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Research Article

Demiclosed Principle for Asymptotically Nonexpansive Mappings in CAT(0) Spaces

B. Nanjaras and B. Panyanak

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Correspondence should be addressed to B. Panyanak, banchap@chiangmai.ac.th

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We prove the demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces. As a consequence, we obtain a Δ-convergence theorem of the Krasnosel'skii-Mann iteration for asymptotically nonexpansive mappings in this setting. Our results extend and improve many results in the literature.

1. Introduction

One of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's demiclosed principle [1] which states that if X is a uniformly convex Banach space, then C is a nonempty closed convex subset of X, and if $T : C \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at each $y \in X$, that is, for any sequence $\{x_n\}$ in C conditions $x_n \to x$ weakly and $(I - T)(x_n) \to y$ strongly imply that $(I - T)(x) = y$ (where I is the identity mapping of X). It is known that the demiclosed principle plays important role in studying the asymptotic behavior for nonexpansive mappings (see, e.g., [2–10]). In [11], Xu proved the demiclosed principle for asymptotically nonexpansive mappings in the setting of a uniformly convex Banach space. The purpose of this paper is to extend Xu's result to a special kind of metric spaces, namely, CAT(0) spaces, which will be defined in the next section. We also apply our result to obtain a Δ-convergence theorem of the Krasnosel'skii-Mann iteration for asymptotically nonexpansive mappings in the CAT(0) space setting.

2. CAT(0) Spaces

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [12]), Euclidean buildings (see [13]), the complex Hilbert ball with a hyperbolic metric (see [14]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [12]. Burago et al. [15] provide a somewhat more elementary treatment and Gromov [16] presents a deeper study.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [17, 18]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and many papers have appeared (see, e.g., [19–34]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$ (see [12, page 165]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if, for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$, by [26, Lemma 2.1(iv)] for each $t \in [0, 1]$, then there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t d(x, y), \qquad d(y, z) = (1 - t) d(x, y). \tag{2.2}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.2). By using this notation, Dhompongsa and Panyanak [26] obtained the following lemma which will be used frequently in the proof of our main results.

Lemma 2.1. *Let X be a CAT(0) space. Then*

$$d\big((1-t)x \oplus ty, z\big) \le (1-t)d(x,z) + td\big(y,z\big), \tag{2.3}$$

for all $x, y, z \in X$ *and* $t \in [0,1]$.

If x, y_1, y_2 are points in a CAT(0) space and if $y_0 = (1/2)y_1 \oplus (1/2)y_2$, then the CAT(0) inequality implies that

$$d(x,y_0)^2 \le \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [35]. In fact (cf. [12, page 163]), a geodesic space is a CAT(0) space if and only if it satisfies (CN).

The following lemma is a generalization of the (CN) inequality which can be found in [26].

Lemma 2.2. *Let* (X,d) *be a CAT(0) space. Then*

$$d\big((1-t)x \oplus ty, z\big)^2 \le (1-t)d(x,z)^2 + td\big(y,z\big)^2 - t(1-t)d\big(x,y\big)^2, \tag{2.4}$$

for all $t \in [0,1]$ *and* $x, y, z \in X$.

3. Demiclosed Principle

In 1976, Lim [36] introduced a concept of convergence in a general metric space which he called "Δ-convergence". In 2008, Kirk and Panyanak [37] specialized Lim's concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Since then the notion of Δ-convergence has been widely studied and a number of papers have appeared (see, e.g., [26, 29, 31, 32, 34]).

We now give the concept of Δ-convergence and collect some of its basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n). \tag{3.1}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}, \tag{3.2}$$

the *asymptotic radius* $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\}, \tag{3.3}$$

the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}, \tag{3.4}$$

and the *asymptotic center* $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}. \tag{3.5}$$

Recall that a bounded sequence $\{x_n\}$ in X is said to be *regular* if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

The following proposition was proved in [22].

Proposition 3.1. *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space X and if C is a closed convex subset of X, then there exists a unique point $u \in C$ such that*

$$r(u, \{x_n\}) = \inf_{x \in C} r(x, \{x_n\}). \tag{3.6}$$

This fact immediately yields the following proposition.

Proposition 3.2. *Let $\{x_n\}$, C, and X be as in Proposition 3.1. Then $A(\{x_n\})$ and $A_C(\{x_n\})$ are singletons.*

The following lemma can be found in [25].

