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Research Article

Ergodic Retractions for Families of Asymptotically Nonexpansive Mappings

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We prove some theorems for the existence of ergodic retractions onto the set of common fixed points of a family of asymptotically nonexpansive mappings. Our results extend corresponding results of Benavides and Ramírez (2001), and Li and Sims (2002).

1. Introduction

Let E be a Banach space and C a nonempty closed and convex subset of E. We recall some definitions.

Definition 1.1. A mapping $T : C \to C$ is said to be

(i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C; \tag{1.1}$$

(ii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C; \tag{1.2}$$

(iii) *of asymptotically nonexpansive type* if for each x in C, we have

$$\limsup_{n\to\infty}\sup_{y\in C}\left(\|T^n x - T^n y\| - \|x-y\|\right) \le 0; \tag{1.3}$$

(iv) *weakly asymptotically nonexpansive* if it satisfies the condition

$$\limsup_{n\to\infty}\|T^n x - T^n y\| \le \|x-y\|, \quad \forall x, y \in C. \tag{1.4}$$

(v) retraction if $T^2 = T$. A subset F of C is called a nonexpansive retract of C if either $F = \emptyset$ or there exists a retraction of C onto F which is a nonexpansive mapping.

Definition 1.2. We say that a nonempty closed convex subset D of C satisfies property (ω) with respect to

(i) a mapping $T : C \to C$ if $\omega_T(x) \subset D$ for every $x \in D$ where

$$\omega_T(x) = \left\{ y \in C : y = w - \lim_k T^{n_k}(x) \text{ for some } n_k \longrightarrow \infty \right\}, \tag{1.5}$$

(ii) a semigroup of mappings $\varphi = \{T(t) : C \to C : t \ge 0\}$ if $\omega_\varphi(x) \subset D$ for every $x \in D$ where

$$\omega_\varphi(x) = \left\{ y \in C : y = w - \lim_i T(t_i)(x) \text{ for some } t_i \uparrow \infty \right\}. \tag{1.6}$$

Obviously, C itself verifies (ω).

Definition 1.3. (i) A mapping $T : C \to C$ is said to satisfy the (ω)-fixed point property ((ω)-fpp) if T has a fixed point in every nonempty closed convex subset D of C which satisfies (ω) with respect to T.

(ii) A semigroup $\varphi = \{T(t) : C \to C : t \ge 0\}$ is said to satisfy the (ω)-fpp if φ has a common fixed point in every nonempty closed convex subset D of C which satisfies (ω) with respect to the semigroup φ.

(iii) A family $\varphi = \{T_i : C \to C : i \in I\}$ is said to satisfy the (ω)-fpp if φ has a common fixed point in every nonempty closed convex subset D of C which satisfies (ω) with respect to each T_i.

In 1965, Kirk [1] proved that if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive mapping $T : C \to C$ has a fixed point. (A nonempty convex subset C of a normed linear space is said to have normal structure if each bounded convex subset K of C consisting of more than one point contains a nondiametral point). Goebel and Kirk [2] proved that if E is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping T of C has a fixed point. This was extended to mappings of asymptotically nonexpansive type by Kirk in [3]. However, whether normal structure implies the existence of fixed points for mappings of asymptotically nonexpansive type is a natural and still open question. Li and Sims [4] proved the following fixed point result in the case that E has uniform normal structure (It is known that a space with uniform normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure).

Theorem 1.4. *Suppose E is a Banach space with uniform normal structure; C is a nonempty bounded subset of E. Then*

(i) *every continuous and asymptotically nonexpansive type mapping $T : C \to C$ satisfies (ω)-fpp;*

(ii) *every semigroup $\varphi = \{T(t) : C \to C : t \geq 0\}$ of asymptotically nonexpansive type mappings on C such that $T(t)$ is continuous on C for each $t \geq 0$ satisfies (ω)-fpp.*

On the other hand, Bruck [5] initiated the study of the structure of the fixed point set $F(T) = \{x : Tx = x\}$ in a general Banach space E: if C is a weakly compact convex subset of E and $T : C \to C$ is nonexpansive and satisfies a conditional fixed point property, then $F(T)$ is a nonexpansive retract of C. The same author [6] used this fact to derive the existence of fixed points for a commuting family of nonexpansive mappings. See, for example, [7, 8] for some related results.

Benavides and Ramírez [9] studied the structure of the set of fixed points for (weakly) asymptotically nonexpansive mappings.

