x} - (A-\gamma\phi)\tilde{x}, J_q(\hat{x}-\tilde{x})\right\rangle \le 0. \tag{2.8}$$

Noticing that

$$\begin{aligned}\left\langle (A-\gamma\phi)\hat{x} - (A-\gamma\phi)\tilde{x}, J_q(\hat{x}-\tilde{x})\right\rangle &= \langle A(\hat{x}-\tilde{x}), J_q(\hat{x}-\tilde{x})\rangle - \gamma\langle \phi\hat{x}-\phi\tilde{x}, J_q(\hat{x}-\tilde{x})\rangle \\ &\ge \overline{\gamma}\|\hat{x}-\tilde{x}\|^q - \gamma\|\phi\hat{x}-\phi\tilde{x}\|\|\hat{x}-\tilde{x}\|^{q-1} \\ &\ge \overline{\gamma}\|\hat{x}-\tilde{x}\|^q - \gamma r\|\hat{x}-\tilde{x}\|^q \\ &\ge (\overline{\gamma}-\gamma r)\|\hat{x}-\tilde{x}\|^q \\ &\ge (\overline{\gamma}-\gamma r)\varepsilon^q \\ &> 0.\end{aligned} \tag{2.9}$$

Therefore $\hat{x} = \tilde{x}$ and the uniqueness is proved. Below, we use $\tilde{x}$ to denote the unique solution of (2.3).

We observe that $\{x_t\}$ is bounded. Indeed, we may assume, with no loss of generality, $t < \|A\|^{-1}$, for all $p \in F(T)$, fixed ε_1, for each $t \in (0,1)$.

Case 1 ($\|x_t - p\| < \varepsilon_1$). In this case, we can see easily that $\{x_t\}$ is bounded.

Case 2 ($\|x_t - p\| \geq \varepsilon_1$). In this case, by Lemmas 2.7 and 2.8, there is a number r_1 such that

$$
\begin{aligned}
\|\phi(x_t) - \phi(p)\| &< r_1 \|x_t - p\|, \\
\|x_t - p\| &= \|t\gamma\phi(x_t) + (I - tA)Tx_t - p\| \\
&= \|t(\gamma\phi(x_t) - Ap) + (I - tA)(Tx_t - p)\| \\
&\leq t\|\gamma\phi(x_t) - Ap\| + (1 - t\overline{\gamma})\|(x_t - p)\| \\
&\leq t\|\gamma\phi(x_t) - \gamma\phi(p)\| + \|\gamma\phi(p) - Ap\| + (1 - t\overline{\gamma})\|x_t - p\| \\
&\leq t\gamma r_1\|x_t - p\| + t\|\gamma\phi(p) - Ap\| + (1 - t\overline{\gamma})\|x_t - p\|,
\end{aligned}
\tag{2.10}
$$

therefore, $\|x_t - p\| \leq \|\gamma\phi(p) - Ap\| / (\overline{\gamma} - \gamma r_1)$. This implies the $\{x_t\}$ is bounded.

To prove that $x_t \to \tilde{x}$ ($\tilde{x} \in F(T)$) as $t \to 0$.

Since $\{x_t\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup x^*$. By $x_t - Tx_t = t(\gamma\phi(x_t) - ATx_t)$. We have $x_{t_n} - Tx_{t_n} \to 0$, as $t_n \to 0$. Since E satisfies Opial's condition, it follows from Lemma 2.3 that $x^* \in F(T)$. We claim

$$
\|x_{t_n} - x^*\| \longrightarrow 0. \tag{2.11}
$$

By contradiction, there is a number ε_0 and a subsequence $\{x_{t_m}\}$ of $\{x_{t_n}\}$ such that $\|x_{t_m} - x^*\| \geq \varepsilon_0$. From Lemma 2.8, there is a number $r_{\varepsilon_0} > 0$ such that $\|\phi(x_{t_m}) - \phi(x^*)\| \leq r_{\varepsilon_0}\|x_{t_m} - x^*\|$, we write

$$
x_{t_m} - x^* = t_m(\gamma\phi(x_{t_m}) - Ax^*) + (I - t_m A)(Tx_{t_m} - x^*), \tag{2.12}
$$

to derive that

$$
\begin{aligned}
\|x_{t_m} - x^*\|^q &= t_m\langle \gamma\phi(x_{t_m}) - Ax^*, J_q(x_{t_m} - x^*)\rangle + \langle (I - t_m A)(Tx_{t_m} - x^*), J_q(x_{t_m} - x^*)\rangle \\
&\leq t_m\langle \gamma\phi(x_{t_m}) - Ax^*, J_q(x_{t_m} - x^*)\rangle + (1 - t_m\overline{\gamma})\|x_{t_m} - x^*\|^q.
\end{aligned}
\tag{2.13}
$$

It follows that

$$
\begin{aligned}
\|x_{t_m} - x^*\|^q &\leq \frac{1}{\overline{\gamma}}\langle \gamma\phi(x_{t_m}) - Ax^*, J_q(x_{t_m} - x^*)\rangle \\
&= \frac{1}{\overline{\gamma}}\left[\langle \gamma\phi(x_{t_m}) - \gamma\phi(x^*), J_q(x_{t_m} - x^*)\rangle + \langle \gamma\phi(x^*) - Ax^*, J_q(x_{t_m} - x^*)\rangle\right] \\
&\leq \frac{1}{\overline{\gamma}}\left[\gamma r_{\varepsilon_0}\|x_{t_m} - x^*\|^q + \langle \gamma\phi(x^*) - Ax^*, J_q(x_{t_m} - x^*)\rangle\right].
\end{aligned}
\tag{2.14}
$$

Therefore,

$$
\|x_{t_m} - x^*\|^q \leq \frac{\langle \gamma\phi(x^*) - Ax^*, J_q(x_{t_m} - x^*)\rangle}{\overline{\gamma} - \gamma r_{\varepsilon_0}}. \tag{2.15}
$$

Using that the duality map J_q is single valued and weakly sequentially continuous from E to E^*, by (2.15), we get that $x_{t_m} \rightarrow x^*$. It is a contradiction. Hence, we have $x_{t_n} \rightarrow x^*$.