Lemma 3.3. *If C is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Definition 3.4 (see [36, 37]). A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 3.5 (see [37]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

There is another concept of convergence in geodesic spaces; it was introduced by Sosov [38] and was specialized to CAT(0) spaces by Espínola and Fernández-León [29] as follows.

Let X be a CAT(0) space and let p be a point in X. Let $\mathcal{S}$ be the set of all the geodesic segments containing the point p. Given $l \in \mathcal{S}$ and $x \in X$, we define the function $\phi_l : X \to \mathbb{R}$ as $\phi_l(x) = d(p, P_l(x))$ where $P_l(x)$ is the projection of x onto l. The set of all these ϕ_l is denoted by $\Phi_p(X)$.

Definition 3.6. A bounded sequence $\{x_n\}$ in X is said to ϕ_p-converge to a point $x \in X$ if

$$\lim_{n \to \infty} \phi(x_n) = \phi(x) \quad \text{for any } \phi \in \Phi_p(X). \tag{3.7}$$

In [29], the authors showed that Δ-convergence and ϕ-convergence are equivalent as the following result.

Proposition 3.7. *A sequence $\{x_n\}$ in a CAT(0) space Δ-converges to p if and only if it ϕ_p-converges to p.*

Recall that a mapping T on a metric space X is said to be *nonexpansive* if

$$d(T(x), T(y)) \le d(x, y), \quad \forall x, y \in X. \tag{3.8}$$

T is called *asymptotically nonexpansive* if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n(x), T^n(y)) \le k_n d(x, y), \quad \forall n \ge 1,\ x, y \in X. \tag{3.9}$$

A point $x \in X$ is called a fixed point of T if $x = T(x)$. We will denote with $F(T)$ the set of fixed points of T. The existence of fixed points for asymptotically nonexpansive mappings in CAT(0) spaces was proved by Kirk [18] as the following result.

Theorem 3.8. *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be asymptotically nonexpansive. Then T has a fixed point.*

Now, we discuss the notion of asymptotic contractions which was introduced by Kirk [39] as the following statement.

Let Ψ denote the class of all mappings $\psi : [0,\infty) \to [0,\infty)$ satisfying what follows:

(i) ψ is continuous,

(ii) $\psi(s) < s$ for all $s > 0$.

Definition 3.9. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an *asymptotic contraction* (see[39]) if

$$d(T^n(x), T^n(y)) \le \psi_n(d(x, y)) \quad \forall x, y \in X, \tag{3.10}$$

where $\psi_n : [0,\infty) \to [0,\infty)$ and $\psi_n \to \psi \in \Psi$ uniformly on the range of d.

T is called a *pointwise contraction* (see[40]) if there exists a mapping $\alpha : X \to [0,1)$ such that

$$d(T(x), T(y)) \le \alpha(x) d(x, y) \quad \text{for each } y \in X. \tag{3.11}$$

Definition 3.10. Let (X, d) be a metric space. A mapping $T : X \to X$ is called an *asymptotic pointwise mapping* (see[30]) if there exists a sequence of mappings $\alpha_n : X \to [0,\infty)$ such that

$$d(T^n(x), T^n(y)) \le \alpha_n(x) d(x, y) \quad \text{for any } y \in X. \tag{3.12}$$

(i) If $\{\alpha_n\}$ converges pointwise to $\alpha : X \to [0,1)$, then T is called an asymptotic pointwise contraction.

(ii) If $\limsup_{n\to\infty}\alpha_n(x) \leq 1$, then T is called asymptotic pointwise nonexpansive.

(iii) If $\limsup_{n\to\infty}\alpha_n(x) \leq k$, with $0 < k < 1$, then T is called strongly asymptotic pointwise contraction.

It is immediately clear that an asymptotically nonexpansive mapping is asymptotic pointwise nonexpansive. By using the ultrapower technique, Kirk [39] established the existence of fixed points for asymptotic contractions in complete metric spaces. In [40], Kirk and Xu gave simple and elementary proofs for the existence of fixed points for asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings in Banach spaces without the use of ultrapowers. Very recently, Hussain and Khamsi [30] extended Kirk-Xu's results to CAT(0) spaces. Moreover, they introduced a notion of convergence in CAT(0) spaces as follows.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let C be a closed convex subset of X which contains $\{x_n\}$. We denote the notation

$$\{x_n\} \rightharpoonup w \quad \text{iff } \Phi(w) = \inf_{x\in C}\Phi(x), \tag{3.13}$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$. By using this notation, they obtained the demiclosed principle for asymptotic pointwise nonexpansive mappings as the following result.