Theorem 1.5. *Let E be a Banach space and C a nonempty weakly compact convex subset of E. Assume that every asymptotically nonexpansive self-mapping of C satisfies the (ω)-fpp. Then for any commuting family φ of asymptotically nonexpansive self-mappings of C, the common fixed point set of φ, $F(\varphi)$, is a nonempty nonexpansive retract of C.*

In this paper, we prove some theorems to guarantee the existence of nonexpansive retractions onto the common fixed points of some families of (weakly) asymptotically nonexpansive (type) mappings. The results obtained in this paper extend in some sense, for example, Theorems 1.4 and 1.5, above.

2. Nonexpansive Retractions for Families of Weakly Asymptotically Nonexpansive Mappings

Theorem 2.1. *Let C be a nonempty weakly compact convex subset of a Banach space E, and $\varphi = \{T_i : i \in I\}$ a family of weakly asymptotically nonexpansive mappings on C such that $F(\varphi) \neq \emptyset$. Assume one of the following assumptions is satisfied:*

(a) *φ satisfies the (ω)-fpp;*

(b) *$F(\varphi)$ is a nonexpansive retract of C.*

Then for each $\alpha \in I$, there exists a nonexpansive retraction P_α from C onto $F(\varphi)$, the common fixed points of φ, such that $P_\alpha T_\alpha = T_\alpha P_\alpha = P_\alpha$, and every closed convex φ-invariant subset of C is also P_α-invariant.

Proof. Consider C^C with the product topology induced by the weak topology on C. Now, consider an $\alpha \in I$ and define

$$\mathfrak{R} := \Big\{T \in C^C : T \text{ is nonexpansive, } T \circ T_\alpha = T, \\ \text{and every closed convex } \varphi\text{-invariant subset of } C \text{ is also } T\text{-invariant}\Big\} \tag{2.1}$$

By applying an argument similar to that in the proof of [9, Theorem 2], it follows that $\mathfrak{R}$ is compact (the topology on $\mathfrak{R}$ is that of weak pointwise convergence) and there is a minimal element $P_\alpha \in \mathfrak{R}$ in the following sense:

$$\text{if } T \in \mathfrak{R} \text{ and } \|T(x) - T(y)\| \leq \|P_\alpha(x) - P_\alpha(y)\|, \quad \forall x, y \in C, \\ \text{then } \|T(x) - T(y)\| = \|P_\alpha(x) - P_\alpha(y)\|. \tag{*}$$

First, we assume the case (a). We shall prove that $P_\alpha(x) \in F(\varphi)$ for all $x \in C$. For a given $x \in C$, consider the set $K = \{T(P_\alpha(x)) : T \in \mathfrak{R}\}$. Then K is a nonempty weakly compact convex subset of C, because $\mathfrak{R}$ is convex and compact. We will show that for all $i \in I$, K satisfies property (ω) with respect to T_i. Fix $i \in I$ and take $y \in K$ and $z \in C$such as $T_i^{n_k} y \rightharpoonup z$, for some $n_k \to \infty$. There exists $h \in \mathfrak{R}$ such that $y = h(P_\alpha(x))$. Consider a subnet $\{T_i^{n_k(\eta)}\}$ of $\{T_i^{n_k}\}$ such that $S(u) = \omega - \lim_\eta T_i^{n_k(\eta)}(u)$ exists for every $u \in C$. Now, taking $u = h(P_\alpha(x))$, we have $z = S(h(P_\alpha(x)))$. Since S is nonexpansive, $h \in \mathfrak{R}$, and $S \circ h \circ T_\alpha = S \circ h$, it follows that $S \circ h \in \mathfrak{R}$ and then $z = S(h(P_\alpha(x))) \in K$. Thus K satisfies the property (ω) with respect to T_i. Since, φ satisfies the (ω)-fpp (by (a)), it follows that $K \cap F(\varphi) \neq \emptyset$. So, there exists $h \in \mathfrak{R}$ with $h(P_\alpha(x)) \in F(\varphi)$. Let $y = h(P_\alpha(x))$. Then $P_\alpha(y) = h(y) = y$, and by using the minimality of P_α, we have

$$\|P_\alpha(x) - y\| = \|P_\alpha(x) - P_\alpha(y)\| = \|h(P_\alpha(x)) - h(P_\alpha(y))\| = \|h(P_\alpha(x)) - y\| = 0. \tag{2.2}$$

So, we get $P_\alpha(x) = y \in F(\varphi)$. Since this is so for each $x \in C$ and P_α belongs to $\mathfrak{R}$, it follows that $P_\alpha^2 = P_\alpha$ and $P_\alpha T_\alpha = T_\alpha P_\alpha = P_\alpha$.

