

Research Article

Properties WORTH and WORTH*, (1 + δ) Embeddings in Banach Spaces with 1-Unconditional Basis and wFPP

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We will use García-Falset and Lloréns Fuster's paper on the AMC-property to prove that a Banach space X that $(1 + \delta)$ embeds in a subspace X_δ of a Banach space Y with a 1-unconditional basis has the property AMC and thus the weak fixed point property. We will apply this to some results by Cowell and Kalton to prove that every reflexive real Banach space with the property WORTH and its dual have the FPP and that a real Banach space X such that B_{X^*} is w^* sequentially compact and X^* has WORTH* has the wFPP.

1. Introduction

In 1988 Sims [1] introduced the notion of weak orthogonality (WORTH) and asked whether spaces with WORTH have the weak fixed point property (wFPP). Since then several partial answers have been given. For instance, in 1993 García-Falset [2] proved that if X is uniformly nonsquare and has WORTH then it has the wFPP, although Mazcuñán Navarro in her doctoral dissertation [3] showed that uniform nonsquareness is enough. In this work she also showed that WORTH plus 2-UNC implies the wFPP. In both of these cases the space X turns out to be reflexive. In 1994 Sims [4] himself proved that WORTH plus ε_0 -inquadrate in every direction for some $\varepsilon_0 < 2$ implies the wFPP and in 2003 Dalby [5] showed that if X^* has WORTH* and is ε_0 -inquadrate in every direction for some $\varepsilon_0 < 2$, then X has the wFPP.

Recently in 2008 Cowell and Kalton [6] studied properties au and au^* in a Banach space X , where au coincides with WORTH if X is separable and au^* in X coincides with WORTH* in X^* if X is a separable Banach space. Among other things they proved that a real Banach space with au^* embeds almost isometrically in a space with a shrinking 1-unconditional basis and observed that au and au^* are equivalent if X is reflexive.

We proved, using property AMC shown by García-Falset and Lloréns Fuster [7] to imply the wFPP, that spaces that $(1 + \delta)$ embed in a space with a 1-unconditional basis have

the wFPP. Combining this with Cowell and Kalton's results we were able to show that a reflexive real Banach space with WORTH and its dual both have FPP, giving a partial answer to Sims' question. We also showed that a separable space X such that X^* has WORTH* and B_{X^*} is w^* sequentially compact has the wFPP.

2. Notations and Definitions

Let $(X, \|\cdot\|_X)$ be a real Banach space and K a closed nonempty bounded convex subset of X .

Definition 2.1. If $x \in X$, we define

$$R(x, K) = \sup\{\|x - z\|_X : z \in K\}. \quad (2.1)$$

If $x, y \in K$ the set of quasi-midpoints of x and y in K is given by

$$M(x, y) = \left\{ z \in K : \max\left\{ \|z - x\|_X, \|z - y\|_X \right\} \leq \frac{1}{2} \|x - y\|_X \right\}. \quad (2.2)$$

Definition 2.2. $[X]$ is the quotient space $l_\infty(X)/c_0(X)$ endowed with the norm $\|\tilde{z}\| = \limsup_n \|z_n\|_X$ where \tilde{z} is the equivalence class of (z_n) in $l_\infty(X)$, which we also will denote by $[z_n]$. For $x \in X$ we will also denote by x the equivalence class $[(x, x, x, \dots)]$ in $[X]$. If K is as above, let $\tilde{K} = \{[z_n] \in [X] : z_n \in K, n = 1, 2, \dots\}$. If Y is a Banach space and $T_n : X \rightarrow Y$ for $n = 1, 2, \dots$, we define $[T_n] = \tilde{T}, \tilde{T} : [X] \rightarrow [Y]$ by

$$\tilde{T}([z_n]) = [T_n z_n]. \quad (2.3)$$

If $T_n = T$ for $n = 1, 2, \dots$ we denote \tilde{T} by T .

It is known that \tilde{K} is also closed bounded and convex in $[X]$ and that $\|\tilde{T}\| = \limsup_n \|T_n\|$.