Proposition 3.11. *Let C be a closed and convex subset of a complete CAT(0) space X, $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_n d(x_n, T(x_n)) = 0$, and $\{x_n\} \rightharpoonup w$. Then $T(w) = w$.*

We now give a connection between this kind of convergence and Δ-convergence.

Proposition 3.12. *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let C be a closed convex subset of X which contains $\{x_n\}$. Then*

(1) $\Delta - \lim_n x_n = x$ *implies that* $\{x_n\} \rightharpoonup x$,

(2) *the converse of* (2.2) *is true if* $\{x_n\}$ *is regular.*

Proof. (1) Suppose that $\Delta - \lim_n x_n = x$, then $x \in C$ by Lemma 3.3. Since $A(\{x_n\}) = \{x\}$, we have $r(\{x_n\}) = r(x, \{x_n\})$. This implies that $\Phi(x) = \inf_{y\in C}\Phi(y)$. Therefore we obtain the desired result.

(2) Suppose that $\{x_n\}$ is regular and $\{x_n\} \rightharpoonup x$. We note that $\{x_n\} \rightharpoonup x$ if and only if $A_C(\{x_n\}) = \{x\}$. Suppose that $A(\{x_n\}) = \{y\}$, again by Lemma 3.3, we have $y \in C$. Therefore $x = y$, and hence, $A(\{x_n\}) = \{x\}$. By the regularity of $\{x_n\}$, we have $A(\{x_n\}) \subset A(\{u_n\})$ for each subsequence $\{u_n\}$ of $\{x_n\}$. Thus $\Delta - \lim_n x_n = x$ since the asymptotic center of any bounded sequence in X must be a singleton. □

The following example shows that the regularity in Proposition 3.12 is necessary.

Example 3.13. Let $X = \mathbb{R}$, d be the usual metric on X, $C = [-1, 1]$, $\{x_n\} = \{1, -1, 1, -1, \ldots\}$, $\{u_n\} = \{-1, -1, -1, \ldots\}$, and $\{v_n\} = \{1, 1, 1, \ldots\}$. Then $A(\{x_n\}) = A_C(\{x_n\}) = \{0\}$, $A(\{u_n\}) = \{-1\}$, and $A(\{v_n\}) = \{1\}$. This means that $\{x_n\} \rightharpoonup 0$ but it does not have a Δ-limit.

Now, we extend Proposition 3.11 to the case of non-self-mappings. The proof closely follows the proof of Proposition 1 in [30]. For the convenience of the reader we include the details.

Proposition 3.14. *Let C be a closed and convex subset of a complete CAT(0) space X and let $T : C \to X$ be an asymptotic pointwise nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_n d(x_n, T(x_n)) = 0$ and $\{x_n\} \rightharpoonup w$. Then $T(w) = w$.*

Proof. As we have observed in the proof of Proposition 3.12, $\{x_n\} \rightharpoonup w$ if and only if $A_C(\{x_n\}) = \{w\}$. By Lemma 3.3, we have $A(\{x_n\}) = \{w\}$. Since $\lim_n d(x_n, T(x_n)) = 0$, then we have $\Phi(x) = \limsup_{n\to\infty} d(T^m(x_n), x)$ for each $x \in C$ and $m \geq 1$. It follows that $\Phi(T^m(x)) \leq \alpha_m(x)\Phi(x)$. In particular, we have $\Phi(T^m(w)) \leq \alpha_m(w)\Phi(w)$ for all $m \geq 1$. Hence

$$\limsup_{m\to\infty} \Phi(T^m(w)) \leq \Phi(w). \tag{3.14}$$

The (CN) inequality implies that

$$d\left(x_n, \frac{w \oplus T^m(w)}{2}\right)^2 \leq \frac{1}{2}d(x_n, w)^2 + \frac{1}{2}d(x_n, T^m(w))^2 - \frac{1}{4}d(w, T^m(w))^2, \tag{3.15}$$

for any $n, m \geq 1$. By taking $n \to \infty$, we get

$$\Phi\left(\frac{w \oplus T^m(w)}{2}\right)^2 \leq \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(T^m(w))^2 - \frac{1}{4}d(w, T^m(w))^2, \tag{3.16}$$

for any $m \geq 1$. Since $A(\{x_n\}) = \{w\}$, we have

$$\Phi(w)^2 \leq \Phi\left(\frac{w \oplus T^m(w)}{2}\right)^2 \leq \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(T^m(w))^2 - \frac{1}{4}d(w, T^m(w))^2, \tag{3.17}$$

for any $m \geq 1$, which implies that

$$d(w, T^m(w))^2 \leq 2\Phi(T^m(w))^2 - 2\Phi(w)^2. \tag{3.18}$$

By (3.14) and (3.18) we have $\lim_{m\to\infty} d(w, T^m(w)) = 0$. Hence $T(w) = w$ as desired. □

As a consequence, we obtain the following corollary which is a generalization of [37, Proposition 3.7].