Now, we assume the case (b). From (b), there is a nonexpansive retraction R from C onto $F(\varphi)$. Put $\varphi' := \varphi \cup \{R\}$. Since $F(\varphi) = F(\varphi')$, we can replace φ by φ' in the above assertions to obtain a minimal element $P_\alpha \in \mathfrak{R}$ in the sense (*), where $\mathfrak{R}$ ia defined here as

$$\Big\{T \in C^C : T \text{ is nonexpansive,} T \circ T_\alpha = T, \\ \text{and every closed convex } \varphi'\text{-invariant subset of } C \text{ is also } T\text{-invariant}\Big\}. \tag{2.3}$$

We note that $R \circ T \circ T_\alpha = R \circ T$, $(\forall T \in \mathfrak{R})$. Since $R \in \varphi'$, every closed convex φ'-invariant subset of C is also R-invariant and consequently $R \circ T$-invariant, $(\forall T \in \mathfrak{R})$. So it is easy to see that $R \circ T \in \mathfrak{R}$, $(\forall T \in \mathfrak{R})$. Therefore, for every $x \in C$, the set $K = \{T(P_\alpha(x)) : T \in \mathfrak{R}\}$ is an R-invariant subset of C. So, considering the fact that $R(K) \subseteq K \cap R(C) = K \cap F(\varphi)$, we obtain $K \cap F(\varphi) \neq \emptyset$. Now, we can repeat the argument used in the last paragraph to get the desired result. □

A nonexpansive retraction satisfying the thesis of Theorem 2.1 is usually called an ergodic retraction (see e.g., [10, 11]).

Combining Theorem 1.5 [9, Theorem 4] and Theorem 2.1(a), we get the following improvement of Theorem 1.5.

Corollary 2.2. *Let E be a Banach space and C a nonempty weakly compact convex subset of E. Assume that every asymptotically nonexpansive self-mapping of C satisfies (ω)-fpp. Then for any commuting family $\varphi = \{T_i : i \in I\}$ of asymptotically nonexpansive self-mappings of C and for each $i \in I$, there exists a nonexpansive retraction P_i from C onto $F(\varphi)$, such that $P_iT_i = T_iP_i = P_i$, and every closed convex φ-invariant subset of C is also P_i-invariant.*

3. Ergodic Retractions for a Semigroup of Asymptotically Nonexpansive Type

Assume that S is a semigroup and $l^\infty(S)$ is the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$, respectively. An element μ of $l^\infty(S)^*$ is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in l^\infty(S)^*$ and $f \in l^\infty(S)$. A mean μ is said to be invariant if $\mu(l_s f) = \mu(r_s f) = \mu(f)$ for each $s \in S$ and $f \in l^\infty(S)$. S is said to be amenable if there is an invariant mean on $l^\infty(S)$. As is well known, S is amenable when it is a commutative semigroup [12].

The following result which we need is well known (see [13]).

Lemma 3.1. *Let f be a function of a semigroup S into E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact. Then, for any $\mu \in l^\infty(S)^*$, there exists a unique element f_μ in E such that $\langle f_\mu, x^*\rangle = \mu_t\langle f(t),\ x^*\rangle$ for all $x^* \in E^*$. Moreover, if μ is a mean, then $f_\mu \in \overline{\text{co}}\{f(t) : t \in S\}$.*

We can write f_μ by $\int f(t)d\mu(t)$. As a direct consequence of Lemma 3.1, we have the following lemma.

Lemma 3.2. *Let C be a nonempty closed convex subset of a Banach space E, $\varphi = \{T(t) : t \geq 0\}$ a semigroup of weakly asymptotically nonexpansive mappings on C such that weak closure of $\{T(t)x : t \geq 0\}$ is weakly compact for each $x \in C$, and μ a mean on $l^\infty(\mathbb{R}^+)$.*

If we write $T_\mu x$ instead of $\int T(t)x\, d\mu(t)$, then the following hold.