Definition 2.3. Let $\mathcal{S}(\mathbb{N})$ be the set of strictly increasing sequences of natural numbers and K a nonempty bounded convex subset of a Banach space X . A sequence (x_n) in K is called equilateral in K , if for every $\beta, \gamma \in \mathcal{S}(\mathbb{N})$ such that $\beta(n) \neq \gamma(n)$ for every $n \in \mathbb{N}$, the following equality holds in $[X]$. If $\tilde{x}_\beta = [x_{\beta(n)}]$ and $\tilde{x}_\gamma = [x_{\gamma(n)}]$, then

$$\|\tilde{x}_\beta\| = \|\tilde{x}_\gamma\| = \|\tilde{x}_\beta - \tilde{x}_\gamma\| = D(x_n) = \text{diam}(K), \quad (2.4)$$

where $D(x_n) = \limsup_n (\limsup_m \|x_n - x_m\|_X)$.

It is easy to see that if (x_n) is equilateral in K , and $\beta, \gamma \in \mathcal{S}(\mathbb{N})$ are as above, then

$$\|\tilde{x}_\beta - \tilde{x}_\gamma\| = \lim_n \|x_{\beta(n)} - x_{\gamma(n)}\|_X = \lim_n \|x_{\beta(n)}\|_X = \lim_n \|x_{\gamma(n)}\|_X. \quad (2.5)$$

Now we define the property which interests us in this paper, it was given by García-Falset and Lloréns Fuster in 1990 [7].

Definition 2.4. A bounded closed convex subset K of a Banach space $(X, \|\cdot\|_X)$ with $0 \in K$ has the AMC property, if for every weakly null sequence (x_n) which is equilateral in K , there exist $\rho \in (0, 1)$, $x \in K$, $\beta, \gamma \in \mathcal{S}(\mathbb{N})$ with $\beta(n) \neq \gamma(n)$ for every $n \in \mathbb{N}$, such that the set

$$M_\rho((x_n), \beta, \gamma) = M(\widetilde{x}_\beta, \widetilde{x}_\gamma) \cap \{[z_n] : d([z_n], K) \leq \rho \operatorname{diam}(K)\} \quad (2.6)$$

is nonempty and $R(x, M_\rho((x_n), \beta, \gamma)) < \operatorname{diam}(K)$. X is said to have AMC if every weakly compact nonempty subset K of X with $0 \in K$ has the AMC property.

3. Embeddings into Spaces with 1-Unconditional Basis and the wFPP

Lin in [8] showed that if X has an unconditional basis (e_n) with unconditional constant $\lambda < (33^{1/2} - 3)/2$, then X has the wFPP. García-Falset and Lloréns Fuster proved that in fact under these conditions X has the AMC property which in turn implies the wFPP. We will follow the proof of this closely to establish the next theorem.

Theorem 3.1. *Let $(X, \|\cdot\|_X)$ be a Banach space and suppose that there exists a Banach space $(Y, \|\cdot\|_Y)$ with a 1-unconditional basis (e_n) and a subspace X_δ of Y such that $d(X, X_\delta) < (1 + \delta)$ where $\delta < (\sqrt{13} - 3)/2$. Then X has AMC and thus the wFPP.*

Proof. Let $S : X \rightarrow X_\delta$ be an isomorphism with $\|S\| \leq 1 + \delta$ and $\|S^{-1}\| \leq 1$. Let $K \subset X$ be a nonempty weakly compact convex subset of X with $0 \in K$ and $\operatorname{diam}(K) = 1$. We will show that K has the AMC property.

Let (x_n) be a weakly null equilateral sequence in K and let $K_\delta = S(K)$. Then K_δ is weakly compact and (Sx_n) is weakly null in Y . Hence there exists a sequence $\beta \in \mathcal{S}(\mathbb{N})$ and projections with respect to the basis (e_n) in Y with

- (a) $P_n : Y \rightarrow \operatorname{sp}(e_{m_n}, e_{m_n+1}, \dots, e_{r_n})$ where $m_n \leq r_n < m_{n+1}$,
- (b) $\lim_n \|P_n y\|_Y = 0$ for all $y \in Y$,
- (c) $\lim_n \|Sx_{\beta(n)} - P_n Sx_{\beta(n)}\|_Y = 0$.