Corollary 3.15. *Let C be a closed and convex subset of a complete CAT(0) space X and let $T : C \to X$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_n d(x_n, T(x_n)) = 0$ and $\Delta - \lim_n x_n = w$. Then $T(w) = w$.*

4. Hyperbolic Spaces

We begin this section by talking about hyperbolic spaces. This class contains the class of CAT(0) spaces (see Lemma 4.4 below).

Definition 4.1 (see [41]). A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is such that

(W1) $d(z, W(x, y, \alpha)) \leq (1-\alpha)d(z, x) + \alpha d(z, y)$,

(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,

(W3) $W(x, y, \alpha) = W(y, x, 1-\alpha)$,

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$ for all $x, y, z, w \in X, \alpha, \beta \in [0,1]$.

It follows from (W1) that, for each $x, y \in X$ and $\alpha \in [0,1]$,

$$d(x, W(x, y, \alpha)) \leq \alpha d(x, y), \qquad d(y, W(x, y, \alpha)) \leq (1-\alpha)d(x, y). \tag{4.1}$$

In fact, we have

$$d(x, W(x, y, \alpha)) = \alpha d(x, y), \qquad d(y, W(x, y, \alpha)) = (1-\alpha)d(x, y), \tag{4.2}$$

since if

$$d(x, W(x, y, \alpha)) < \alpha d(x, y) \quad \text{or} \quad d(y, W(x, y, \alpha)) < (1-\alpha)d(x, y), \tag{4.3}$$

then we get

$$\begin{aligned} d(x, y) &\leq d(x, W(x, y, \alpha)) + d(W(x, y, \alpha), y) \\ &< \alpha d(x, y) + (1-\alpha)d(x, y) \\ &= d(x, y), \end{aligned} \tag{4.4}$$

which is a contradiction. By comparing between (2.2) and (4.2), we can also use the notation $(1-\alpha)x \oplus \alpha y$ for $W(x, y, \alpha)$ in a hyperbolic space (X, d, W).

Definition 4.2 (see [41]). The hyperbolic space (X, d, W) is called *uniformly convex* if for any $r > 0$ and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$ such that, for all $a, x, y \in X$,

$$\left.\begin{aligned} d(x, a) &\leq r \\ d(y, a) &\leq r \\ d(x, y) &\geq \varepsilon r \end{aligned}\right\} \Longrightarrow d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r. \tag{4.5}$$

A mapping $\eta : (0, \infty) \times (0,2] \to (0,1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called *a modulus of uniform convexity.*

Lemma 4.3 (see [41, Lemma 7]). *Let (X,d,W) be a uniformly convex hyperbolic space with modulus of uniform convexity η. For any $r>0$, $\varepsilon\in(0,2]$, $\lambda\in[0,1]$, and $a,x,y\in X$,*

$$\left.\begin{array}{r} d(x,a)\le r \\ d(y,a)\le r \\ d(x,y)\ge \varepsilon r \end{array}\right\} \Longrightarrow d((1-\lambda)x\oplus\lambda y,a)\le(1-2\lambda(1-\lambda)\eta(r,\varepsilon))r. \tag{4.6}$$

Lemma 4.4 (see [41, Proposition 8]). *Assume that X is a CAT(0) space. Then X is uniformly convex, and*

$$\eta(r,\varepsilon)=\frac{\varepsilon^2}{8} \tag{4.7}$$

is a modulus of uniform convexity.

The following result is a characterization of uniformly convex hyperbolic spaces which is an analog of Schu [42, Lemma 1.3]. It can be applied to a CAT(0) space as well.

