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Research Article

A Continuation Method for Weakly Kannan Maps

David Ariza-Ruiz and Antonio Jiménez-Melado

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain

Correspondence should be addressed to Antonio Jiménez-Melado, melado@uma.es

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The first continuation method for contractive maps in the setting of a metric space was given by Granas. Later, Frigon extended Granas theorem to the class of weakly contractive maps, and recently Agarwal and O'Regan have given the corresponding result for a certain type of quasicontractions which includes maps of Kannan type. In this note we introduce the concept of weakly Kannan maps and give a fixed point theorem, and then a continuation method, for this class of maps.

1. Introduction

Suppose that (X,d) is a metric space and that $f : D \subset X \to X$ is a map. We say that f is contractive if there exists $\alpha \in [0,1)$ such that $d(f(x),f(y)) \leq \alpha d(x,y)$ for all $x,y \in D$. The well-known Banach fixed point theorem states that f has a fixed point if $D = X$ and (X,d) is complete. In 1962, Rakotch [1] obtained an extension of Banach theorem replacing the constant α by a function of $d(x,y)$, $\alpha = \alpha(d(x,y))$, provided that α is nonincreasing and $0 \leq \alpha(t) < 1$ for all $t > 0$ (for a recent refinement of this result see [2]). A similar generalization of the contractive condition was considered by Dugundji and Granas [3], who extended Banach theorem to the class of weakly contractive mappings (i.e., $\alpha = \alpha(x,y)$, with $\sup\{\alpha(x,y) : a \leq d(x,y) \leq b\} < 1$ for all $0 < a \leq b$).

Another focus of attention in Fixed Point Theory is to establish fixed point theorems for non-self mappings. In the setting of a Banach space, Gatica and Kirk [4] proved that if $f : \overline{U} \to X$ is contractive, with U an open neighborhood of the origin, then f has a fixed point if it satisfies the well-known Leray-Schauder condition:

$$f(x) \neq \lambda x, \quad \text{for } x \in \partial U,\ \lambda > 1. \tag{L-S}$$

Recently, Kirk [5] has extended this result to the abstract setting of a certain class of metric spaces: the CAT(0) spaces. In the proof, the author uses a homotopy result due to

Granas [6], which is known as continuation method for contractive maps. In fact, the jump from a Banach space setting to the metric space setting was given by Granas himself in [6] (for more information on this topic see, for instance, [7–9]). After Granas, Frigon [8] gave a similar result for weakly contractive maps.

A variant of the Banach contraction principle was given by Kannan [10], who proved that a map $f : X \to X$, where (X, d) is a complete metric space, has a unique fixed point if f is what we call a Kannan map, that is, there exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \frac{\alpha}{2}\left[d(x, f(x)) + d(y, f(y))\right]. \tag{1.1}$$

In this note, following the pattern of Dugundji and Granas [3], we extend Kannan theorem to the class of weakly Kannan maps (i.e., $\alpha = \alpha(x, y)$, with $\sup\{\alpha(x, y) : a \leq d(x, y) \leq b\} < 1$ for all $0 < a \leq b$). This is done in Section 2. In Section 3 we use a local version of the previous result to obtain a continuation method for weakly Kannan maps.

2. Weakly Kannan Maps

In this section we follow the pattern of Dugundji and Granas [3] to introduce the concept of weakly Kannan maps.

Definition 2.1. Let (X, d) be a metric space, $D \subset X$, and $f : D \to X$. Therefore f is a weakly Kannan map if there exists $\alpha : D \times D \to [0, 1]$, with $\theta(a, b) := \sup\{\alpha(x, y) : a \leq d(x, y) \leq b\} < 1$ for every $0 < a \leq b$ such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \frac{\alpha(x, y)}{2}\left[d(x, f(x)) + d(y, f(y))\right]. \tag{2.1}$$

Remark 2.2. Clearly, any weakly Kannan map f has at most one fixed point: if $x = f(x)$ and $y = f(y)$, then

$$d(x, y) = d(f(x), f(y)) \leq \frac{1}{2}\left[d(x, f(x)) + d(y, f(y))\right] = 0. \tag{2.2}$$