Let $\gamma \in \mathcal{S}(\mathbb{N})$ be given by $\gamma(n) = \beta(n + 1)$. Then clearly $\beta(n) \neq \gamma(n)$ for every $n \in \mathbb{N}$. Let $\tilde{P}, \tilde{Q} : [Y] \rightarrow [Y]$ be given by $\tilde{P} = [P_n]$ and $\tilde{Q} = [P_{n+1}]$ and let $\tilde{S} : [X] \rightarrow [Y]$ be $\tilde{S} = [(S, S, \dots)]$. Recall that we will write S instead of \tilde{S} . By (a), (b), and (c) and since (x_n) is equilateral we have that

- (1) $\|S\tilde{x}_\beta\| \leq 1 + \delta$, $\|S\tilde{x}_\gamma\| \leq 1 + \delta$ and $\|S\tilde{x}_\beta - S\tilde{x}_\gamma\| \leq 1 + \delta$,
- (2) $\tilde{P}S\tilde{x}_\beta = S\tilde{x}_\beta$, $\tilde{Q}S\tilde{x}_\gamma = Sx_\gamma$,
- (3) $\tilde{Q}S\tilde{x}_\beta = 0$, $\tilde{P}S\tilde{x}_\gamma = 0$,
- (4) for all $\tilde{y} \in [Y]$, $\tilde{P}\tilde{y} = \tilde{Q}\tilde{y} = 0$.

Therefore, since (e_n) is 1-unconditional

$$\|S\tilde{x}_\beta + S\tilde{x}_\gamma\| = \|\tilde{Q}S\tilde{x}_\beta + \tilde{P}S\tilde{x}_\gamma\| = \|\tilde{Q}S\tilde{x}_\beta - \tilde{P}S\tilde{x}_\gamma\| = \|S\tilde{x}_\beta - S\tilde{x}_\gamma\| \leq 1 + \delta. \quad (3.1)$$

Thus $\|\tilde{x}_\beta + \tilde{x}_\gamma\| = \|S^{-1}S(\tilde{x}_\beta + \tilde{x}_\gamma)\| \leq 1 + \delta$. Since by hypothesis $\|\tilde{x}_\beta - \tilde{x}_\gamma\| \leq 1$, we obtain that $(\tilde{x}_\beta + \tilde{x}_\gamma)/2 \in M_\rho((x_n), \beta, \gamma)$ if $\delta < 1$ and $\rho = (1 + \delta)/2$. Next we will show that $R(0, M_\rho((x_n), \beta, \gamma)) < 1$.

To this effect let $\tilde{w} = [w_n] \in M_\rho((x_n), \beta, \gamma)$. Define $\tilde{u} = (\tilde{P} + \tilde{Q})S\tilde{w}$ and $\tilde{r} = (I - (\tilde{P} + \tilde{Q}))S\tilde{w}$. Then for every $x \in K$

$$2\tilde{u} = (\tilde{u} + \tilde{r} - Sx) + (\tilde{u} - \tilde{r} + Sx). \quad (3.2)$$

By the unconditionality of (e_n) and since by (4) we have that $\tilde{P}Sx = \tilde{Q}Sx = 0$,

$$\begin{aligned} \|\tilde{u} - \tilde{r} + Sx\| &= \left\| (\tilde{P} + \tilde{Q})(S\tilde{w} + Sx) - (I - (\tilde{P} + \tilde{Q}))(S\tilde{w} - Sx) \right\| \\ &= \left\| (\tilde{P} + \tilde{Q})(S\tilde{w} + Sx) + (I - (\tilde{P} + \tilde{Q}))(S\tilde{w} - Sx) \right\| = \|\tilde{u} + \tilde{r} - Sx\|. \end{aligned} \quad (3.3)$$