We next prove that x^* solves the variational inequality (2.3). Since

$$x_t = t\gamma\phi(x_t) + (I - tA)Tx_t, \tag{2.16}$$

we derive that

$$(A - \gamma\phi)x_t = -\frac{1}{t}(I - tA)(I - T)x_t. \tag{2.17}$$

Notice

$$\begin{aligned}\langle (I-T)x_t - (I-T)z, J_q(x_t - z)\rangle &\geq \|x_t - z\|^q - \|Tx_t - Tz\|\|x_t - z\|^{q-1} \\ &\geq \|x_t - z\|^q - \|x_t - z\|^q \\ &= 0.\end{aligned} \tag{2.18}$$

It follows that, for $z \in F(T)$,

$$\begin{aligned}\langle (A-\gamma\phi)x_t, J_q(x_t - z)\rangle &= -\frac{1}{t}\langle (I-tA)(I-T)x_t, J_q(x_t - z)\rangle \\ &= -\frac{1}{t}\langle (I-T)x_t - (I-T)z, J_q(x_t - z)\rangle + \langle A(I-T)x_t, J_q(x_t - z)\rangle \\ &\leq \langle A(I-T)x_t, J_q(x_t - z)\rangle.\end{aligned} \tag{2.19}$$

Now replacing t in (2.19) with t_n and letting $n \rightarrow \infty$, noticing $(I-T)x_{t_n} \rightarrow (I-T)x^* = 0$ for $x^* \in F(T)$, we obtain $\langle (A - \gamma\phi)x^*, J_q(x^* - z)\rangle \leq 0$. That is, $x^* \in F(T)$ is a solution of (2.3); Hence $\tilde{x} = x^*$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (at $t \rightarrow 0$) equals $\tilde{x}$, therefore, $x_t \rightarrow \tilde{x}$ as $t \rightarrow 0$. □

Lemma 2.10 (see, e.g., Mitrinović [19, page 63]). *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{q/(q-1)}, \tag{2.20}$$

for arbitrary positive real numbers a, b.

Lemma 2.11. *Let E be a q-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping J_q from E to E^* and C be a nonempty convex subset of E. Assume that $T_i : C \rightarrow E$ is a countable family of λ_i-strict pseudocontraction for some $0 < \lambda_i < 1$ and $\inf\{\lambda_i : i \in \mathbb{N}\} > 0$ such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\eta_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} \eta_i = 1$. Then $\sum_{i=1}^{\infty} \eta_i T_i : C \rightarrow E$ is a λ-strict pseudocontraction with $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\}$ and $F(\sum_{i=1}^{\infty} \eta_i T_i) = F$.*

Proof. Let

$$G_n x = \eta_1 T_1 x + \eta_2 T_2 x + \cdots + \eta_n T_n x \tag{2.21}$$

and $\sum_{i=1}^{n} \eta_i = 1$. Then, $G_n : C \to E$ is a λ_i-strict pseudocontraction with $\lambda = \min\{\lambda_i : 1 \le i \le n\}$. Indeed, we can firstly see the case of $n = 2$.

$$\begin{aligned}
&\left\langle (I - G_2)x - (I - G_2)y, J_q(x - y) \right\rangle \\
&\quad = \langle \eta_1 (I - T_1)x + \eta_2 (I - T_2)x - \eta_1 (I - T_1)y - \eta_2 (I - T_2)y, J_q(x - y) \rangle \\
&\quad = \eta_1 \left\langle (I - T_1)x - (I - T_1)y, J_q(x - y) \right\rangle + \eta_2 \left\langle (I - T_2)x - (I - T_2)y, J_q(x - y) \right\rangle \\
&\quad \ge \eta_1 \lambda_1 \left\| (I - T_1)x - (I - T_1)y \right\|^q + \eta_2 \lambda_2 \left\| (I - T_2)x - (I - T_2)y \right\|^q \\
&\quad \ge \lambda \left[\eta_1 \left\| (I - T_1)x - (I - T_1)y \right\|^q + \eta_2 \left\| (I - T_2)x - (I - T_2)y \right\|^q \right] \\
&\quad \ge \lambda \left\| (I - G_2)x - (I - G_2)y \right\|^q,
\end{aligned} \tag{2.22}$$

which shows that $G_2 : C \to E$ is a λ-strict pseudocontraction with $\lambda = \min\{\lambda_i : i = 1, 2\}$. By the same way, our proof method easily carries over to the general finite case.