Remark 2.3. Notice that if $f : D \subset X \to X$ is a weakly Kannan map and we define $\alpha_f(x, y)$ on $D \times D$ as

$$\alpha_f(x, y) = \begin{cases} \dfrac{2d(f(x), f(y))}{d(x, f(x)) + d(y, f(y))} & \text{if } d(x, f(x)) + d(y, f(y)) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

then α_f is well defined, takes values in $[0, 1]$, satisfies $\sup\{\alpha_f(x, y) : a \leq d(x, y) \leq b\} < 1$ for all $0 < a \leq b$ (for α_f is smaller than any α associated to f), and also satisfies (2.1), with α replaced by α_f, for all $x, y \in D$. Conversely, if α_f is defined as in (2.3) and satisfies the above set of conditions, then f is a weakly Kannan map, establishing in this way an equivalent definition for Kannan maps.

Remark 2.4. Although Kannan showed that the concept of Kannan map is independent of the concept of contractive map, Janos [11] observed that any contractive map $f : D \subset X \to X$ whose Lipschitz constant defined by

$$L(f) = \sup\left\{ \frac{d(f(x), f(y))}{d(x,y)} : x, y \in X, x \neq y \right\} \tag{2.4}$$

is less than $1/3$ is a Kannan map. Next, we exhibit an example of a weakly Kannan map f, with $L(f) = 1/3$, which is not a Kannan map, thus showing that the constant $1/3$ in the aforementioned result by Janos is sharp.

Example 2.5. Consider the metric space $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$, and let $f : X \to X$ be the function defined as $f(x) = (1/3)\log(1 + e^x)$. Then, $L(f) = 1/3$ and f is a weakly Kannan map, but not a Kannan map.

The equality $L(f) = 1/3$ follows from the fact that $|f'(x)| < 1/3$ for all $x \in [0, \infty)$ together with

$$\lim_{x \to \infty} \frac{d(f(x), f(0))}{d(x, 0)} = \frac{1}{3}. \tag{2.5}$$

We also have that f is not a Kannan map because

$$\lim_{x \to \infty} \frac{2d(f(x), f(0))}{d(x, f(x)) + d(0, f(0))} = 1. \tag{2.6}$$

To check that f is a weakly Kannan map, consider the function $\alpha : X \times X \to [0, \infty)$ given by (2.3). This function is well defined and also takes values in $[0, 1]$ since $L(f) = 1/3$. Next, assume that $0 < a \le b$ and let us see that $\theta(a, b) = \sup\{\alpha(x, y) : \ a \le d(x, y) \le b\} < 1$. To see this, observe that $|u - f(u)| \to \infty$ as $u \to \infty$, so there is $M > 0$ such that $|u - f(u)| > b$ for all $u > M$. Observe also that f_M, the restriction of f to $[0, M]$, is a Kannan map with constant $\alpha_M \in [0, 1)$, due to the fact that $L(f_M) < 1/3$, for f_M is continuously differentiable on $[0, M]$ and $|f'(u)| < 1/3$ for all $u \in [0, M]$. We will see $\theta(a, b) \le \max\{2/3, \alpha_M\}$. To do it, suppose that $x, y \in [0, \infty)$ with $a \le |x - y| \le b$ and $0 \le x < y$. Then, if $y > M$, use $|y - f(y)| > b$ and that $L(f) \le 1/3$ to obtain $\alpha(x, y) \le 2/3$. Otherwise, we would have $0 \le x < y \le M$ and then $\alpha(x, y) \le \alpha_M$.

Although the way we have introduced the concept of weakly Kannan map has been by analogy with the work done by Dugundji and Granas in [3], we would like to mention that this extension may be done in some different ways. For instance, Pathak et al. [12, Theorem 3.1] have proved the following result.

Theorem A. *Let (X, d) be a complete metric space and suppose that $f : X \to X$ is a map such that*

$$d(f(x), f(y)) \le \alpha_1(d(x, f(x)))d(x, f(x)) + \alpha_2(d(y, f(y)))d(y, f(y)), \tag{2.7}$$

for all $x, y \in X$, where $\alpha_i : \mathbb{R} \to [0, 1)$. If, in addition, there exists a sequence $\{x_n\}$ in X with $d(x_n, f(x_n)) \to 0$, then f has a fixed point in X.

