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Research Article

Fixed Point Theorems on Spaces Endowed with Vector-Valued Metrics

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The purpose of this work is to present some (local and global) fixed point results for singlevalued and multivalued generalized contractions on spaces endowed with vector-valued metrics. The results are extensions of some theorems given by Perov (1964), Bucur et al. (2009), M. Berinde and V. Berinde (2007), O'Regan et al. (2007), and so forth.

1. Introduction

The classical Banach contraction principle was extended for contraction mappings on spaces endowed with vector-valued metrics by Perov in 1964 (see [1]).

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

$(d1)$ $d(x,y) \geq 0$ for all $x, y \in X$; if $d(x,y) = 0$, then $x = y$;

$(d2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;

$(d3)$ $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

If $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$, and $c \in \mathbb{R}$, by $\alpha \leq \beta$ (resp., $\alpha < \beta$) we mean that $\alpha_i \leq \beta_i$ (resp., $\alpha_i < \beta_i$) for $i \in \{1, 2, \ldots, m\}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$ for $i \in \{1, 2, \ldots, m\}$.

A set X equipped with a vector-valued metric d is called a generalized metric space. We will denote such a space with (X, d). For the generalized metric spaces, the notions of convergent sequence, Cauchy sequence, completeness, open subset, and closed subset are similar to those for usual metric spaces.

If (X,d) is a generalized metric space, $x_0 \in X$ and $r = (r_i)_{i=1}^m \in \mathbb{R}^m$, with $r_i > 0$ for each $i \in \{1,2,\dots,m\}$, then we will denote by

$$B(x_0,r) := \{x \in X \mid d(x_0,x) < r\} \tag{1.1}$$

the open ball centered in x_0 with radius r, by $\overline{B(x_0,r)}$ the closure (in (X,d)) of the open ball, and by

$$\widetilde{B}(x_0,r) := \{x \in X \mid d(x_0,x) \le r\} \tag{1.2}$$

the closed ball centered in x_0 with radius r.

If $f : X \to X$ is a singlevalued operator, then we denote by $\mathrm{Fix}(f)$ the set of all fixed points of f; that is, $\mathrm{Fix}(f) := \{x \in X \mid x = f(x)\}$.

For the multivalued operators we use the following notations:

$$\begin{gathered} P(X) := \{Y \subset X \mid Y \neq \emptyset\}; \\ P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}; \\ P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}. \end{gathered} \tag{1.3}$$

Now, if $F : X \to P(X)$ is a multivalued operator, then we denote by $\mathrm{Fix}(F)$ the fixed points set of F, that is, $\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}$.

The set $\mathrm{Graph}(F) = \{(x,y) \in X \times X \mid y \in F(x)\}$ is called the graph of the multivalued operator F.

In the context of a metric space (X,d), if $A,B \in P(X)$, then we will use the following notations:

(a) the gap functional $D : P(X) \times P(X) \to \mathbb{R}_+$:

$$D(A,B) := \inf\{d(a,b) \mid a \in A,\ b \in B\}; \tag{1.4}$$

(b) the generalized excess functional $\rho : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$:

$$\rho(A,B) := \sup\{D(a,B) \mid a \in A\}; \tag{1.5}$$

(c) the generalized Pompeiu-Hausdorff functional $H : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$:

$$H(A,B) := \max\{\rho(A,B), \rho(B,A)\}. \tag{1.6}$$

It is well known that H is a generalized metric, in the sense that if $A,B \in P_{cl}(X)$, then $H(A,B) \in \mathbb{R}_+ \cup \{+\infty\}$.

Throughout this paper we denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by Θ the zero $m \times m$ matrix, and by I the identity $m \times m$ matrix. If

$A \in M_{m,m}(\mathbb{R}_+)$, then the symbol A^τ stands for the transpose matrix of A. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

Recall that a matrix A is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$ (see Varga [2]).

Notice that, for the proof of the main results, we need the following theorem, part of which being a classical result in matrix analysis; see, for example, [3, Lemma 3.3.1, page 55], [4, page 37], and [2, page 12]. For the assertion (iv) see [5].

Theorem 1.1. *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following are equivalents.*

(i) *A is convergent towards zero.*

(ii) *$A^n \to 0$ as $n \to \infty$.*

(iii) *The eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with* $\det(A - \lambda I) = 0$.

(iv) *The matrix $I - A$ is nonsingular and*

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots . \tag{1.7}$$

(v) *The matrix $I - A$ is nonsingular and $(I - A)^{-1}$ has nonnenegative elements.*

(vi) *$A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.*

Remark 1.2. Some examples of matrix convergent to zero are

(a) any matrix $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

(b) any matrix $A := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

(c) any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

For other examples and considerations on matrices which converge to zero, see Rus [4], Turinici [6], and so forth.

Main result for self contractions on generalized metric spaces is Perov's fixed point theorem; see [1].