Next, we prove the infinite case. From the definition of λ-strict pseudocontraction, we know

$$\left\langle (I - T_n)x - (I - T_n)y, J_q(x - y) \right\rangle \ge \lambda \left\| (I - T_n)x - (I - T_n)y \right\|^q. \tag{2.23}$$

Hence, we can get

$$\left\| (I - T_n)x - (I - T_n)y \right\| \le \left(\frac{1}{\lambda} \right)^{1/(q-1)} \left\| x - y \right\|. \tag{2.24}$$

Taking $p \in F(T_n)$, from (2.24), we have

$$\| (I - T_n)x \| = \left\| (I - T_n)x - (I - T_n)p \right\| \le \left(\frac{1}{\lambda} \right)^{1/(q-1)} \left\| x - p \right\|. \tag{2.25}$$

Consquently, for all $x \in E$, if $F = \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$, $\eta_i > 0$ $(i \in \mathbb{N})$ and $\sum_{i=1}^{\infty} \eta_i = 1$, then $\sum_{i=1}^{\infty} \eta_i T_i$ strongly converges. Let

$$Tx = \sum_{i=1}^{\infty} \eta_i T_i x, \tag{2.26}$$

we have

$$Tx = \sum_{i=1}^{\infty} \eta_i T_i x = \lim_{n \to \infty} \sum_{i=1}^{n} \eta_i T_i x = \lim_{n \to \infty} \frac{1}{\sum_{i=1}^{n} \eta_i} \sum_{i=1}^{n} \eta_i T_i x. \tag{2.27}$$

Hence,

$$
\begin{aligned}
&\langle (I-T)x-(I-T)y, J_q(x-y)\rangle \\
&= \lim_{n\to\infty}\left\langle \left(I-\frac{1}{\sum_{i=1}^{n}\eta_i}\sum_{i=1}^{n}\eta_i T_i\right)x+\left(I-\frac{1}{\sum_{i=1}^{n}\eta_i}\sum_{i=1}^{n}\eta_i T_i\right)y, J_q(x-y)\right\rangle \\
&= \lim_{n\to\infty}\frac{1}{\sum_{i=1}^{n}\eta_i}\sum_{i=1}^{n}\eta_i\langle (I-T_i)x-(I-T_i)y, J_q(x-y)\rangle \\
&\geq \lim_{n\to\infty}\frac{1}{\sum_{i=1}^{n}\eta_i}\sum_{i=1}^{n}\eta_i\lambda\|(I-T_i)x-(I-T_i)y\|^q \\
&\geq \lambda\lim_{n\to\infty}\left\|\left(I-\frac{1}{\sum_{i=1}^{n}\eta_i}\sum_{i=1}^{n}\eta_i T_i\right)x-\left(I-\frac{1}{\sum_{i=1}^{n}\eta_i}\sum_{i=1}^{n}\eta_i T_i\right)y\right\|^q \\
&= \lambda\|(I-T)x-(I-T)y\|^q.
\end{aligned}
\tag{2.28}
$$

So, we get T is λ-strict pseudocontraction.

Finally, we show $F(\sum_{i=1}^{\infty}\eta_i T_i)=F$. Suppose that $x=\sum_{i=1}^{\infty}\eta_i T_i x$, it is sufficient to show that $x\in F$. Indeed, for $p\in F$, we have

$$
\begin{aligned}
\|x-p\|^q &= \langle x-p, J_q(x-p)\rangle \\
&= \left\langle \sum_{i=1}^{\infty}\eta_i T_i x-p, J_q(x-p)\right\rangle \\
&= \sum_{i=1}^{\infty}\eta_i\langle T_i x-p, J_q(x-p)\rangle \\
&\leq \|x-p\|^q-\lambda\sum_{i=1}^{\infty}\eta_i\|x-T_i x\|^q,
\end{aligned}
\tag{2.29}
$$

where $\lambda=\inf\{\lambda_i : i\in\mathbb{N}\}$. Hence, $x=T_i x$ for each $i\in\mathbb{N}$, this means that $x\in F$. □

3. Main Results

Lemma 3.1. *Let E be a real q-uniformly smooth, strictly convex Banach space and C be a closed convex subset of E such that $C\pm C\subset C$. Let C be also a sunny nonexpansive retraction of E. Let $\phi : C\to C$ be a MKC. Let $A : C\to C$ be a strongly positive linear bounded operator with the coefficient $\overline{\gamma}>0$ such that $0<\gamma<\overline{\gamma}$ and $T_i : C\to E$ be λ_i-strictly pseudo-contractive non-self-mapping such that $F=\bigcap_{i=1}^{\infty}F(T_i)\neq\emptyset$. Let $\lambda=\inf\{\lambda_i : i\in\mathbb{N}\}>0$. Let $\{x_n\}$ be a sequence of C generated by (1.12) with the sequences $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$, assume for each n, $\{\eta_i^{(n)}\}$ be an infinity sequence of positive number such that $\sum_{i=1}^{\infty}\eta_i^{(n)}=1$ for all n and $\eta_i^{(n)}>0$. The following control conditions are satisfied*

(i) $\sum_{i=1}^{\infty}\alpha_n=\infty, \lim_{n\to\infty}\alpha_n=0$,

(ii) $1-\alpha\leq 1-\beta_n\leq\mu$, $\mu=\min\{1,\{q\lambda/C_q\}^{1/(q-1)}\}$ *for some $\alpha\in(0,1)$ and for all $n\geq 0$,*

(iii) $\lim_{n\to\infty}(\beta_{n+1}-\beta_n)=0$, $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\eta_i^{n+1}-\eta_i^{n}|=0$,

(iv) $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$.

