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Research Article

Strong and Weak Convergence of the Modified Proximal Point Algorithms in Hilbert Space

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For a monotone operator T, we shall show weak convergence of Rockafellar's proximal point algorithm to some zero of T and strong convergence of the perturbed version of Rockafellar's to $P_Z u$ under some relaxed conditions, where P_Z is the metric projection from H onto $Z = T^{-1}0$. Moreover, our proof techniques are simpler than some existed results.

1. Introduction

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let I be on identity operator in H. We shall denote by $\mathbb{N}$ the set of all positive integers, by Z the set of all zeros of T, that is, $Z = T^{-1}0 = \{x \in D(T); 0 \in Tx\}$ and by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{x \in E; Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Let T be an operator with domain $D(T)$ and range $R(T)$ in H. Recall that T is said to be *monotone* if

$$\left\langle x - y, x' - y' \right\rangle \geq 0, \quad \forall x, y \in D(T),\ x' \in Tx,\ y' \in Ty. \tag{1.1}$$

A monotone operator T is said to be *maximal monotone* if T is monotone and $R(I + rT) = H$ for all $r > 0$.

In fact, theory of monotone operator is very important in nonlinear analysis and is connected with theory of differential equations. It is well known (see [1]) that many physically significant problems can be modeled by the initial-value problems of the form

$$x\prime(t) + Tx(t) = 0,$$
$$x(0) = x_0, \tag{1.2}$$

where T is a monotone operator in an appropriate space. Typical examples where such evolution equations occur can be found in the heat and wave equations or Schrodinger equations. On the other hand, a variety of problems, including convex programming and variational inequalities, can be formulated as finding a zero of monotone operators. Then the problem of finding a solution $x \in H$ with $0 \in Tx$ has been investigated by many researchers; see, for example, Bruck [2], Rockafellar [3], Brézis and Lions [4], Reich [5, 6], Nevanlinna and Reich [7], Bruck and Reich [8], Jung and Takahashi [9], Khang [10], Minty [11], Xu [12], and others. Some of them dealt with the weak convergence of (1.4) and others proved strong convergence theorems by imposing strong assumptions on T.

One popular method of solving $0 \in Tx$ is the proximal point algorithm of Rockafellar [3] which is recognized as a powerful and successful algorithm in finding a zero of monotone operators. Starting from any initial guess $x_0 \in H$, this proximal point algorithm generates a sequence $\{x_k\}$ given by

$$x_{k+1} = J_{c_k}^{T}(x_k + e_k), \tag{1.3}$$

where $J_r^T = (I + rT)^{-1}$ for all $r > 0$ is the resolvent of T on the space H. Rockafellar's [3] proved the weak convergence of his algorithm (1.3) provided that the regularization sequence $\{c_k\}$ remains bounded away from zero and the error sequence $\{e_k\}$ satisfies the condition $\sum_{k=0}^{+\infty} \|e_k\| < \infty$. Güler's example [13] however shows that in an infinite-dimensional Hilbert space, Rochafellar's algorithm (1.3) has only weak convergence. Recently several authors proposed modifications of Rochafellar's proximal point algorithm (1.3) to have strong convergence. For examples, Solodov and Svaiter [14] and Kamimura and Takahashi [15] studied a modified proximal point algorithm by an additional projection at each step of iteration. Lehdili and Moudafi [16] obtained the convergence of the sequence $\{x_k\}$ generated by the algorithm

$$x_{k+1} = J_{\lambda_k}^{T_k} x_k, \quad k \geq 0, \tag{1.4}$$

where $T_k = \mu_k I + T$, $\mu_k > 0$, is viewed as a Tikhonov regularization of T. Using the technique of variational distance, Lehdili and Moudafi [16] were able to prove convergence theorems for the algorithm (1.4) and its perturbed version, under certain conditions imposed upon the sequences $\{\lambda_k\}$ and $\{\mu_k\}$. For a maximal monotone operator T, Xu [12] and Song and Yang [17] used the technique of nonexpansive mappings to get convergence theorems for $\{x_k\}$ defined by the perturbed version of the algorithm (1.4):

$$x_{k+1} = J_{r_k}^{T}(t_k u + (1 - t_k)x_k). \tag{1.5}$$

In this paper, under more relaxed conditions on the sequences $\{r_k\}$ and $\{t_k\}$, we shall show that the sequence $\{x_k\}$ generated by (1.5) converges strongly to $P_Z u \in T^{-1}0$ (where P_Z is the metric projection from H onto Z) and the sequence $\{x_k\}$ generated by (1.3) weakly converges to some $x^* \in T^{-1}0$. Moreover, our proof techniques are simpler than those of Lehdili and Moudafi [16], Xu [12], and Song and Yang [17].

2. Preliminaries and Basic Results

Let T be a monotone operator with $Z \neq \emptyset$. We use J_r^T and A_r to denote the resolvent and Yosida's approximation of T, respectively. Namely,

$$J_r^T = (I + rT)^{-1}, \qquad A_r = \frac{I - J_r^T}{r}, \quad r > 0. \tag{2.1}$$

For J_r^T and A_r, the following is well known. For more details, see [18, Pages 369–400] or [3, 19].