Theorem 1.3 (Perov [3]). *Let (X, d) be a complete generalized metric space and the mapping $f : X \to X$ with the property that there exists a matrix $A \in M_{m,m}(\mathbb{R})$ such that $d(f(x), f(y)) \leq Ad(x, y)$ for all $x, y \in X$.*

If A is a matrix convergent towards zero, then

(1) $\mathrm{Fix}(f) = \{x^*\}$;

(2) *the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$ is convergent and it has the limit x^*, for all $x_0 \in X$;*

(3) *one has the following estimation:*

$$d(x_n, x^*) \leq A^n (I - A)^{-1} d(x_0, x_1); \tag{1.8}$$

(4) *if* $g : X \to X$ *satisfies the condition* $d(f(x), g(x)) \leq \eta$, *for all* $x \in X$, $\eta \in \mathbb{R}^m$ *and considering the sequence* $y_n = g^n(x_0)$ *one has*

$$d(y_n, x^*) \leq (I - A)^{-1}\eta + A^n(I - A)^{-1}d(x_0, x_1). \tag{1.9}$$

On the other hand, notice that the evolution of macrosystems under uncertainty or lack of precision, from control theory, biology, economics, artificial intelligence, or other fields of knowledge, is often modeled by semilinear inclusion systems:

$$\begin{aligned} x_1 &\in T_1(x_1, x_2), \\ x_2 &\in T_1(x_1, x_2), \end{aligned} \tag{1.10}$$

(where $T_i : X \times X \to P(X)$ for $i \in \{1, 2\}$ are multivalued operators; here $P(X)$ stands for the family of all nonempty subsets of a Banach space X). The system above can be represented as a fixed point problem of the form

$$x \in T(x) \quad \left(\text{where } T := (T_1, T_2) : X^2 \longrightarrow P\left(X^2\right),\ x = (x_1, x_2)\right). \tag{1.11}$$

Hence, it is of great interest to give fixed point results for multivalued operators on a set endowed with vector-valued metrics or norms. However, some advantages of a vector-valued norm with respect to the usual scalar norms were already pointed out by Precup in [5]. The purpose of this work is to present some new fixed point results for generalized (singlevalued and multivalued) contractions on spaces endowed with vector-valued metrics. The results are extensions of the theorems given by Perov [1], O'Regan et al. [7], M. Berinde and V. Berinde [8], and by Bucur et al. [9].

2. Main Results

We start our considerations by a local fixed point theorem for a class of generalized singlevalued contractions.

Theorem 2.1. *Let* (X, d) *be a complete generalized metric space,* $x_0 \in X$, $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ *with* $r_i > 0$ *for each* $i \in \{1, 2, \ldots, m\}$ *and let* $f : \widetilde{B}(x_0, r) \to X$ *having the property that there exist* $A, B \in M_{m,m}(\mathbb{R}_+)$ *such that*

$$d(f(x), f(y)) \leq Ad(x, y) + Bd(y, f(x)) \tag{2.1}$$

for all $x, y \in \widetilde{B}(x_0, r)$. *We suppose that*

(1) A *is a matrix that converges toward zero;*

(2) *if* $u \in \mathbb{R}_+^m$ *is such that* $u(I - A)^{-1} \leq (I - A)^{-1}r$, *then* $u \leq r$;

(3) $d(x_0, f(x_0))(I - A)^{-1} \leq r$.

Then $\mathrm{Fix}(f) \neq \emptyset$.

In addition, if the matrix $A + B$ *converges to zero, then* $\mathrm{Fix}(f) = \{x^*\}$.

Proof. We consider $(x_n)_{n\in\mathbb{N}}$ the sequence of successive approximations for the mapping f, defined by

$$\begin{aligned} x_{n+1} &= f(x_n), \quad \forall n \in \mathbb{N}, \\ x_0 &\in X, \text{ be arbitrary}. \end{aligned} \tag{2.2}$$

Using (3), we have $d(x_0, x_1)(I-A)^{-1} = d(x_0, f(x_0))(I-A)^{-1} \leq r \leq (I-A)^{-1}r$.

Thus, by (2) we get that $d(x_0, x_1) \leq r$ and hence $x_1 \in \widetilde{B}(x_0, r)$. Similarly, $d(x_1, x_2)(I-A)^{-1} = d(f(x_0), f(x_1))(I-A)^{-1} \leq Ad(x_0, x_1)(I-A)^{-1} + Bd(x_1, f(x_0))(I-A)^{-1} \leq Ar$.

Since $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$, by (2) we get

$$\begin{aligned} d(x_0, x_2)(I-A)^{-1} &\leq d(x_0, x_1)(I-A)^{-1} + d(x_1, x_2)(I-A)^{-1} \\ &\leq Ir + Ar \leq \left(I + A + A^2 + \cdots\right)r = (I-A)^{-1}r. \end{aligned} \tag{2.3}$$

Thus $d(x_0, x_2) \leq r$ and hence $x_2 \in \widetilde{B}(x_0, r)$.

Inductively, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{B}(x_0, r)$ satisfying, for all $n \in \mathbb{N}$, the following conditions:

(i) $x_{n+1} = f(x_n)$;

(ii) $d(x_0, x_n)(I-A)^{-1} \leq (I-A)^{-1}r$;

(iii) $d(x_n, x_{n+1})(I-A)^{-1} \leq A^n r$.