Observe that relation (2.7) can be written in the following more general form:

$$d(f(x), f(y)) \leq A_1(x)d(x, f(x)) + A_2(y)d(y, f(y)), \tag{2.8}$$

for all $x, y \in X$, where $A_i : X \to [0, 1)$, $i = 1, 2$, and notice that any map satisfying (2.8) also satisfies the relation (2.1) with $\alpha(x, y) = 2\max\{A_1(x), A_2(y)\}$. In fact, the arguments used by the authors in the proof of Theorem A are also valid for this class of maps. Next, we state this slightly more general result and include the proof for the sake of completeness. Then, we obtain, as a consequence, a fixed point theorem for weakly Kannan maps.

Theorem 2.6. *Let (X, d) be a complete metric space and assume that $A : X \times X \to [0, \infty)$ is a bounded function satisfying the following condition: for any sequence $\{x_n\}$ in X and $u \in X$,*

$$x_n \longrightarrow u \Longrightarrow \limsup A(x_n, u) < 1. \tag{$*$}$$

Assume also that $f : X \to X$ is a map such that

$$d(f(x), f(y)) \leq A(x, y)\left[d(x, f(x)) + d(y, f(y))\right], \tag{2.9}$$

for all $x, y \in X$. If there exists a sequence $\{x_n\}$ in X with $d(x_n, f(x_n)) \to 0$, then f has a unique fixed point u in X, and $x_n \to u$.

Proof. Since A is bounded, there exists $M > 0$ such that $|A(x, y)| \leq M$ for all $x, y \in X$. Suppose that $\{x_n\}$ is a sequence in X with $d(x_n, f(x_n)) \to 0$ and use (2.9) to obtain that, for all $n, m \in \mathbb{N}$,

$$d(f(x_n), f(x_m)) \leq M\left[d(x_n, f(x_n)) + d(x_m, f(x_m))\right]. \tag{2.10}$$

This implies that $\{f(x_n)\}$ is a Cauchy sequence. Since (X, d) is complete, the sequence $\{f(x_n)\}$ is convergent, say to $u \in X$. Then $x_n \to u$ because $d(x_n, f(x_n)) \to 0$. Thus, by $(*)$, $\limsup A(x_n, u) < 1$.

That $u = f(u)$ is a consequence of the following relation and the fact that $\limsup A(x_n, u) < 1$, then

$$\begin{aligned} d(u, f(u)) &= \lim d(f(x_n), f(u)) \leq \limsup A(x_n, u)\left[d(x_n, f(x_n)) + d(u, f(u))\right] \\ &= d(u, f(u)) \limsup A(x_n, u). \end{aligned} \tag{2.11}$$

Finally, u is the unique fixed point of f because if $z = f(z)$:

$$d(u, z) = d(f(u), f(z)) \leq A(u, z)\left[d(u, f(u)) + d(z, f(z))\right] = 0. \tag{2.12}$$

□

Corollary 2.7. *Let (X, d) be a complete metric space and suppose that $f : X \to X$ is a weakly Kannan map. Then, f has a unique fixed point $u \in X$ and, for any $x_0 \in X$, the sequence of iterates $\{f^n(x_0)\}$ converges to u.*

Proof. Since f is a weakly Kannan map, there exists a function $\alpha : X \times X \to [0,1]$ with $\theta(a,b) := \sup\{\alpha(x,y) : a \leq d(x,y) \leq b\} < 1$ for all $0 < a \leq b$, satisfying (2.1) for all $x, y \in X$. Hence, the function $A : X \times X \to [0,1/2]$ given as $A(x,y) = (1/2)\alpha(x,y)$ is bounded and satisfies the conditions $(*)$ and (2.9).

Consider any $x_0 \in X$ and define $x_n = f(x_{n-1})$, $n = 1,2,\ldots$. We may assume that $d(x_0,x_1) > 0$ because otherwise we have finished. We will prove that $d(x_n, f(x_n)) \to 0$ and hence, by Theorem 2.6, $\{x_n\}$ will converge to a point u which is the unique fixed point of f.