(i) *$T_\mu x = x$ for each $x \in F(\varphi)$.*

(ii) *$T_\mu x \in \overline{\text{co}}\{T(t)x : t \geq 0\}$ for each $x \in C$.*

(iii) *If μ is invariant, then $T_\mu T(t) = T_\mu$ for each $t \geq 0$ and T_μ is a nonexpansive mapping from C into itself.*

Proof. We only need to prove that T_μ is nonexpansive: consider $x, y \in C$ and $x^* \in J(T_\mu x - T_\mu y)$. Then for each $s \geq 0$, we have

$$\begin{aligned}\|T_\mu x - T_\mu y\|^2 &= \langle T_\mu x - T_\mu y, x^*\rangle = \mu_t\langle T(t)x - T(t)y, x^*\rangle = \mu_t\langle T(t+s)x - T(t+s)y, x^*\rangle \\ &\leq \|T_\mu x - T_\mu y\| \sup_{t\geq 0}\|T(t+s)x - T(t+s)y\|.\end{aligned} \tag{3.1}$$

Therefore, $\|T_\mu x - T_\mu y\| \le \sup_{t\ge 0}\|T(t+s)x - T(t+s)y\|$, for every $s \ge 0$. Consequently, we get

$$\|T_\mu x - T_\mu y\| \le \inf_{s\ge 0}\sup_{t\ge 0}\|T(t+s)x - T(t+s)y\| \le \|x-y\|. \tag{3.2}$$

□

The following is our main result which is an improvement of Theorem 1.4 [4, Theorem 2.2].

Theorem 3.3. *Suppose E is a Banach space with uniform normal structure; C is a nonempty bounded closed and convex subset of E; $\varphi = \{T(t) : t \ge 0\}$ is a semigroup of asymptotically nonexpansive type mappings on C such that $T(t)$ is continuous on C for each $t \ge 0$. Then there exists a nonexpansive retraction P from C onto $F(\varphi)$, such that $PT(t) = T(t)P = P$ for each $t \in S$, and every closed convex φ-invariant subset of C is also P-invariant.*

Proof. Consider C^C with the product topology induced by the weak topology on C. Now, define

$$\begin{aligned}\mathfrak{R} := \Big\{&T \in C^C : T \text{ is nonexpansive, } T\circ T(t) = T,\ \forall t \ge 0 \\ &\text{and every closed convex } \varphi\text{-invariant subset of } C \text{ is also } T\text{-invariant}\Big\}.\end{aligned} \tag{3.3}$$

We note that $\mathfrak{R} \ne \emptyset$, because the mapping T_μ in Lemma 3.2 belongs to $\mathfrak{R}$. By applying an argument similar to that in the proof of [9, Theorem 2] (see also the proof of [7, Lemma 3.1]), it follows that $\mathfrak{R}$ is compact and there is a minimal element $P \in \mathfrak{R}$ in the following sense:

$$\begin{aligned}&\text{if } T \in \mathfrak{R} \text{ and } \|T(x) - T(y)\| \le \|P(x) - P(y)\|, \quad \forall x, y \in C,\\ &\text{then } \|T(x) - T(y)\| = \|P(x) - P(y)\|.\end{aligned} \tag{3.4}$$

We will prove that $P(x) \in F(\varphi)$ for all $x \in C$. For a given $x \in C$, consider the set $K = \{T(P(x)) : T \in \mathfrak{R}\}$. Then K is a nonempty weakly compact convex subset of C, because $\mathfrak{R}$ is convex and compact. Take $y \in K$ and $z \in C$such as $T(t_i)y \rightharpoonup z$, for some $t_i \uparrow \infty$. There exists $h \in \mathfrak{R}$, such that $y = h(P(x))$. Consider a subnet $\{T(t_{i(\eta)})\}$ of $\{T(t_i\}$ such that $S(u) = \omega - \lim_\eta T(t_{i(\eta)})(u)$ exists for every $u \in C$. Now, taking $u = h(P(x))$, we have $z = S(h(P(x)))$. Since S is nonexpansive, $h \in \mathfrak{R}$ and $S\circ h\circ T(t) = S\circ h$ for every $t \ge 0$, it follows that $S\circ h \in \mathfrak{R}$ and then $z = S(h(P(x))) \in K$. Thus K satisfies the property (ω) with respect to the semigroup φ. Now, from Theorem 1.4, it follows that $K \cap F(\varphi) \ne \emptyset$. From this and the argument used in the proof of Theorem 2.1, we obtain $P(x) \in F(\varphi)$. Since this holds for each $x \in C$, $P^2 = P$. □

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