Lemma 4.5. *Let (X,d,W) be a uniformly convex hyperbolic space with modulus of convexity η, and let $x\in X$. Suppose that η increases with r (for a fixed ε), that $\{t_n\}$ is a sequence in $[b,c]$ for some $b,c\in(0,1)$, and $\{x_n\}$, $\{y_n\}$ are sequences in X such that* $\limsup_n d(x_n,x)\le r$, $\limsup_n d(y_n,x)\le r$, *and* $\lim_n d((1-t_n)x_n\oplus t_n y_n,x)=r$ *for some $r\ge 0$. Then*

$$\lim_{n\to\infty} d(x_n,y_n)=0. \tag{4.8}$$

Proof. The case $r=0$ is trivial. Now suppose that $r>0$. If it is not the case that $d(x_n,y_n)\to 0$ as $n\to\infty$, then there are subsequences, denoted by $\{x_n\}$ and $\{y_n\}$, such that

$$\inf_n d(x_n,y_n)>0. \tag{4.9}$$

Choose $\varepsilon\in(0,1]$ such that

$$d(x_n,y_n)\ge\varepsilon(r+1)>0\quad\forall n\in\mathbb{N}. \tag{4.10}$$

Since $0<b(1-c)\le 1/2$ and $0<\eta(r,\varepsilon)\le 1, 0<2b(1-c)\eta(r,\varepsilon)\le 1$. This implies that $0\le 1-2b(1-c)\eta(r,\varepsilon)<1$. Choose $R\in(r,r+1)$ such that

$$(1-2b(1-c)\eta(r,\varepsilon))R<r. \tag{4.11}$$

Since

$$\limsup_n d(x_n,x)\le r,\quad \limsup_n d(y_n,x)\le r,\quad r<R, \tag{4.12}$$

there are further subsequences again denoted by $\{x_n\}$ and $\{y_n\}$ such that

$$d(x_n, x) \le R, \quad d(y_n, x) \le R, \quad d(x_n, y_n) \ge \varepsilon R \quad \forall n \in \mathbb{N}. \tag{4.13}$$

Then by Lemma 4.3 and (4.11),

$$\begin{aligned} d((1-t_n)x_n \oplus t_n y_n, x) &\le (1 - 2t_n(1-t_n)\eta(R, \varepsilon))R \\ &\le (1 - 2b(1-c)\eta(r, \varepsilon))R < r, \end{aligned} \tag{4.14}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) < r, \tag{4.15}$$

which contradicts the hypothesis. □

5. Δ-Convergence Theorem

We now give an application of Corollary 3.15. The following concept for Banach spaces is due to Schu [42]. Let C be a nonempty closed convex subset of a CAT(0) space X and let $T : C \to C$ be an asymptotically nonexpansive mapping. The Krasnoselski-Mann iteration starting from $x_1 \in C$ is defined by

$$x_{n+1} = \alpha_n T^n(x_n) \oplus (1-\alpha_n)x_n, \quad n \ge 1, \tag{5.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Recall that a mapping T from a subset C of a CAT(0) space X into itself is called *uniformly Lipschitzian* [43] if there exists $L > 0$ such that

$$d(T^n(x), T^n(y)) \le L d(x, y) \quad \forall x, y \in C, \; n \in \mathbb{N}. \tag{5.2}$$

In this case, we call T a uniformly L-Lipschitzian mapping. We also note from (3.9) and (5.2) that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian for some $L > 0$.

The following lemma can be found in [44].

Lemma 5.1. *Let C be a nonempty convex subset of a CAT(0) space X and let $T : C \to C$ be uniformly L-Lipschitzian for some $L > 0$, $\{\alpha_n\} \subset [0, 1]$, and $x_1 \in C$. Suppose that $\{x_n\}$ is given by* (5.1)*, and set $c_n = d(T^n(x_n), x_n)$ for all $n \in \mathbb{N}$. Then*

$$d(x_n, T(x_n)) \le c_n + c_{n-1} L\left(1 + 3L + 2L^2\right) \quad \forall n \in \mathbb{N}. \tag{5.3}$$

The following lemma can be found in [45].

Lemma 5.2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + b_n) a_n \quad \forall n \geq 1. \tag{5.4}$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists. In particular, if there is a subsequence of $\{a_n\}$ which converges to 0, then $\lim_{n\to\infty} a_n = 0$.

The following lemmas are also needed.