From (iii) we get, for all $n \in \mathbb{N}$ and $p \in \mathbb{N}$, $p > 0$, that

$$\begin{aligned} d(x_n, x_{n+p})(I-A)^{-1} &= d(x_n, x_{n+1})(I-A)^{-1} + d(x_{n+1}, x_{n+2})(I-A)^{-1} \\ &\quad + \cdots + d(x_{n+p-1}, x_{n+p})(I-A)^{-1} \\ &\leq A^n r + A^{n+1} r + \cdots + A^{n+p-1} r \\ &\leq A^n\left(I + A + A^2 + \cdots + A^{p-1} + \cdots\right)r \\ &\leq A^n (I-A)^{-1} r \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.4}$$

Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Using the fact that $(\widetilde{B}(x_0, r), d)$ is a complete metric space, we get that $(x_n)_{n\in\mathbb{N}}$ is convergent in the closed set $\widetilde{B}(x_0, r)$. Thus, there exists $x^* \in \widetilde{B}(x_0, r)$ such that $x^* = \lim_{n\to\infty} x_n$.

Next, we show that $x^* \in \operatorname{Fix}(f)$.

Indeed, we have the following estimation:

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, x_n) + d(x_n, f(x^*)) = d(x^*, x_n) + d(f(x_{n-1}), f(x^*)) \\ &\leq d(x^*, x_n) + Ad(x_{n-1}, x^*) + Bd(x^*, x_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.5}$$

Hence $x^* \in \mathrm{Fix}(f)$. In addition, letting $p \to \infty$ in the estimation of $d(x_n, x_{n+p})$, we get

$$d(x_n, x^*) \le A^n (I - A)^{-1} d(x_0, x_1). \tag{2.6}$$

We show now the uniqueness of the fixed point.

Let $x^*, y^* \in \mathrm{Fix}(f)$ with $x^* \neq y^*$. Then

$$d(x^*, y^*) = d\big(f(x^*), f(y^*)\big) \le Ad(x^*, y^*) + Bd\big(y^*, f(x^*)\big) = (A + B)d(x^*, y^*), \tag{2.7}$$

which implies $(I - A - B)d(x^*, y^*) \le 0 \in \mathbb{R}^m$. Taking into account that $I - A - B$ is nonsingular and $(I - A - B)^{-1} \in M_{m,m}(\mathbb{R}_+)$ we deduce that $d(x^*, y^*) \le 0$ and thus $x^* = y^*$. □

Remark 2.2. By similitude to [10], a mapping $f : Y \subseteq X \to X$ satisfying the condition

$$d\big(f(x), f(y)\big) \le Ad(x, y) + Bd\big(y, f(x)\big), \quad \forall x, y \in Y, \tag{2.8}$$

for some matrices $A, B \in M_{m,m}(\mathbb{R}_+)$ with A a matrix that converges toward zero, could be called an almost contraction of Perov type.

We have also a global version of Theorem 2.1, expressed by the following result.

Corollary 2.3. *Let (X, d) be a complete generalized metric space. Let $f : X \to X$ be a mapping having the property that there exist $A, B \in M_{m,m}(\mathbb{R}_+)$ such that*

$$d\big(f(x), f(y)\big) \le Ad(x, y) + Bd\big(y, f(x)\big), \quad \forall x, y \in X. \tag{2.9}$$

If A is a matrix that converges towards zero, then

(1) $\mathrm{Fix}(f) \neq \emptyset$;

(2) *the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n := f^n(x_0)$ converges towards a fixed point of f, for all $x_0 \in X$;*

(3) *one has the estimation*

$$d(x_n, x^*) \le A^n (I - A)^{-1} d(x_0, x_1), \tag{2.10}$$

where $x^* \in \mathrm{Fix}(f)$.

In addition, if the matrix $A + B$ converges to zero, then $\mathrm{Fix}(f) = \{x^*\}$.

Remark 2.4. Any matrix $A = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, where $a, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$, satisfies the assumptions (1)-(2) in Theorem 2.1.

Remark 2.5. Let us notice here that some advantages of a vector-valued norm with respect to the usual scalar norms were very nice pointed out, by several examples, in Precup in [5]. More precisely, one can show that, in general, the condition that A is a matrix convergent to zero is weaker than the contraction conditions for operators given in terms of the scalar norms on X of the following type:

$\|x\|_M := \|x_1\| + \|x_2\|$
$\|x\|_C := \max\{\|x_1\|, \|x_2\|\}$ or
$\|x\|_E := (\|x_1\|^2 + \|x_2\|^2)^{1/2}$.

As an application of the previous results we present an existence theorem for a system of operatorial equations.

Theorem 2.6. *Let $(X, |\cdot|)$ be a Banach space and let $f_1, f_2 : X \times X \to X$ be two operators. Suppose that there exist $a_{ij}, b_{ij} \in \mathbb{R}_+$, $i, j \in \{1, 2\}$ such that, for each $x := (x_1, x_2)$, $y := (y_1, y_2) \in X \times X$, one has:*

(1) $|f_1(x_1,x_2)-f_1(y_1,y_2)| \le a_{11}|x_1-y_1|+a_{12}|x_2-y_2|+b_{11}|x_1-f_1(y_1,y_2)|+b_{12}|x_2-f_2(y_1,y_2)|$,

(2) $|f_2(x_1,x_2)-f_2(y_1,y_2)| \le a_{21}|x_1-y_1|+a_{22}|x_2-y_2|+b_{21}|x_1-f_1(y_1,y_2)|+b_{22}|x_2-f_2(y_1,y_2)|$.