Therefore

$$2\|\tilde{u}\| \leq 2\|\tilde{u} + \tilde{r} - Sx\| = 2\|S\tilde{w} - Sx\|. \quad (3.4)$$

Now let $v \in K$ be such that $\|\tilde{w} - v\| \leq \rho$. Such an element exists since $\tilde{w} \in M_\rho((x_n), \beta, \gamma)$. Recalling that $\rho = (1 + \delta)/2$, we obtain that

$$2\|\tilde{u}\| \leq 2\|S\tilde{w} - Sv\| \leq (1 + \delta)^2. \quad (3.5)$$

Using again that $\tilde{w} \in M_\rho((x_n), \beta, \gamma)$, we get

$$\begin{aligned} 2\|\tilde{w}\| &\leq \left\| \tilde{w} + \frac{\tilde{x}_\beta + \tilde{x}_\gamma}{2} \right\| + \left\| \frac{\tilde{w} - \tilde{x}_\beta}{2} \right\| + \left\| \frac{\tilde{w} - \tilde{x}_\gamma}{2} \right\| \\ &\leq \left\| \tilde{w} + \frac{\tilde{x}_\beta + \tilde{x}_\gamma}{2} \right\| + \frac{1}{2} \end{aligned} \quad (3.6)$$

or equivalently

$$\left\| \tilde{w} + \frac{\tilde{x}_\beta + \tilde{x}_\gamma}{2} \right\| \geq 2\|\tilde{w}\| - \frac{1}{2}. \quad (3.7)$$

On the other hand, since by (2) $\tilde{P}Sx_\beta = Sx_\beta$, we have

$$\begin{aligned} \|S\tilde{w} + S\tilde{x}_\beta - 2\tilde{P}S\tilde{w}\| &= \|S\tilde{w} + \tilde{P}S\tilde{x}_\beta - 2\tilde{P}S\tilde{w}\| = \|(I - \tilde{P})S\tilde{w} + \tilde{P}(S\tilde{x}_\beta - S\tilde{w})\| \\ &= \|(I - \tilde{P})S\tilde{w} - \tilde{P}(S\tilde{x}_\beta - S\tilde{w})\| = \|S\tilde{w} - \tilde{P}S\tilde{x}_\beta\| = \|S\tilde{w} - S\tilde{x}_\beta\| \\ &\leq (1 + \delta)\|\tilde{w} - \tilde{x}_\beta\| \leq \frac{1 + \delta}{2}. \end{aligned} \quad (3.8)$$

Similarly

$$\|S\tilde{w} + S\tilde{x}_\gamma - 2\tilde{Q}S\tilde{w}\| \leq \frac{1+\delta}{2}. \quad (3.9)$$

By (3.8) and (3.9), and (3.5) we obtain

$$\begin{aligned} 2\left\|S\tilde{w} + \frac{S\tilde{x}_\beta + S\tilde{x}_\gamma}{2}\right\| &\leq (1+\delta) + 2\left\|(\tilde{P} + \tilde{Q})S\tilde{w}\right\| = (1+\delta) + 2\|\tilde{u}\| \\ &\leq (1+\delta) + (1+\delta)^2. \end{aligned} \quad (3.10)$$

Hence

$$\left\|\tilde{w} + \frac{\tilde{x}_\beta + \tilde{x}_\delta}{2}\right\| \leq \frac{(1+\delta)(2+\delta)}{2}. \quad (3.11)$$

Finally, from (3.7) and (3.11) we have $2\|\tilde{w}\| - 1/2 \leq (1+\delta)(2+\delta)/2$ and

$$\|\tilde{w}\| \leq \frac{\delta^2 + 3\delta + 3}{4}. \quad (3.12)$$

Therefore, if $\delta < (\sqrt{13} - 3)/2$ we conclude that $R(0, M_\rho((x_n), \beta, \gamma)) < 1$ and thus X has the AMC property. \square

Remark 3.2. It is evident that if the space Y has a $(1 + \lambda)$ unconditional basis, if λ is small enough, the above result remains true for some δ .