First of all, observe that the inequality

$$d(x_{n+1},x_n) \leq \alpha(x_n,x_{n-1})d(x_n,x_{n-1}) \tag{2.13}$$

holds for all $n \geq 1$. In fact, it is a consequence of the following one, which is true by (2.1):

$$d(x_{n+1},x_n) \leq \frac{\alpha(x_n,x_{n-1})}{2}\left[d(x_{n+1},x_n) + d(x_n,x_{n-1})\right]. \tag{2.14}$$

From (2.13) we obtain that the sequence $\{d(x_n,x_{n-1})\}$ is nonincreasing, for $0 \leq \alpha(x_n,x_{n-1}) \leq 1$, and then it is convergent to the real number

$$d = \inf\{d(x_n,x_{n-1}) : n = 1,2,\ldots\}. \tag{2.15}$$

To prove that $d = 0$, suppose that $d > 0$ and arrive to a contradiction as follows: use

$$0 < d \leq d(x_n,x_{n-1}) \leq d(x_1,x_0) \tag{2.16}$$

and the definition of $\theta = \theta(d, d(x_1,x_0))$ to obtain $\alpha(x_n,x_{n-1}) \leq \theta$ for all $n = 1,2,\ldots$. This, together with (2.13), gives that

$$d \leq d(x_{n+1},x_n) \leq \theta^n d(x_1,x_0), \tag{2.17}$$

for all $n = 1,2,\ldots$, which is impossible since $d > 0$ and $0 \leq \theta < 1$. □

Remark 2.8. We do not know whether Theorem A is, or not, a particular case of Theorem 2.6, although that is the case if the functions α_1, α_2 satisfy the additional assumption $\sup\{\alpha_1(t) + \alpha_2(t) : t \geq 0\} < 2$. To see this, suppose that the map $f : X \to X$ is in the conditions of Theorem A, that is, f satisfies relation (2.7) for some given functions $\alpha_i : \mathbb{R} \to [0,1)$, $i = 1,2$, and suppose also that the functions α_1, α_2 satisfy in addition $\sup\{\alpha_1(t) + \alpha_2(t) : t \geq 0\} < 2$. Define $A : X \times X \to [0,\infty)$ as $A(x,y) = \max\{a(x), a(y)\}$, where $a : X \to [0,1)$ is given by

$$a(z) = \frac{1}{2}\left[\alpha_1(d(z,f(z))) + \alpha_2(d(z,f(z)))\right]. \tag{2.18}$$

Let us see that, with this function A, f satisfies the hypotheses of Theorem 2.6. Indeed, A is clearly bounded and also satisfies $(*)$; if $\{x_n\}$ is a sequence in X and $u \in X$, with $x_n \to u$, then

$$\sup\{a(x_n) : n = 1,2,\dots\} \leq \sup\left\{\frac{\alpha_1(t)+\alpha_2(t)}{2} : t \geq 0\right\} < 1. \tag{2.19}$$

Since we also have that $a(u) < 1$, we obtain that $\sup\{A(x_n,u) : n = 1,2,\dots\} < 1$.

Finally, to see that f satisfies relation (2.9), use relation (2.7) with $x, y \in X$, together with the same relation interchanging the roles of x and y, and the fact that $d(f(x), f(y)) = d(f(y), f(x))$, to obtain that

$$d(f(x), f(y)) \leq a(x)d(x, f(x)) + a(y)d(y, f(y)), \tag{2.20}$$

from which the result follows.

To prove the homotopy result of the next section, we will need the following local version of Corollary 2.7.