In addition, assume that the matrix $A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ converges to 0.

Then, the system

$$u_1 = f_1(u_1, u_2), \qquad u_2 = f_1(u_1, u_2) \tag{2.11}$$

has at least one solution $x^* \in X \times X$. Moreover, if, in addition, the matrix $A + B$ converges to zero, then the above solution is unique.

Proof. Consider $E := X \times X$ and the operator $f : E \to P_{cl}(E)$ given by the expression $f(x_1, x_2) := (f_1(x_1, x_2), f_2(x_1, x_2))$. Then our system is now represented as a fixed point equation of the following form: $x = f(x)$, $x \in E$. Notice also that the conditions (1) + (2) can be jointly represented as follows:

$$\|f(x) - f(y)\| \le A \cdot \|x - y\| + B \cdot \|x - f(y)\|, \quad \text{for each } x, y \in E := X \times X. \tag{2.12}$$

Hence, Corollary 2.3 applies in (E, d), with $d(u, v) := \|u - v\| := \begin{pmatrix} |u_1-v_1| \\ |u_2-v_2| \end{pmatrix}$. □

We present another result in the case of a generalized metric space but endowed with two metrics.

Theorem 2.7. *Let X be a nonempty set and let d, ρ be two generalized metrics on X. Let $f : X \to X$ be an operator. We assume that*

(1) *there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that $d(f(x), f(y)) \le \rho(x, y) \cdot C$;*

(2) *(X, d) is a complete generalized metric space;*

(3) *$f : (X, d) \to (X, d)$ is continuous;*

(4) *there exists $A, B \in M_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ one has*

$$\rho(f(x), f(y)) \le A\rho(x, y) + B\rho(y, f(x)). \tag{2.13}$$

If the matrix A converges towards zero, then $\mathrm{Fix}(f) \ne \emptyset$.

In addition, if the matrix $A+B$ converges to zero, then $\operatorname{Fix}(f)=\{x^*\}$.

Proof. We consider the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$ defined recurrently by $x_{n+1}=f(x_n)$, $x_0\in X$ being arbitrary. The following statements hold:

$$\begin{aligned}
&\rho(x_1,x_2)=\rho\big(f(x_0),f(x_1)\big)\le A\rho(x_0,x_1)+B\rho\big(x_1,f(x_0)\big)=A\rho(x_0,x_1),\\
&\rho(x_2,x_3)=\rho\big(f(x_1),f(x_2)\big)\le A\rho(x_1,x_2)+B\rho\big(x_2,f(x_1)\big)\le A^2\rho(x_0,x_1),\\
&\qquad\vdots\\
&\rho(x_n,x_{n+1})\le A^n\rho(x_0,x_1),\quad \forall n\in\mathbb{N},\ n\ge 1.
\end{aligned}\tag{2.14}$$

Now, let $p\in\mathbb{N}$, $p>0$. We estimate

$$\begin{aligned}
\rho\big(x_n,x_{n+p}\big)&\le\rho(x_n,x_{n+1})+\rho(x_{n+1},x_{n+2})+\cdots+\rho\big(x_{n+p-1},x_{n+p}\big)\\
&\le A^n\rho(x_0,x_1)+A^{n+1}\rho(x_0,x_1)+\cdots+A^{n+p-1}\rho(x_0,x_1)\\
&\le A^n\Big(I+A+A^2+\cdots+A^{p-1}+\cdots\Big)\rho(x_0,x_1)\\
&=A^n(I-A)^{-1}\rho(x_0,x_1).
\end{aligned}\tag{2.15}$$

Letting $n\to\infty$ we obtain that $\rho(x_n,x_{n+p})\to 0\in\mathbb{R}^m$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ.

On the other hand, using the statement (1), we get

$$\begin{aligned}
d\big(x_n,x_{n+p}\big)&=d\big(f(x_{n-1}),f\big(x_{n+p-1}\big)\big)\le\rho\big(x_{n-1},x_{n+p-1}\big)\cdot C\\
&\le A^{n-1}(I-A)^{-1}\rho(x_0,x_1)C\longrightarrow 0,\quad\text{as } n\longrightarrow\infty.
\end{aligned}\tag{2.16}$$

Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Since (X,d) is complete, one obtains the existence of an element $x^*\in X$ such that $x^*=\lim_{n\to\infty}x_n$ with respect to d.

We prove next that $x^*=f(x^*)$, that is, $\operatorname{Fix}(f)\neq\emptyset$. Indeed, since $x_{n+1}=f(x_n)$, for all $n\in\mathbb{N}$, letting $n\to\infty$ and taking into account that f is continuous with respect to d, we get that $x^*=f(x^*)$.