Then, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Proof. Write, for each $n\ge 0$, $B_n=\sum_{i=1}^{\infty}\eta_i^{(n)}T_i$. By Lemma 2.11, each B_n is a λ-strict pseudocontraction on C and $F(B_n)=F$ for all n and the algorithm (1.12) can be rewritten as

$$\begin{gathered} x_1=x\in C,\\ y_n=P_C\left[\beta_n x_n+(1-\beta_n)B_n x_n\right],\\ x_{n+1}=\alpha_n\gamma\phi(x_n)+\gamma_n x_n+\left((1-\gamma_n)I-\alpha_n A\right)y_n,\quad n\ge 1. \end{gathered}\tag{3.1}$$

The rest of the proof will now be split into two parts.

Step 1. First, we show that sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Define a mapping

$$L_n x:=P_C\left[\beta_n x+(1-\beta_n)B_n x\right].\tag{3.2}$$

Then, from the control condition (ii), Lemmas 2.5 and 2.6, we obtain $L_n: C\to C$ is nonexpansive. Taking a point $p\in F$, by Lemma 2.4, we can get $L_n p=p$. Hence, we have

$$\|y_n-p\|=\|L_n x_n-p\|\le\|x_n-p\|.\tag{3.3}$$

From definition of MKC and Lemma 2.8, for each $\varepsilon>0$ there is a number $r_\varepsilon\in(0,1)$, if $\|x_n-z\|<\varepsilon$ then $\|\phi(x_n)-\phi(z)\|<\varepsilon$; If $\|x_n-z\|\ge\varepsilon$ then $\|\phi(x_n)-\phi(z)\|\le r_\varepsilon\|x_n-z\|$. It follow (3.1)

$$\begin{aligned} \|x_{n+1}-p\| &= \left\|\alpha_n\gamma\phi(x_n)+\gamma_n x_n+\left((1-\gamma_n)I-\alpha_n A\right)y_n-p\right\|\\ &= \left\|\alpha_n(\gamma\phi(x_n)-Ap)+\gamma_n(x_n-p)+\left((1-\gamma_n)I-\alpha_n A\right)(y_n-p)\right\|\\ &\le (1-\gamma_n-\alpha_n\overline{\gamma})\|x_n-p\|+\gamma_n\|x_n-p\|+\alpha_n\|\gamma\phi(x_n)-Ap\|\\ &\le (1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n\gamma\max\{r_\varepsilon\|x_n-p\|,\varepsilon\}+\alpha_n\|\gamma\phi(p)-Ap\|\\ &= \max\{(1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n\gamma r_\varepsilon\|x_n-p\|+\alpha_n\|\gamma\phi(p)-Ap\|,\\ &\qquad (1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n\gamma\varepsilon+\alpha_n\|\gamma\phi(p)-Ap\|\}\\ &= \max\{(1-\alpha_n\overline{\gamma}+\alpha_n\gamma r_\varepsilon)\|x_n-p\|+\alpha_n\|\gamma\phi(p)-Ap\|,(1-\alpha_n\overline{\gamma})\|x_n-p\|\\ &\qquad +\alpha_n\gamma\varepsilon+\alpha_n\|\gamma\phi(p)-Ap\|\}\\ &= \max\{[1-(\alpha_n\overline{\gamma}-\alpha_n\gamma r_\varepsilon)]\|x_n-p\|+\alpha_n\|\gamma\phi(p)-Ap\|,(1-\alpha_n\overline{\gamma})\|x_n-p\|\\ &\qquad +\alpha_n\gamma\varepsilon+\alpha_n\|\gamma\phi(p)-Ap\|\}. \end{aligned}\tag{3.4}$$

By induction, we have

$$\|x_n - p\| \le \max\left\{ \|x_0 - p\|, \frac{\|\gamma\phi(p) - Ap\|}{\overline{\gamma} - \gamma r_\varepsilon}, \frac{\gamma\varepsilon + \|\gamma\phi(p) - Ap\|}{\overline{\gamma}} \right\}, \quad n \ge 1, \tag{3.5}$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{L_n x_n\}$.

Step 2. In this part, we shall claim that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. From (3.1), we get

$$x_{n+1} = \alpha_n \gamma\phi(x_n) + \gamma_n x_n + \left[(1-\gamma_n)I - \alpha_n A\right] L_n x_n. \tag{3.6}$$

Define

$$x_{n+1} = (1-\gamma_n) l_n + \gamma_n x_n, \quad \forall n \ge 0, \tag{3.7}$$

where

$$l_n = \frac{x_{n+1} - \gamma_n x_n}{1-\gamma_n}. \tag{3.8}$$

It follows that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}\gamma\phi(x_{n+1}) + \gamma_{n+1}x_{n+1} + \left[(1-\gamma_{n+1})I - \alpha_{n+1}A\right]L_{n+1}x_{n+1} - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\ &\quad - \frac{\alpha_n\gamma\phi(x_n) + \gamma_n x_n + \left[(1-\gamma_n)I - \alpha_n A\right]L_n x_n - \gamma_n x_n}{1-\gamma_n} \\ &= \frac{\alpha_{n+1}\left[\gamma\phi(x_{n+1}) - AL_{n+1}x_{n+1}\right]}{1-\gamma_{n+1}} - \frac{\alpha_n\left[\gamma\phi(x_n) - AL_n x_n\right]}{1-\gamma_n} + L_{n+1}x_{n+1} - L_n x_n, \end{aligned} \tag{3.9}$$