Corollary 2.9. *Assume that* (X,d) *is a complete metric space,* $x_0 \in X$, $r > 0$, *and* $f : \overline{B(x_0,r)} \to X$ *is a weakly Kannan map with associated function* α *satisfying* (2.1). *If* θ *is defined as usual, and*

$$d(x_0, f(x_0)) < \frac{1}{3}\min\left\{\frac{r}{2}, r\left[1-\theta\left(\frac{r}{2}, r\right)\right]\right\}, \tag{2.21}$$

then f *has a fixed point.*

Proof. In view of Corollary 2.7, it suffices to show that the closed ball $\overline{B(x_0,r)}$ is invariant under f. To prove it, consider any $x \in \overline{B(x_0,r)}$ and obtain the relation

$$\begin{aligned} d(x_0, f(x)) &\leq d(x_0, f(x_0)) + d(f(x_0), f(x)) \\ &\leq d(x_0, f(x_0)) + \frac{\alpha(x_0,x)}{2}\left[d(x_0, f(x_0)) + d(x, f(x))\right] \\ &\leq d(x_0, f(x_0)) + \frac{\alpha(x_0,x)}{2}\left[d(x_0, f(x_0)) + d(x, x_0) + d(x_0, f(x))\right], \end{aligned} \tag{2.22}$$

from which, having in mind that $\alpha(x_0, x) \leq 1$,

$$d(x_0, f(x)) \leq 3d(x_0, f(x_0)) + \alpha(x_0,x)d(x_0,x). \tag{2.23}$$

To end the proof, obtain that $d(x_0, f(x)) \leq r$ through the above inequality by considering two cases: if $d(x_0,x) \leq r/2$, then $d(x_0, f(x)) \leq r$ because $d(x_0, f(x_0)) \leq r/6$. Otherwise, we would have $r/2 \leq d(x_0,x) \leq r$, and consequently $\alpha(x_0,x) \leq \theta(r/2, r)$, from which

$$d(x_0, f(x)) \le r\left[1 - \theta\left(\frac{r}{2}, r\right)\right] + r\theta\left(\frac{r}{2}, r\right) = r. \tag{2.24}$$

□

3. A Homotopy Result

In 1974 Ćirić [13] introduced the concept of quasicontractions and proved the following fixed point theorem: suppose that (X, d) is a complete metric space and that $f : X \to X$ is a quasicontraction, that is, there exists $q \in [0, 1)$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \le q \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}. \tag{3.1}$$

Then, f has a fixed point in X.

Observe that any contractive map, as well as any Kannan map, is a quasicontraction; thus, the theorem by Ćirić generalizes the well known fixed point theorems by Banach and Kannan.

On the other hand, Agarwal and O'Regan [14] considered a certain class of quasi-contractions: those maps $f : X \to X$, where (X, d) is a metric space, for which there exists $q \in (0, 1)$ such that, for all $x, y \in X$,

$$\begin{aligned} &d(f(x), f(y)) \\ &\quad \le q \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}[d(x, f(y)) + d(y, f(x))]\right\}, \end{aligned} \tag{Q}$$

and gave the following homotopy result.

Theorem B. *Let (X, d) be a complete metric space, U an open subset of X, and $H : \overline{U} \times [0, 1] \to X$ satisfying the following properties:*

(i) *$H(x, \lambda) \neq x$ for all $x \in \partial U$ and all $\lambda \in [0, 1]$,*

(ii) *there exists $q \in (0, 1)$ such that for all $x, y \in \overline{U}$ and $\lambda \in [0, 1]$ we have*

$$\begin{aligned} &d(H(x, \lambda), H(y, \lambda)) \\ &\quad \le q \max\left\{d(x, y), d(x, H(x, \lambda)), d(y, H(y, \lambda)), \frac{1}{2}[d(x, H(y, \lambda)), d(y, H(x, \lambda))]\right\}, \end{aligned} \tag{3.2}$$

(iii) *$H(x, \lambda)$ is continuous in λ, uniformly for $x \in \overline{U}$.*

If $H(\cdot, 0)$ has a fixed point in U, then $H(\cdot, \lambda)$ also has a fixed point in U for all $\lambda \in [0, 1]$.

The above homotopy result includes the corresponding one for the class of Kannan maps, and in the following theorem we show that an analogous result is true for the wider class of weakly Kannan maps.