The uniqueness of the fixed point x^* is proved below.

Let $x^*,y^*\in\operatorname{Fix}(f)$ such that $x^*\neq y^*$. We estimate

$$\rho\big(x^*,y^*\big)=\rho\big(f(x^*),f(y^*)\big)\le A\rho\big(x^*,y^*\big)+B\rho\big(y^*,f(x^*)\big)=(A+B)\rho\big(x^*,y^*\big).\tag{2.17}$$

Thus, using the additional assumption on the matrix $A+B$, we have that

$$(I-A-B)\rho\big(x^*,y^*\big)\le 0\Longrightarrow\rho\big(x^*,y^*\big)\le 0\Longrightarrow x^*=y^*.\tag{2.18}$$

□

In what follows, we will present some results for the case of multivalued operators.

Theorem 2.8. *Let (X,d) be a complete generalized metric space and let $x_0 \in X$, $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1,2,\ldots,m\}$. Consider $F : \widetilde{B}(x_0,r) \to P_{cl}(X)$ a multivalued operator. One assumes that*

(i) *there exist $A, B \in M_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in \widetilde{B}(x_0,r)$ and $u \in F(x)$ there exists $v \in F(y)$ with*

$$d(u,v) \leq Ad(x,y) + Bd(y,u); \tag{2.19}$$

(ii) *there exists $x_1 \in F(x_0)$ such that $d(x_0,x_1)(I-A)^{-1} \leq r$;*

(iii) *if $u \in \mathbb{R}_+^m$ is such that $u(I-A)^{-1} \leq (I-A)^{-1}r$, then $u \leq r$.*

If A is a matrix convergent towards zero, then $\mathrm{Fix}(F) \neq \emptyset$.

Proof. By (ii) and (iii), there exists $x_1 \in F(x_0)$ such that

$$d(x_0,x_1)(I-A)^{-1} \leq r \leq (I-A)^{-1}r \Longrightarrow d(x_0,x_1) \leq r \Longrightarrow x_1 \in \widetilde{B}(x_0,r). \tag{2.20}$$

For $x_1 \in F(x_0)$, there exists $x_2 \in F(x_1)$ with

$$d(x_1,x_2)(I-A)^{-1} \leq Ad(x_0,x_1)(I-A)^{-1} + Bd(x_1,x_1)(I-A)^{-1} \leq Ar. \tag{2.21}$$

Hence

$$\begin{aligned} d(x_0,x_2)(I-A)^{-1} &\leq d(x_0,x_1)(I-A)^{-1} + d(x_1,x_2)(I-A)^{-1} \\ &\leq Ir + Ar \leq \left(I + A + A^2 + \cdots\right)r = (I-A)^{-1}r \\ &\Longrightarrow d(x_0,x_2) \leq r \Longrightarrow x_2 \in \widetilde{B}(x_0,r). \end{aligned} \tag{2.22}$$

Next, for $x_2 \in F(x_1)$, there exists $x_3 \in F(x_2)$ with

$$d(x_2,x_3)(I-A)^{-1} \leq Ad(x_1,x_2)(I-A)^{-1} + Bd(x_2,x_2)(I-A)^{-1} \leq A^2 r, \tag{2.23}$$

and hence

$$\begin{aligned} d(x_0,x_3)(I-A)^{-1} &\leq d(x_0,x_1)(I-A)^{-1} + d(x_1,x_2)(I-A)^{-1} + d(x_2,x_3)(I-A)^{-1} \\ &\leq Ir + Ar + A^2 r \leq \left(I + A + A^2 + \cdots\right)r = (I-A)^{-1}r \\ &\Longrightarrow d(x_0,x_3) \leq r \Longrightarrow x_3 \in \widetilde{B}(x_0,r). \end{aligned} \tag{2.24}$$

By induction, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{B}(x_0,r)$ such that, for all $n\in\mathbb{N}$, we have

(1) $x_{n+1}\in F(x_n)$;

(2) $d(x_0,x_n)(I-A)^{-1}\leq (I-A)^{-1}r$;

(3) $d(x_n,x_{n+1})(I-A)^{-1}\leq A^n r$.

By a similar approach as before (see the proof of Theorem 2.1), we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $(\widetilde{B}(x_0,r),d)$. Hence $(x_n)_{n\in\mathbb{N}}$ is convergent in $\widetilde{B}(x_0,r)$. Thus, there exists $x^*\in\widetilde{B}(x_0,r)$ such that $x^*=\lim_{n\to\infty}x_n$.

Next we show that $x^*\in F(x^*)$.