Lemma 5.3. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be asymptotically nonexpansive with a sequence $\{k_n\}$ in $[1, \infty)$ for which $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $x_1 \in C$, $\{\alpha_n\} \subset [0,1]$, and $\{x_n\}$ is given by (5.1). Then $\lim_{n\to\infty} d(x_n, x)$ exists for each $x \in F(T)$.*

Proof. Let $x \in F(T)$, then we have

$$\begin{aligned} d(x_{n+1}, x) &\leq d(\alpha_n T^n(x_n) \oplus (1 - \alpha_n) x_n, x) \\ &\leq \alpha_n d(T^n(x_n), x) + (1 - \alpha_n) d(x_n, x) \\ &\leq \alpha_n d(T^n(x_n), T^n(x)) + (1 - \alpha_n) d(x_n, x) \\ &\leq \alpha_n k_n d(x_n, x) + (1 - \alpha_n) d(x_n, x) \\ &\leq [\alpha_n k_n + (1 - \alpha_n)] d(x_n, x) \\ &\leq [1 + \alpha_n (k_n - 1)] d(x_n, x). \end{aligned} \tag{5.5}$$

Hence

$$d(x_{n+1}, x) \leq [1 + \alpha_n (k_n - 1)] d(x_n, x). \tag{5.6}$$

Since $\{d(x_n, x)\}$ is bounded and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, by Lemma 5.2, we get that $\lim_{n\to\infty} d(x_n, x)$ exists. This completes the proof. $\square$

Lemma 5.4. *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be asymptotically nonexpansive with a sequence $\{k_n\}$ in $[1, \infty)$ for which $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$. Suppose that $x_1 \in C$ and that $\{x_n\}$ is given by (5.1). Then*

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0. \tag{5.7}$$

Proof. It follows from Theorem 3.8 that T has a fixed point $x \in C$. In view of Lemma 5.3 we can let $\lim_n d(x_n, x) = c$ for some c in $\mathbb{R}$. Since

$$d(T^n(x_n), x) = d(T^n(x_n), T^n(x)) \leq k_n d(x_n, x), \tag{5.8}$$

for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} d(T^n(x_n), x) \le c. \tag{5.9}$$

On the other hand, since

$$\begin{aligned} d(x_{n+1}, x) &\le \alpha_n d(T^n(x_n), x) + (1-\alpha_n) d(x_n, x) \\ &\le \alpha_n d(T^n(x_n), T^n(x)) + (1-\alpha_n) d(x_n, x) \\ &\le [\alpha_n k_n + (1-\alpha_n)] d(x_n, x) \\ &\le k_n d(x_n, x), \end{aligned} \tag{5.10}$$

then

$$d(x_{n+1}, x) \le d(\alpha_n T^n(x_n) \oplus (1-\alpha_n) x_n, x) \le k_n d(x_n, x). \tag{5.11}$$

Hence

$$\lim_{n\to\infty} (d(\alpha_n T^n(x_n) \oplus (1-\alpha_n) x_n, x)) = c. \tag{5.12}$$

By Lemma 4.5, we have $\lim_{n\to\infty} d(T^n(x_n), x_n) = 0$. As we have observed, every asymptotically nonexpansive mapping is also uniformly L-Lipschitzian for some $L > 0$; it follows from Lemma 5.1 that $\lim_{n\to\infty} d(T(x_n), x_n) = 0$. This completes the proof. □

Lemma 5.5 (see [26]). *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 5.6. *Let C be a closed convex subset of a complete CAT(0) space X and let $T : C \to X$ be an asymptotically nonexpansive mapping. Suppose that $\{x_n\}$ is a bounded sequence in C such that $\lim_n d(x_n, T(x_n)) = 0$ and $d(x_n, v)$ converges for each $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) = \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 3.5 and 3.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. By Corollary 3.15, we have $v \in F(T)$. By Lemma 5.5, $u = v$. This shows that $\omega_w(x_n) \subset F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, $\{d(x_n, u)\}$ converges. By Lemma 5.5, $x = u$. This completes the proof. □

We are now ready to prove our main result.

Theorem 5.7. *Let C be a bounded closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be asymptotically nonexpansive with a sequence $\{k_n\} \subset [1, \infty)$ for which $\sum_{n=1}^{\infty} (k_n - 1) < \infty$.*

Suppose that $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[a,b]$ for some $a,b \in (0,1)$. Then the sequence $\{x_n\}$ given by (5.1) *Δ-converges to a fixed point of T.*

Proof. It follows from Theorem 3.8 that $F(T)$ is nonempty. By Lemma 5.3, $\{d(x_n,v)\}$ is convergent for each $v \in F(T)$. By Lemma 5.4, we have $\lim_{n\to\infty} d(x_n,T(x_n)) = 0$. By Lemma 5.6, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. □

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