Theorem 3.1. *Let (X,d) be a complete metric space, U an open subset of X, and $H:\overline{U}\times[0,1]\to X$ satisfying the following properties:*

(P1) *$H(x,\lambda)\neq x$ for all $x\in\partial U$ and all $\lambda\in[0,1]$,*

(P2) *there exists $\alpha:\overline{U}\times\overline{U}\to[0,1]$ such that for all $x,y\in\overline{U}$ and $\lambda\in[0,1]$ one has*

$$d(H(x,\lambda),H(y,\lambda))\leq\frac{\alpha(x,y)}{2}\left[d(x,H(x,\lambda))+d(y,H(y,\lambda))\right],\tag{3.3}$$

and $\theta(a,b)=\sup\{\alpha(x,y):a\leq d(x,y)\leq b\}<1$ for all $0<a\leq b$,

(P3) *there exists a continuous function $\phi:[0,1]\to\mathbb{R}$ such that, for every $x\in\overline{U}$ and $t,s\in[0,1]$, $d(H(x,t),H(x,s))\leq|\phi(t)-\phi(s)|$.*

If $H(\cdot,0)$ has a fixed point in U, then $H(\cdot,\lambda)$ also has a fixed point in U for all $\lambda\in[0,1]$.

Proof. Consider the nonempty set

$$A=\{\lambda\in[0,1]:H(x,\lambda)=x\text{ for some }x\in U\}.\tag{3.4}$$

We will prove that $A=[0,1]$, and for this it suffices to show that A is both closed and open in $[0,1]$.

We start showing that A is closed in $[0,1]$: suppose that $\{\lambda_n\}$ is a sequence in A converging to $\lambda\in[0,1]$ and let us show that $\lambda\in A$. By definition of A, there exists a sequence $\{x_n\}$ in U with $x_n=H(x_n,\lambda_n)$. We will prove that $\{x_n\}$ converges to a point $x_0\in U$ with $H(x_0,\lambda)=x_0$, thus showing that $\lambda\in A$.

That $\{x_n\}$ is a Cauchy sequence is a consequence of the following relation, where we have used (P2), (P3), and the fact that $x_m=H(x_m,\lambda_m)$:

$$\begin{aligned}
d(x_n,x_m)&=d(H(x_n,\lambda_n),H(x_m,\lambda_m))\\
&\leq d(H(x_n,\lambda_n),H(x_n,\lambda_m))+d(H(x_n,\lambda_m),H(x_m,\lambda_m))\\
&\leq\left|\phi(\lambda_n)-\phi(\lambda_m)\right|+\frac{\alpha(x_n,x_m)}{2}\left[d(x_n,H(x_n,\lambda_m))+d(x_m,H(x_m,\lambda_m))\right]\\
&=\left|\phi(\lambda_n)-\phi(\lambda_m)\right|+\frac{\alpha(x_n,x_m)}{2}d(H(x_n,\lambda_n),H(x_n,\lambda_m))\\
&\leq\frac{3}{2}\left|\phi(\lambda_n)-\phi(\lambda_m)\right|.
\end{aligned}\tag{3.5}$$

Write $x_0 = \lim x_n$ and let us see that $x_0 \in U$ and also that $x_0 = H(x_0, \lambda)$. That $x_0 = H(x_0, \lambda)$ is a consequence of the following relation:

$$\begin{aligned} d(x_0, H(x_0,\lambda)) &\le d(x_0, x_n) + d(x_n, H(x_0,\lambda)) \\ &\le d(x_0,x_n) + d(H(x_n,\lambda_n), H(x_n,\lambda)) + d(H(x_n,\lambda), H(x_0,\lambda)) \\ &\le d(x_0,x_n) + \left|\phi(\lambda_n) - \phi(\lambda)\right| + \frac{1}{2}[d(x_n, H(x_n,\lambda)) + d(x_0, H(x_0,\lambda))] \\ &\le d(x_0,x_n) + \frac{3}{2}\left|\phi(\lambda_n) - \phi(\lambda)\right| + \frac{1}{2} d(x_0, H(x_0,\lambda)), \end{aligned} \tag{3.6}$$

and that $x_0 \in U$ is straightforward from (P1).