which yields that

$$\begin{aligned} \|l_{n+1} - l_n\| &\le \frac{\alpha_{n+1}\left\|\gamma\phi(x_{n+1}) - AL_{n+1}x_{n+1}\right\|}{1-\gamma_{n+1}} + \frac{\alpha_n\left\|\gamma\phi(x_n) - AL_n x_n\right\|}{1-\gamma_n} + \|L_{n+1}x_{n+1} - L_n x_n\| \\ &\le \frac{\alpha_{n+1}\left\|\gamma\phi(x_{n+1}) - AL_{n+1}x_{n+1}\right\|}{1-\gamma_{n+1}} + \frac{\alpha_n\left\|\gamma\phi(x_n) - AL_n x_n\right\|}{1-\gamma_n} + \|L_{n+1}x_{n+1} - L_{n+1}x_n\| \\ &\quad + \|L_{n+1}x_n - L_n x_n\| \\ &\le \frac{\alpha_{n+1}\left\|\gamma\phi(x_{n+1}) - AL_{n+1}x_{n+1}\right\|}{1-\gamma_{n+1}} + \frac{\alpha_n\left\|\gamma\phi(x_n) - AL_n x_n\right\|}{1-\gamma_n} + \|x_{n+1} - x_n\| \\ &\quad + \|L_{n+1}x_n - L_n x_n\|. \end{aligned} \tag{3.10}$$

Next, we estimate $\|L_{n+1}x_n - L_nx_n\|$. Notice that

$$\begin{aligned}
\|L_{n+1}x_n - L_nx_n\| &= \left\| P_C\left[\beta_{n+1}x_n + (1-\beta_{n+1})B_{n+1}x_n\right] - P_C\left[\beta_nx_n + (1-\beta_n)B_nx_n\right] \right\| \\
&\leq \left\| \left[\beta_{n+1}x_n + (1-\beta_{n+1})B_{n+1}x_n\right] - \left[\beta_nx_n + (1-\beta_n)B_nx_n\right] \right\| \\
&\leq \left|\beta_{n+1} - \beta_n\right| \|x_n - B_{n+1}x_n\| + (1-\beta_n)\|B_{n+1}x_n - B_nx_n\| \\
&\leq \left|\beta_{n+1} - \beta_n\right| \|x_n - B_{n+1}x_n\| + (1-\beta_n)\sum_{i-1}^{\infty}\left|\eta_i^{(n+1)} - \eta_i^{(n)}\right| \|T_ix_n\|.
\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we have

$$\begin{aligned}
\|l_{n+1} - l_n\| \leq{}& \frac{\alpha_{n+1}\left\|\gamma\phi(x_{n+1}) - AL_{n+1}x_{n+1}\right\|}{1-\gamma_{n+1}} + \frac{\alpha_n\left\|\gamma\phi(x_n) - AL_nx_n\right\|}{1-\gamma_n} + \|x_{n+1} - x_n\| \\
&+ \left|\beta_{n+1} - \beta_n\right| \|x_n - B_{n+1}x_n\| + (1-\beta_n)\sum_{i=1}^{\infty}\left|\eta_i^{(n+1)} - \eta_i^{(n)}\right| \|T_ix_n\|.
\end{aligned} \tag{3.12}$$

Hence, we have

$$\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}\left\|\gamma\phi(x_{n+1}) - AL_{n+1}x_{n+1}\right\|}{1-\gamma_{n+1}} + \frac{\alpha_n\left\|\gamma\phi(x_n) - AL_nx_n\right\|}{1-\gamma_n} \\
&+ \|x_n - B_{n+1}x_n\|\left|\beta_{n+1} - \beta_n\right| + (1-\beta_n)\sum_{i=1}^{\infty}\left|\eta_i^{(n+1)} - \eta_i^{(n)}\right| \|T_ix_n\|.
\end{aligned} \tag{3.13}$$

Observing conditions (i), (iii), (iv), and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{T_nx_n\}$, $\{T_ny_n\}$ it follows that

$$\limsup_{n\to\infty}\{\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\} \leq 0. \tag{3.14}$$

Thus by Lemma 2.1, we have $\lim_{n\to\infty}\|l_n - x_n\| = 0$.

From (3.7), we have

$$x_{n+1} - x_n = (1-\gamma_n)(l_n - x_n). \tag{3.15}$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.16}$$

□

Theorem 3.2. *Let E be a real q-uniformly smooth, strictly convex Banach space which admits a weakly sequentially continuous duality mapping J_q from E to E^* and C be a closed convex subset of E which be also a sunny nonexpansive retraction of E such that $C \pm C \subset C$. Let $\phi : C \to C$ be* *a MKC. Let $A : C \to C$ be a strongly positive linear bounded operator with the coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}$ and $T_i : C \to E$ be λ_i-strictly pseudo-contractive non-self-mapping such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$. Let $\{x_n\}$ be a sequence of C generated by* (1.12) *with the sequences $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$, assume for each n, $\Sigma_{i=1}^{\infty}\eta_i^{(n)} = 1$ for all n and $\eta_i^{(n)} > 0$ for all $i \in \mathbb{N}$. They satisfy the conditions (i), (ii), (iii), (iv) of Lemma 3.1 and (v) $\lim_{n\to\infty}\beta_n = \alpha$, $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\eta_i^n - \eta_i| = 0$ and $\sum_{i=1}^{\infty}\eta_i = 1$. Then $\{x_n\}$ converges strongly to $\widetilde{x} \in F$, which also solves the following variational inequality*

$$\left\langle \gamma\phi(\widetilde{x}) - A\widetilde{x}, J_q\left(p - \widetilde{x}\right)\right\rangle \leq 0, \quad \forall p \in F. \tag{3.17}$$