Using (i) and the fact that $x_n\in F(x_{n-1})$, for all $n\in\mathbb{N}$ $n\geq 1$, we get, for each $n\in\mathbb{N}$, the existence of $u_n\in F(x^*)$ such that

$$d(x_n,u_n)\leq Ad(x_{n-1},x^*)+Bd(x^*,x_n). \tag{2.25}$$

On the other hand

$$\begin{aligned} d(x^*,u_n)&\leq d(x^*,x_n)+d(x_n,u_n)\\ &\leq d(x^*,x_n)+Ad(x_{n-1},x^*)+Bd(x^*,x_n). \end{aligned} \tag{2.26}$$

Letting $n\to\infty$, we get $d(x^*,u_n)\to 0$. Hence, we have $\lim_{n\to\infty}u_n=x^*$ and since $u_n\in F(x^*)$ and $F(x^*)$ is closed set, we get that $x^*\in F(x^*)$. □

Remark 2.9. From the proof of the above theorem, we also get the following estimation:

$$d(x_n,x^*)\leq A^n(I-A)^{-1}d(x_0,x_1),\quad \text{for each } n\in\mathbb{N} \text{ with } n\geq 1, \tag{2.27}$$

where x^* is a fixed point for the multivalued operator F, and the pair $(x_0,x_1)\in\operatorname{Graph}(F)$ is arbitrary.

We have also a global variant for the Theorem 2.8 as follows.

Corollary 2.10. *Let (X,d) be a complete generalized metric space and $F:X\to P_{\mathrm{cl}}(X)$ a multivalued operator. One supposes that there exist $A,B\in M_{m,m}(\mathbb{R}_+)$ such that for each $x,y\in X$ and all $u\in F(x)$, there exists $v\in F(y)$ with*

$$d(u,v)\leq Ad(x,y)+Bd(y,u). \tag{2.28}$$

If A is a matrix convergent towards zero, then $\operatorname{Fix}(F)\neq\emptyset$.

Remark 2.11. By a similar approach to that given in Theorem 2.6, one can obtain an existence result for a system of operatorial inclusions of the following form:

$$\begin{aligned} x_1&\in T_1(x_1,x_2),\\ x_2&\in T_1(x_1,x_2), \end{aligned} \tag{2.29}$$

where $T_1, T_2 : X \times X \to P_{cl}(X)$ are multivalued operators satisfying a contractive type condition (see also [9]).

The following results are obtained in the case of a set X endowed with two metrics.

Theorem 2.12. *Let (X, d) be a complete generalized metric space and ρ another generalized metric on X. Let $F : X \to P(X)$ be a multivalued operator. One assumes that*

(i) *there exists a matrix $C \in M_{m,m}(\mathbb{R}_+)$ such that $d(x, y) \le \rho(x, y) \cdot C$, for all $x, y \in X$;*

(ii) *$F : (X, d) \to (P(X), H_d)$ has closed graph;*

(iii) *there exist $A, B \in M_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ and $u \in F(x)$, there exists $v \in F(y)$ with*

$$\rho(u, v) \le A\rho(x, y) + B\rho(y, u). \tag{2.30}$$

If A is a matrix convergent towards zero, then $\mathrm{Fix}(F) \neq \emptyset$.

Proof. Let $x_0 \in X$ such that $x_1 \in F(x_0)$.

For $x_1 \in F(x_0)$, there exists $x_2 \in F(x_1)$ such that

$$\rho(x_1, x_2) \le A\rho(x_0, x_1) + B\rho(x_1, x_1) = A\rho(x_0, x_1). \tag{2.31}$$

For $x_2 \in F(x_1)$, there exists $x_3 \in F(x_2)$ such that

$$\rho(x_2, x_3) \le A\rho(x_1, x_2) + B\rho(x_2, x_2) \le A^2\rho(x_0, x_1). \tag{2.32}$$

Consequently, we construct by induction the sequence $(x_n)_{n\in\mathbb{N}}$ in X which satisfies the following properties:

(1) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;

(2) $\rho(x_n, x_{n+1}) \le A^n\rho(x_0, x_1)$, for all $n \in \mathbb{N}$.

We show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X with respect to ρ. In order to do that, let $p \in \mathbb{N}$, $p > 0$. One has the estimation

$$\begin{aligned}
\rho(x_n, x_{n+p}) &\le \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+p-1}, x_{n+p}) \\
&\le A^n\rho(x_0, x_1) + A^{n+1}\rho(x_0, x_1) + \cdots + A^{n+p-1}\rho(x_0, x_1) \\
&\le A^n\left(I + A + \cdots + A^{p-1} + \cdots\right)\rho(x_0, x_1) \\
&= A^n(I - A)^{-1}\rho(x_0, x_1).
\end{aligned} \tag{2.33}$$

Since the matrix A converges towards zero, one has $A^n \to \Theta$ as $n \to \infty$. Letting $n \to \infty$ one get $\rho(x_n, x_{n+p}) \to 0$ which implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ.

Using (i), we obtain that $d(x_n, x_{n+p}) \le \rho(x_n, x_{n+p}) \cdot C \to 0$ as $n \to \infty$. Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d too.

Since (X,d) is complete, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in X. Thus there exists $x^* \in X$ such that $x^* = \lim_{n\to\infty} x_n$ with respect to d.

Finally, we show that $x^* \in F(x^*)$.

Since $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$ and F has closed graph, by using the limit presented above, we get that $x^* \in F(x^*)$, that is, $\mathrm{Fix}(F) \neq \emptyset$. □

Remark 2.13. (1) Theorem 2.12 holds even if the assumption (iii) is replaced by

(iii′) there exist $A, B \in M_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ and $u \in F(x)$, there exists $v \in F(y)$ such that$\rho(u,v) \le A\rho(x,y) + Bd(y,u)$.