Next we prove that A is open in $[0,1]$: suppose that $\lambda_0 \in A$ and let us show that $(\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1] \subset A$, for some $\delta > 0$. Since $\lambda_0 \in A$, there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Consider $r > 0$ with $\overline{B(x_0, r)} \subset U$ and use the continuity of ϕ to obtain $\delta > 0$ such that

$$\left|\phi(\lambda) - \phi(\lambda_0)\right| < \min\left\{\frac{r}{2}, r\left[1 - \theta\left(\frac{r}{2}, r\right)\right]\right\}, \tag{3.7}$$

for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$.

To show now that any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$ is also in A, it suffices to prove that the map $H(\cdot, \lambda) : \overline{B(x_0, r)} \to X$ has a fixed point. And this is true by Corollary 2.9, since

$$\begin{aligned} d(x_0, H(x_0,\lambda)) &= d(H(x_0,\lambda_0), H(x_0,\lambda)) \\ &\le \left|\phi(\lambda_0) - \phi(\lambda)\right| \\ &< \min\left\{\frac{r}{2}, r\left[1 - \theta\left(\frac{r}{2}, r\right)\right]\right\}. \end{aligned} \tag{3.8}$$

□

Remark 3.2. A careful reading of the proof shows that hypothesis (P3) in Theorem 3.1 can be easily replaced by the weaker hypothesis (iii) in Theorem B.

Remark 3.3. The counterpart to Theorem 3.1 for weakly contractive maps was proved by Frigon [8]. In that result, it was assumed, in place of our (3.3), an equivalent formulation of the following condition (H'):

$$d\big(H(x,\lambda), H(y,\lambda)\big) \le \alpha(x,y) d(x,y). \tag{H'}$$

Observe that condition (H') means that all the maps $H(\cdot, \lambda) : \overline{U} \to X$, $\lambda \in [0,1]$ are weakly contractive, and with the same function α. Our condition (3.3) is no surprise then. It also means that all the maps $H(\cdot, \lambda)$ are of weakly Kannan type, and with the same function α.

We end the section with an example of a homotopy H satisfying (P1), (P2), and (P3) but not the hypotheses of Theorem B. In fact, the function $f = H(\cdot, 1)$ will be of weakly Kannan type, but will not satisfy the quasicontractivity condition (Q) (hence, it will not be of Kannan type since any Kannan map satisfies (Q)). Moreover, f will not be of weakly contractive type.

Example 3.4. Consider the metric space (X, d), where $X = [-1, 1]$ and $d(x, y) = |x - y|$, and let $f : [-1, 1] \to [-1, 1]$ be the map given as

$$f(x) = \begin{cases} -\sin(x), & -1 \leq x < 1, \\ 0, & x = 1. \end{cases} \tag{3.9}$$

First of all, we will see that the map f does not satisfy condition (Q). Define, for $x, y \in [-1, 1]$,

$$\beta(x, y) = \max\left\{ d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}[d(x, f(y)) + d(y, f(x))] \right\}. \tag{3.10}$$

Then, for $x \in (0, 1)$, we have that $\beta(x, -x) = 2|x|$, since $|\sin(x)| \leq |x|$. Hence,

$$\lim_{x \to 0} \frac{d(f(x), f(-x))}{\beta(x, -x)} = \lim_{x \to 0} \frac{|\sin(x)|}{|x|} = 1, \tag{3.11}$$

showing that no $q \in (0, 1)$ can be found to satisfy (Q).

Secondly, observe that f is not weakly contractive, since any weakly contractive map is continuous.

Next, let us check that f is a weakly Kannan map. Since f has 0 as unique fixed point then, the function $\alpha : [-1, 1] \times [-1, 1] \to [0, \infty)$ given by $\alpha(x, y) = 2d(f(x), f(y))/(d(x, f(x)) + d(y, f(y)))$ if $(x, y) \neq (0, 0)$, $\alpha(0, 0) = 0$, is well defined. We have to check that α only takes values in $[0, 1]$ and that $\theta(a, b) = \sup\{\alpha(x, y) : a \leq |x - y| \leq b, x, y \in [-1, 1]\} < 1$ for all $0 < a \leq b$. In fact, all this will follow if we just show that, for $0 < a \leq 2$,

$$\theta(a, 2) \leq \max\left\{ \frac{2}{3}, 1 - \frac{a}{8}\left(1 - \cos\left(\frac{a}{4}\right)\right) \right\}. \tag{3.12}$$