Proof. From (3.1), we obtain

$$\begin{aligned}\|L_n x_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - L_n x_n\| \\ &= \|x_n - x_{n+1}\| + \left\|\alpha_n\gamma\phi(x_n) + \gamma_n(x_n - L_n x_n) - \alpha_n A L_n x_n\right\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\left(\left\|\gamma\phi(x_n)\right\| + \|AL_n x_n\|\right) + \gamma_n\|x_n - L_n x_n\|.\end{aligned} \tag{3.18}$$

So $\|L_n x_n - x_n\| \leq 1/(1-\gamma_n)(\|x_n - x_{n+1}\| + \alpha_n(\|\gamma\phi(x_n)\| + \|AL_n x_n\|)$, which together with the condition (i), (iv) and Lemma 3.1 implies

$$\lim_{n\to\infty}\|L_n x_n - x_n\| = 0. \tag{3.19}$$

Define $B = \sum_{i=1}^{\infty}\eta_i T_i$, then $B : C \to E$ is a λ-strict pseudocontraction such that $F(B) = \bigcap_{i=1}^{\infty} F(T_i) = F$ by Lemma 2.11, furthermore $B_n x \to Bx$ as $n \to \infty$ for all $x \in C$. Defines $T : C \to E$ by

$$Tx = \alpha x + (1-\alpha)Bx. \tag{3.20}$$

Then, T is nonexpansive with $F(T) = F(B)$ by Lemma 2.5. It follows from Lemma 2.4 that $F(P_C T) = F(T) = F$. Notice that

$$\begin{aligned}\|P_C T x_n - x_n\| &\leq \|x_n - L_n x_n\| + \|L_n x_n - P_C T x_n\| \\ &\leq \|x_n - L_n x_n\| + \left\|\beta_n x_n + (1-\beta_n)B_n x_n - [\alpha x_n + (1-\alpha)Bx_n]\right\| \\ &\leq \|x_n - L_n x_n\| + \left\|(\beta_n - \alpha)(x_n - B_n x_n) + (1-\alpha)(B_n x_n - Bx_n)\right\| \\ &\leq \|x_n - L_n x_n\| + (\beta_n - a)\|x_n - B_n x_n\| + (1-\alpha)\|B_n x_n - Bx_n\|\end{aligned} \tag{3.21}$$

which combines with (3.19) yielding that

$$\lim_{n\to\infty}\|P_C T x_n - x_n\| = 0. \tag{3.22}$$

Next, we show that

$$\limsup_{n\to\infty}\left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J_q(x_n - \tilde{x})\right\rangle \leq 0, \tag{3.23}$$

where $\tilde{x} = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto t\gamma\phi(x) + (1 - tA)P_C T x. \tag{3.24}$$

To see this, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J(x_n - \tilde{x})\right\rangle = \lim_{k\to\infty}\left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J(x_{n_k} - \tilde{x})\right\rangle. \tag{3.25}$$

We may also assume that $x_{n_k} \rightharpoonup q$. Note that $q \in F(T)$ in virtue of Lemma 2.3 and (3.22). It follow from the Lemma 2.9 and J_q is weak weakly sequentially continuous duality mapping that

$$\begin{aligned}\limsup_{n\to\infty}\left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J_q(x_n - \tilde{x})\right\rangle &= \lim_{k\to\infty}\left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J_q(x_{n_k} - \tilde{x})\right\rangle \\ &= \left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J_q(q - \tilde{x})\right\rangle \leq 0.\end{aligned} \tag{3.26}$$

Hence, we have

$$\limsup_{n\to\infty}\left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J_q(x_n - \tilde{x})\right\rangle \leq 0. \tag{3.27}$$

Finally, We show $\|x_n - \tilde{x}\| \to 0$. By contradiction, there is a number ε_0 such that

$$\limsup_{n\to\infty}\|x_n - \tilde{x}\| \geq \varepsilon_0. \tag{3.28}$$

Case 1. Fixed ε_1 ($\varepsilon_1 < \varepsilon_0$), if for some $n \geq N \in \mathbb{N}$ such that $\|x_n - \tilde{x}\| \geq \varepsilon_0 - \varepsilon_1$, and for the other $n \geq N \in \mathbb{N}$ such that $\|x_n - \tilde{x}\| < \varepsilon_0 - \varepsilon_1$.

Let

$$M_n = \frac{q\left\langle \gamma\phi(\tilde{x}) - A\tilde{x}, J(x_{n+1} - \tilde{x})\right\rangle}{(\varepsilon_0 - \varepsilon_1)^q}. \tag{3.29}$$

From (3.23), we know $\limsup_{n\to\infty} M_n \le 0$. Hence, there is a number N, when $n > N$, we have $M_n \le \overline{\gamma} - \gamma$. We extract a number $n_0 \ge N$ stastifying $\|x_{n_0} - \widetilde{x}\| < \varepsilon_0 - \varepsilon_1$, then we estimate $\|x_{n_0+1} - \widetilde{x}\|$.