4. Some Consequences

There has always been the conjecture that a space with property WORTH has the wFPP. We show here that this is correct as long as X is reflexive. We also show that property WORTH* in X^* implies the wFPP in Banach spaces X so that B_{X^*} is w^* sequentially compact and that WORTH together with WABS implies the wFPP as well. All these results are consequences of some theorems by Cowell and Kalton [6]. First we need to recall some definitions.

Definition 4.1. A Banach space X has the WORTH property if for every weakly null sequence $(x_n) \subset X$ and every $x \in X$, the following equality holds:

$$\lim_n (\|x_n - x\| - \|x_n + x\|) = 0. \quad (4.1)$$

This definition was given by Sims in [1]. The next definition was stated by Dalby [5].

Definition 4.2. A Banach space X^* has the WORTH* property if for every weak* null sequence $(x_n^*) \subset X^*$ and every $x^* \in X^*$, the following equality holds:

$$\lim_n (\|x_n^* - x^*\| - \|x_n^* + x^*\|) = 0. \quad (4.2)$$

If X is separable and X^* has WORTH*, this coincides with the property au^* defined in [6].

Definition 4.3. A Banach space X has the Weak Alternating Banach-Saks (WABS) property if every bounded sequence (x_n) in X has a convex block sequence (y_n) such that

$$\lim_n \sup_{r_1 < r_2 < \dots < r_n} \left\| \frac{1}{n} \sum_{j=1}^n (-1)^j y_{r_j} \right\| = 0. \quad (4.3)$$

Cowell and Kalton in [6] proved the following three results.

Theorem 4.4. *If X is a separable real Banach space, then X^* has the property WORTH* if and only if for any $\delta > 0$ there is a Banach space Y with a shrinking 1-unconditional basis and a subspace X_δ of Y such that $d(X, X_\delta) < 1 + \delta$.*

Dalby [5] observed that property WORTH* in a space X^* implies property WORTH in X and it follows that if X is reflexive, then both properties are equivalent. From this and another theorem we are not going to mention here, Cowell and Kalton obtained the next theorem.

Theorem 4.5. *If X is a separable real reflexive space, then X has property WORTH if and only if for any $\delta > 0$ there is a reflexive Banach space Y with a 1-unconditional basis and a subspace X_δ of Y such that $d(X, X_\delta) < 1 + \delta$.*

The third result we are going to use is as follows.

Theorem 4.6. *If X is a separable real Banach space, then X has both the properties WORTH and WABS if and only if for any $\delta > 0$ there is a Banach space Y with a shrinking 1-unconditional basis and a subspace X_δ of Y such that $d(X, X_\delta) < 1 + \delta$.*

From this and our previous work it follows directly the following:

Theorem 4.7. *If X is a real separable space such that either*

- (I) X^* has property WORTH*,
- (II) X is reflexive and has property WORTH, or
- (III) X has both the properties WORTH and WABS,

then X has the property AMC and thus the wFPP.

It is known that reflexivity implies WABS, and thus (II) implies (III), but we want to include (II) in order to deduce the next corollary. Properties WORTH and WABS are inherited by subspaces, and if X^* has property WORTH* and B_{X^*} is w^* sequentially compact, then the dual of any subspace of X also has this property. Hence we have the following result.

Corollary 4.8. *Let X be a real Banach space.*

- (1) *If X is reflexive and has property WORTH, then X and X^* both have the FPP.*
- (2) *If X has properties WORTH and WABS, then X has the wFPP.*
- (3) *If X^* has WORTH* and B_{X^*} is w^* sequentially compact, then X has the wFPP.*

Proof. If X is a Banach space that satisfies (1), (2), or (3), every separable subspace has the wFPP and hence, since the wFPP is separably determined, X has the wFPP. If X is separable and reflexive and has property WORTH, then it has property WORTH* as well and this implies by definition that X^* also has property WORTH. Therefore both have the FPP and hence the result follows for nonseparable reflexive spaces. \square

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