(2) Letting $p \to \infty$ in the estimation of $\rho(x_n, x_{n+p})$, presented in the proof of Theorem 2.12, we get

$$\rho(x_n, x^*) \le A^n (I-A)^{-1} \rho(x_0, x_1). \tag{2.34}$$

Using the relation between the generalized metrics d and ρ, one has immediately

$$d(x_n, x^*) \le CA^n (I-A)^{-1} \rho(x_0, x_1). \tag{2.35}$$

Theorem 2.14. *Let (X,d) be a complete generalized metric space and ρ another generalized metric on X. Let $x_0 \in X$, $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1,2,\ldots,m\}$ and let $F : \widetilde{B}_\rho(x_0,r) \to P(X)$ be a multivalued operator. Suppose that*

(i) *there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that $d(x,y) \le C\rho(x,y)$, for all $x, y \in X$;*

(ii) *$F : (\widetilde{B}_\rho(x_0,r), d) \to (P_b(X), H_d)$ has closed graph;*

(iii) *there exist $A, B \in M_{m,m}(\mathbb{R}_+)$ such that A is a matrix that converges to zero and for all $x, y \in \widetilde{B}_\rho(x_0,r)$ and $u \in F(x)$, there exists $v \in F(y)$ such that*

$$\rho(u,v) \le A\rho(x,y) + B\rho(y,u); \tag{2.36}$$

(iv) *if $u \in \mathbb{R}_+^m$ is such that $u(I-A)^{-1} \le (I-A)^{-1} r$, then $u \le r$;*

(v) *$\rho(x_0,x_1)(I-A)^{-1} \le r$.*

Then $\mathrm{Fix}(F) \neq \emptyset$.

Proof. Let $x_0 \in X$ such that $x_1 \in F(x_0)$. By (v) one has

$$\rho(x_0,x_1)(I-A)^{-1} \le r \le (I-A)^{-1} r, \tag{2.37}$$

which implies $x_1 \in \widetilde{B}_\rho(x_0,r)$.

Since $x_1 \in F(x_0)$, there exists $x_2 \in F(x_1)$ such that

$$\rho(x_1,x_2)(I-A)^{-1} \le A\rho(x_0,x_1)(I-A)^{-1} + B\rho(x_1,x_1)(I-A)^{-1} \le Ar. \tag{2.38}$$

Hence,

$$\begin{aligned}\rho(x_0,x_2)(I-A)^{-1} &\le \rho(x_0,x_1)(I-A)^{-1}+\rho(x_1,x_2)(I-A)^{-1}\\ &\le Ir+Ar\le (I+A+\cdots+A^n+\cdots)r\le (I-A)^{-1}r,\end{aligned} \tag{2.39}$$

which implies that $\rho(x_0,x_2)\le r$, that is, $x_2\in \widetilde{B}_\rho(x_0,r)$.

For $x_2\in F(x_1)$, there exists $x_3\in F(x_2)$ such that

$$\rho(x_2,x_3)(I-A)^{-1}\le A\rho(x_1,x_2)(I-A)^{-1}+B\rho(x_2,x_2)(I-A)^{-1}\le A^2r. \tag{2.40}$$

Then the following estimation holds:

$$\begin{aligned}\rho(x_0,x_3)(I-A)^{-1} &\le \rho(x_0,x_1)(I-A)^{-1}+\rho(x_1,x_2)(I-A)^{-1}+\rho(x_2,x_3)(I-A)^{-1}\\ &\le Ir+Ar+A^2r\le (I-A)^{-1}r,\end{aligned} \tag{2.41}$$

and thus $\rho(x_0,x_3)\le r$, that is, $x_3\in\widetilde{B}_\rho(x_0,r)$.

Inductively, we can construct the sequence $(x_n)_{x\in\mathbb{N}}$ which has its elements in the closed ball $\widetilde{B}_\rho(x_0,r)$ and satisfies the following conditions:

(1) $x_{n+1}\in F(x_n)$, for all $n\in\mathbb{N}$;

(2) $\rho(x_n,x_{n+1})(I-A)^{-1}\le A^n r$, for all $n\in\mathbb{N}$.

By a similar approach as in the proof of Theorem 2.12, the conclusion follows. □

A homotopy result for multivalued operators on a set endowed with a vector-valued metric is the following.

Theorem 2.15. *Let (X,d) be a generalized complete metric space in Perov sense, let U be an open subset of X, and let V be a closed subset of X, with $U\subset V$. Let $G: V\times[0,1]\to P(X)$ be a multivalued operator with closed (with respect to d) graph, such that the following conditions are satisfied:*

(a) *$x\notin G(x,t)$, for each $x\in V\setminus U$ and each $t\in[0,1]$;*

(b) *there exist $A,B\in\mathcal{M}_{m,m}(\mathbb{R}_+)$ such that the matrix A is convergent to zero such that for each $t\in[0,1]$, for each $x,y\in X$ and all $u\in G(x,t)$), there exists $v\in G(y,t)$ with $d(u,v)\le Ad(x,y)+Bd(y,u)$.*

(c) *there exists a continuous increasing function $\phi:[0,1]\to\mathbb{R}^m$ such that for all $t,s\in[0,1]$, each $x\in V$ and each $u\in G(x,t)$ there exists $v\in G(x,s)$ such that $d(u,v)\le|\phi(t)-\phi(s)|$;*

(d) *if $v,r\in\mathbb{R}^m_+$ are such that $v\cdot(I-A)^{-1}\le (I-A)^{-1}\cdot r$, then $v\le r$;*

Then $G(\cdot,0)$ has a fixed point if and only if $G(\cdot,1)$ has a fixed point.