$$\begin{aligned}
\|x_{n_0+1} - \widetilde{x}\|^q &= \left\|\alpha_{n_0}\gamma\phi(x_{n_0}) + \gamma_{n_0}x_{n_0} + \left[(1-\gamma_{n_0})I - \alpha_{n_0}A\right]y_{n_0} - \widetilde{x}\right\|^q \\
&= \left\|\left[(1-\gamma_{n_0})I - \alpha_{n_0}A\right](y_{n_0} - \widetilde{x}) + \alpha_{n_0}(\gamma\phi(x_{n_0}) - A\widetilde{x}) + \gamma_{n_0}(x_{n_0} - \widetilde{x})\right\|^q \\
&= \left\langle \left[(1-\gamma_{n_0})I - \alpha_{n_0}A\right](y_{n_0} - \widetilde{x}) + \alpha_{n_0}(\gamma\phi(x_{n_0}) - A\widetilde{x}) + \gamma_{n_0}(x_{n_0} - \widetilde{x}), J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&= \left\langle \left[(1-\gamma_{n_0})I - \alpha_{n_0}A\right](y_{n_0} - \widetilde{x}), J_q(x_{n_0+1} - \widetilde{x})\right\rangle + \left\langle \alpha_{n_0}(\gamma\phi(x_{n_0}) - A\widetilde{x}), J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&\quad + \left\langle \gamma_{n_0}(x_{n_0} - \widetilde{x}), J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&= \left\langle \left[(1-\gamma_{n_0})I - \alpha_{n_0}A\right](y_{n_0} - \widetilde{x}), J_q(x_{n_0+1} - \widetilde{x})\right\rangle + \alpha_{n_0}\gamma\left\langle \phi(x_{n_0}) - \phi(\widetilde{x}), J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&\quad + \alpha_{n_0}\left\langle \gamma\phi(\widetilde{x}) - A\widetilde{x}, J_q(x_{n_0+1} - \widetilde{x})\right\rangle + \left\langle \gamma_{n_0}(x_{n_0} - \widetilde{x}), J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&\le \left(1 - \gamma_{n_0} - \alpha_{n_0}\overline{\gamma}\right)\|x_{n_0} - \widetilde{x}\|\|x_{n_0+1} - \widetilde{x}\|^{q-1} + \alpha_{n_0}\gamma\left\|\phi(x_{n_0}) - \phi(\widetilde{x})\right\|\|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&\quad + \alpha_{n_0}\left\langle \gamma\phi(\widetilde{x}) - A\widetilde{x}, J_q(x_{n_0+1} - \widetilde{x})\right\rangle + \gamma_{n_0}\|x_{n_0} - \widetilde{x}\|\|x_{n_0+1} - \widetilde{x}\|^{q-1} \\
&< \left[1 - \alpha_{n_0}(\overline{\gamma} - \gamma)\right](\varepsilon_0 - \varepsilon_1)\|x_{n_0+1} - \widetilde{x}\|^{q-1} + \alpha_{n_0}\left\langle \gamma\phi(\widetilde{x}) - A\widetilde{x}, J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&\le \frac{1}{q}\left[1 - \alpha_{n_0}(\overline{\gamma} - \gamma)\right]^q(\varepsilon_0 - \varepsilon_1)^q + \frac{q-1}{q}\|x_{n_0+1} - \widetilde{x}\|^q \\
&\quad + \alpha_{n_0}\left\langle \gamma\phi(\widetilde{x}) - A\widetilde{x}, J_q(x_{n_0+1} - \widetilde{x})\right\rangle \quad \text{by Lemma 2.10,}
\end{aligned} \tag{3.30}$$

which implies that

$$\begin{aligned}
\|x_{n_0+1} - \widetilde{x}\|^q &< \left[1 - \alpha_{n_0}(\overline{\gamma} - \gamma)\right]^q(\varepsilon_0 - \varepsilon_1)^q + q\alpha_{n_0}\left\langle \gamma\phi(\widetilde{x}) - A\widetilde{x}, J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&< \left[1 - \alpha_{n_0}(\overline{\gamma} - \gamma)\right](\varepsilon_0 - \varepsilon_1)^q + q\alpha_{n_0}\left\langle \gamma\phi(\widetilde{x}) - A\widetilde{x}, J_q(x_{n_0+1} - \widetilde{x})\right\rangle \\
&= \left[1 - \alpha_{n_0}(\overline{\gamma} - \gamma - M_n)\right](\varepsilon_0 - \varepsilon_1)^q \\
&\le (\varepsilon_0 - \varepsilon_1)^q.
\end{aligned} \tag{3.31}$$

Hence, we have

$$\|x_{n_0+1} - \widetilde{x}\| < \varepsilon_0 - \varepsilon_1. \tag{3.32}$$

In the same way, we can get

$$\|x_n - \widetilde{x}\| < \varepsilon_0 - \varepsilon_1, \quad \forall n \ge n_0. \tag{3.33}$$

It contradict the $\limsup_{n\to\infty}\|x_n - \widetilde{x}\| \ge \varepsilon_0$.

Case 2. Fixed ε_1 ($\varepsilon_1 < \varepsilon_0$), if $\|x_n - \tilde{x}\| \geq \varepsilon_0 - \varepsilon_1$ for all $n \geq N \in \mathbb{N}$, from Lemma 2.8, there is a number r, $(0 < r < 1)$ such that

$$\left\|\phi(x_n) - \phi(\tilde{x})\right\| \leq r\|x_n - \tilde{x}\|, \quad n \geq N. \tag{3.34}$$