Thus, take $0 < a \leq 2$ and assume that $x, y \in [-1, 1]$, with $a \leq |x - y|$. If any of the points x, y equals 1, for example $y = 1$, then use $|x + \sin(x)| = |x| + |\sin(x)|$ and $|\sin(x)| \leq |x|$ to obtain that

$$\alpha(x, y) = \frac{2|\sin(x)|}{|x + \sin(x)| + 1} = \frac{2|\sin(x)|}{|x| + |\sin(x)| + 1} \leq \frac{2}{3}. \tag{3.13}$$

Otherwise, we would have that $x, y \in [-1, 1)$. In this case, since $|x - y| \geq a$, then we may assume additionally that $|x| \geq a/2$, and we claim that

$$\alpha(x, y) \leq 1 - \frac{a}{8}\left(1 - \cos\left(\frac{a}{4}\right)\right). \tag{3.14}$$

To be convinced of this, check the following chain of inequalities having in mind that $|z+\sin(z)| = |z| + |\sin(z)|$ for all $z \in [-1,1]$, that $|\sin(z)| \le |z|$, and also that $|\cos(x/2)| \le \cos(a/4)$:

$$\begin{aligned}
\alpha(x,y) &= \frac{2\left|\sin(x)-\sin(y)\right|}{|x+\sin(x)| + \left|y+\sin(y)\right|} \\
&\le \frac{2|\sin(x)| + 2\left|\sin(y)\right|}{|x| + |\sin(x)| + \left|y\right| + \left|\sin(y)\right|} \\
&= 1 - \frac{|x| - |\sin(x)| + \left|y\right| - \left|\sin(y)\right|}{|x| + |\sin(x)| + \left|y\right| + \left|\sin(y)\right|} \\
&\le 1 - \frac{|x| - |\sin(x)|}{4} \\
&\le 1 - \frac{1}{4}\left(|x| - 2\left|\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right|\right) \\
&\le 1 - \frac{|x|}{4}\left(1 - \left|\cos\left(\frac{x}{2}\right)\right|\right) \\
&\le 1 - \frac{a}{8}\left(1 - \left|\cos\left(\frac{a}{4}\right)\right|\right).
\end{aligned} \tag{3.15}$$

Next, define $H : [-1,1] \times [0,1] \to [-1,1]$ by $H(x,\lambda) = \lambda f(x)$ and let us see that H satisfies (P1), (P2), and (P3).

It is obvious that H satisfies (P1). To check (P2), observe that

$$\left|x - \lambda f(x)\right| = |x| + \left|\lambda f(x)\right|, \tag{3.16}$$

for all $\lambda \in [0,1]$ and all $x \in [-1,1]$, and hence, if $\alpha(x,y)$ is the function previously defined, we have that, for all $\lambda \in [0,1]$ and all $x, y \in [-1,1]$,

$$\begin{aligned}
d(H(x,\lambda), H(y,\lambda)) &= \lambda\left|f(x) - f(y)\right| \\
&\le \lambda\frac{\alpha(x,y)}{2}\left[\left|x - f(x)\right| + \left|y - f(y)\right|\right] \\
&= \lambda\frac{\alpha(x,y)}{2}\left[|x| + \left|f(x)\right| + \left|y\right| + \left|f(y)\right|\right] \\
&\le \frac{\alpha(x,y)}{2}\left[|x| + \left|\lambda f(x)\right| + \left|y\right| + \left|\lambda f(y)\right|\right] \\
&= \frac{\alpha(x,y)}{2}\left[\left|x - \lambda f(x)\right| + \left|y - \lambda f(y)\right|\right] \\
&= \frac{\alpha(x,y)}{2}\left[d(x, H(x,\lambda)) + d(y, H(y,\lambda))\right].
\end{aligned} \tag{3.17}$$

Finally, (P3) is trivially satisfied with $\phi(t) = t$.

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