Proof. Suppose that $G(\cdot,0)$ has a fixed point z. From (a) we have that $z\in U$. Define

$$Q:=\{(t,x)\in[0,1]\times U \mid x\in G(x,t)\}. \tag{2.42}$$

Clearly $Q \neq \emptyset$, since $(0, z) \in Q$. Consider on Q a partial order defined as follows:

$$(t, x) \leq (s, y) \quad \text{iff } t \leq s, \qquad d(x, y) \leq 2[\phi(s) - \phi(t)] \cdot (I - A)^{-1}. \tag{2.43}$$

Let M be a totally ordered subset of Q and consider $t^* := \sup\{t \mid (t, x) \in M\}$. Consider a sequence $(t_n, x_n)_{n \in \mathbb{N}^*} \subset M$ such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ for each $n \in \mathbb{N}^*$ and $t_n \to t^*$, as $n \to +\infty$. Then

$$d(x_m, x_n) \leq 2[\phi(t_m) - \phi(t_n)] \cdot (I - A)^{-1}, \quad \text{for each } m, n \in \mathbb{N}^*, \ m > n. \tag{2.44}$$

When $m, n \to +\infty$, we obtain $d(x_m, x_n) \to 0$ and, thus, $(x_n)_{n \in \mathbb{N}^*}$ is d-Cauchy. Thus $(x_n)_{n \in \mathbb{N}^*}$ is convergent in (X, d). Denote by $x^* \in X$ its limit. Since $x_n \in G(x_n, t_n), n \in \mathbb{N}^*$ and since G is d-closed, we have that $x^* \in G(x^*, t^*)$. Thus, from (a), we have $x^* \in U$. Hence $(t^*, x^*) \in Q$. Since M is totally ordered we get that $(t, x) \leq (t^*, x^*)$, for each $(t, x) \in M$. Thus (t^*, x^*) is an upper bound of M. By Zorn's Lemma, Q admits a maximal element $(t_0, x_0) \in Q$. We claim that $t_0 = 1$. This will finish the proof.

Suppose $t_0 < 1$. Choose $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ with $r_i > 0$ for each $i \in \{1, 2, \ldots, m\}$ and $t \in]t_0, 1]$ such that $B(x_0, r) \subset U$, where $r := 2[\phi(t) - \phi(t_0)] \cdot (I - A)^{-1}$. Since $x_0 \in G(x, t_0)$, by (c), there exists $x_1 \in G(x_0, t)$ such that $d(x_0, x_1) \leq |\phi(t) - \phi(t_0)|$. Thus, $d(x_0, x_1)(I - A)^{-1} \leq |\phi(t) - \phi(t_0)| \cdot (I - A)^{-1} < r$.

Since $\overline{B(x_0, r)} \subset V$, the multivalued operator $G(\cdot, t) : \overline{B(x_0, r)} \to P_{\mathrm{cl}}(X)$ satisfies, for all $t \in [0, 1]$, the assumptions of Theorem 2.1 Hence, for all $t \in [0, 1]$, there exists $x \in \overline{B(x_0, r)}$ such that $x \in G(x, t)$. Thus $(t, x) \in Q$. Since $d(x_0, x) \leq r = 2[\phi(t) - \phi(t_0)](I - A)^{-1}$, we immediately get that $(t_0, x_0) < (t, x)$. This is a contradiction with the maximality of (t_0, x_0).

Conversely, if $G(\cdot, 1)$ has a fixed point, then putting $t := 1 - t$ and using first part of the proof we get the conclusion. □

Remark 2.16. Usually in the above result, we take $Q = \overline{U}$. Notice that in this case, condition (a) becomes

(a′) $x \notin G(x, t)$, for each $x \in \partial U$ and each $t \in [0, 1]$.

Remark 2.17. If in the above results we consider $m = 1$, then we obtain, as consequences, several known results in the literature, as those given by M. Berinde and V. Berinde [8], Precup [5], Petruşel and Rus [11], and Feng and Liu [12]. Notice also that the theorems presented here represent extensions of some results given Bucur et al. [9], O'Regan and Precup [13], O'Regan et al. [7], Perov [1], and so forth.

Remark 2.18. Notice also that since $\mathbb{R}_+^n$ is a particular type of cone in a Banach space, it is a nice direction of research to obtain extensions of these results for the case of operators on K-metric (or K-normed) spaces (see Zabrejko [14]). For other similar results, open questions, and research directions see [7, 11–13, 15–18].

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