It follow (3.1) that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^q &= \left\|\alpha_n\gamma\phi(x_n) + \gamma_n x_n + \left[(1-\gamma_n)I - \alpha_n A\right]y_n - \tilde{x}\right\|^q \\
&= \left\|\left[(1-\gamma_n)I - \alpha_n A\right](y_n - \tilde{x}) + \alpha_n(\gamma\phi(x_n) - A\tilde{x}) + \gamma_n(x_n - \tilde{x})\right\|^q \\
&= \left\langle \left[(1-\gamma_n)I - \alpha_n A\right](y_n - \tilde{x}) + \alpha_n(\gamma\phi(x_n) - A\tilde{x}) + \gamma_n(x_n - \tilde{x}), J_q(x_{n+1} - \tilde{x})\right\rangle \\
&= \left\langle \left[(1-\gamma_n)I - \alpha_n A\right](y_n - \tilde{x}), J_q(x_{n+1} - \tilde{x})\right\rangle + \left\langle \alpha_n(\gamma\phi(x_n) - A\tilde{x}), J_q(x_{n+1} - \tilde{x})\right\rangle \\
&\quad + \left\langle \gamma_n(x_n - \tilde{x}), J_q(x_{n+1} - \tilde{x})\right\rangle \\
&= \left\langle \left[(1-\gamma_n)I - \alpha_n A\right](y_n - \tilde{x}), J_q(x_{n+1} - \tilde{x})\right\rangle + \left\langle \alpha_n(\gamma\phi(x_n) - \phi(\tilde{x})), J_q(x_{n+1} - \tilde{x})\right\rangle \\
&\quad + \left\langle \alpha_n(\gamma\phi(\tilde{x} - A\tilde{x}), J_q(x_{n+1} - \tilde{x})\right\rangle + \left\langle \gamma_n(x_n - \tilde{x}), J_q(x_{n+1} - \tilde{x})\right\rangle \\
&\leq (1 - \gamma_n - \alpha_n\overline{\gamma})\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\|^{q-1} + \alpha_n\gamma r\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\|^{q-1} \\
&\quad + \alpha_n\langle\gamma\phi(\tilde{x}) - A\tilde{x}, J_q(x_{n+1} - \tilde{x})\rangle + \gamma_n\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\|^{q-1} \\
&= \left[1 - \alpha_n(\overline{\gamma} - \gamma r)\right]\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\|^{q-1} + \alpha_n\langle\gamma\phi(\tilde{x}) - A\tilde{x}, J_q(x_{n+1} - \tilde{x})\rangle \\
&\leq \left[1 - \alpha_n(\overline{\gamma} - \gamma r)\right]\frac{1}{q}\|x_n - \tilde{x}\|^q + \frac{q-1}{q}\|x_{n+1} - \tilde{x}\|^q + \alpha_n\langle\gamma\phi(\tilde{x}) \\
&\quad - A\tilde{x}, J_q(x_{n+1} - \tilde{x})\rangle \quad \text{by Lemma 2.10,}
\end{aligned} \tag{3.35}$$

which implies that

$$\|x_{n+1} - \tilde{x}\|^q \leq \left[1 - \alpha_n(\overline{\gamma} - \gamma r)\right]\|x_n - \tilde{x}\|^q + q\alpha_n\langle\gamma\phi(\tilde{x}) - A\tilde{x}, J_q(x_{n+1} - \tilde{x})\rangle. \tag{3.36}$$

Apply Lemma 2.2 to (3.36) to conclude $x_n \to \tilde{x}$ as $n \to \infty$. It contradict the $\|x_n - \tilde{x}\| \geq \varepsilon_0 - \varepsilon_1$. This completes the proof. □

Corollary 3.3. *Let D be a closed convex subset of a Hilbert space H such that $D \pm D \subset D$ and $f \in D$ with the coefficient $0 < \alpha < 1$. Let $A : C \to C$ be a strongly positive linear bounded operator with the coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}$ and $T_i : C \to E$ be λ_i-strictly pseudo-contractive non-self-mapping such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$. Let $\{x_n\}$ be a sequence of C generated by (1.12) with the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $[0,1]$, assume for each n, $\Sigma_{i=1}^{\infty}\eta_i^{(n)} = 1$ for all n and $\eta_i^{(n)} > 0$ for all $i \in \mathbb{N}$. They satisfy the conditions (i), (ii), (iii), (iv) of Lemma 3.1 and (v)*

$\lim_{n\to\infty}\beta_n = \alpha$, $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\eta_i^n - \eta_i| = 0$ *and* $\sum_{i=1}^{\infty}\eta_i = 1$. *Then* $\{x_n\}$ *converges strongly to* $\tilde{x} \in F$, *which also solves the following variational inequality*

$$\langle \gamma\phi(\tilde{x}) - A\tilde{x}, p - \tilde{x}\rangle \leq 0, \quad \forall p \in F. \tag{3.37}$$

Remark 3.4. We conclude the paper with the following observations.

(i) Theorem 3.2 improve and extends Theorem 3.1 of Zhang and Su [17], Theorem 1 of Yao et al. [11], and Theorem 2.2 of Cai and Hu [12]. Corollary 3.3 also improve and extend Theorem 2.1 of Choa et al. [20], Theorem 2.1 of Jung [21], Theorem 2.1 of Qin et al. [22] and includes those results as special cases. Especially, Our results extends above results form contractions to more general Meir-Keeler contraction (MKC, for short). Our iterative scheme studied in present paper can be viewed as a refinement and modification of the iterative methods in [12, 13, 17, 22]. On the other hand, our iterative schemes concern an infinite countable family of λ_i-strict pseudocontractions mappings, in this respect, they can be viewed as an another improvement.

(ii) The advantage of the results in this paper is that less restrictions on the parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\eta_i^n\}$ are imposed. Our results unify many recent results including the results in [12, 17, 22].

(iii) It is worth noting that we obtained two strong convergence results concerning an infinite countable family of λ_i-strict pseudocontractions mappings. Our result is new and the proofs are simple and different from those in [11, 12, 17, 19–25].

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