(i) $A_r x \in T J_r^T x$ for all $x \in R(I + rT)$;

(ii) $\|A_r x\| \leq |Tx| = \inf\{\|y\|; y \in Tx\}$ for all $x \in D(T) \cap R(I + rT)$;

(iii) $J_r^T : R(I + rT) \to D(I + rT) = D(T)$ is a single-valued nonexpansive mapping for each $r > 0$ (i.e., $\|J_r^T x - J_r^T y\| \leq \|x - y\|$ for all $x, y \in R(I + rT)$);

(iv) $Z = T^{-1}0 = F(J_r^T) = \{x \in D(J_r); J_r^T x = x\}$ is closed and convex;

(v) (The Resolvent Identity) For $r > 0$ and $t > 0$ and $x \in E$,

$$J_r^T x = J_t^T \left(\frac{t}{r} x + \left(1 - \frac{t}{r}\right) J_r^T x \right). \tag{2.2}$$

In the rest of this paper, it is always assumed that Z is nonempty so that the metric projection P_Z from H onto Z is well defined. It is known that P_Z is nonexpansive and characterized by the inequality: given $x \in H$ and $v \in Z$; then $v = P_Z x$ if and only if

$$\langle x - v, y - v \rangle \leq 0, \quad \forall y \in Z. \tag{2.3}$$

In order to facilitate our investigation in the next section we list a useful lemma.

Lemma 2.1 (see Xu [20, Lemma 2.5]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{k+1} \leq (1 - \lambda_k) a_k + \lambda_k \beta_k + \sigma_k, \quad \forall k \geq 0, \tag{2.4}$$

where $\{\lambda_k\}$, $\{\beta_k\}$, and $\{\sigma_k\}$ satisfy the conditions (i) $\sum_{k=0}^{\infty} \lambda_k = \infty$; (ii) either $\limsup_{k\to\infty} \beta_k \leq 0$ or $\sum_{k=0}^{\infty} |\lambda_k \beta_k| < \infty$; (iii) $\sigma_k \geq 0$ for all k and $\sum_{k=0}^{\infty} \sigma_k < \infty$. Then $\{a_k\}$ converges to zero as $k \to \infty$.

3. Strongly Convergence Theorems

Let T be a monotone operator on a Hilbert space H. Then J_r^T is a single-valued nonexpansive mapping from $R(I+rT)$ to $D(I+rT)=D(T)\cap D(I)=D(T)$. When K is a nonempty closed convex subset of H such that $\overline{D(T)}\subset K\subset R(I+rT)$ for all $r>0$ (here $\overline{D(T)}$ is closure of $D(T)$), then we have $t_k u+(1-t_k)x_k\in K\subset R(I+r_kT)$ for $u, x_k\in K$ and all $k\in\mathbb{N}$, and hence the following iteration is well defined

$$x_{k+1}=J_{r_k}^T(t_k u+(1-t_k)x_k). \tag{3.1}$$

Next we will show strong convergence of $\{x_k\}$ defined by (3.1) to find a zero of T. For reaching this objective, we always assume $Z=T^{-1}0\neq\emptyset$ in the sequel.

Theorem 3.1. *Let T be a monotone operator on a Hilbert space H with $Z=T^{-1}0\neq\emptyset$. Assume that K is a nonempty closed convex subset of H such that $\overline{D(T)}\subset K\subset R(I+rT)$ for all $r>0$ and for an anchor point $u\in K$ and an initial value $x_0\in K$, $\{x_k\}$ is iteratively defined by* (3.1)*. If $\{t_k\}\subset(0,1)$ and $\{r_k\}\subset(0,+\infty)$ satisfy*

(i) $\lim_{k\to\infty} t_k=0$;

(ii) $\sum_{k=0}^{+\infty} t_k=\infty$;

(iii) $\lim_{k\to\infty} r_k=\infty$,

then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.

Proof. The proof consists of the following steps:

Step 1. The sequence $\{x_k\}$ is bounded. Let $y_k=t_k u+(1-t_k)x_k$, then $x_{k+1}=J_{r_k}^T y_k$ and for some $z\in T^{-1}0=F(J_r^T)$, we have

$$\begin{aligned}\|x_{k+1}-z\| &= \left\|J_{r_k}^T y_k-z\right\| \le \|y_k-z\| = \|t_k u+(1-t_k)x_k-z\|\\ &\le t_k\|u-z\|+(1-t_k)\|x_k-z\|\\ &\le \max\{\|x_k-z\|,\|u-z\|\}\\ &\ \ \vdots\\ &\le \max\{\|x_0-z\|,\|u-z\|\}.\end{aligned} \tag{3.2}$$

So, the sequences $\{x_k\}$, $\{y_k\}$, and $\{J_{r_k}^T y_k\}$ are bounded.

Step 2. $\lim_{k\to\infty}\|x_k-J_r^T x_k\|=0$ for each $r>0$. Since

$$\begin{aligned}\left\|x_{k+1}-J_r^T x_{k+1}\right\| &= \left\|J_{r_k}^T y_k-J_r^T J_{r_k}^T y_k\right\| = \left\|\left(I-J_r^T\right)J_{r_k}^T y_k\right\|\\ &= r\left\|A_r J_{r_k}^T y_k\right\| \le r\left|T J_{r_k}^T y_k\right| \le r\left\|A_{r_k} y_k\right\|\\ &= r\frac{\left\|y_k-J_{r_k}^T y_k\right\|}{r_k}\longrightarrow 0\quad (k\longrightarrow\infty),\end{aligned} \tag{3.3}$$

we have

$$\lim_{k\to\infty}\left\|x_k - J_r^T x_k\right\| = 0. \tag{3.4}$$

Step 3. $\limsup_{k\to\infty}\langle u - P_Z u, x_k - P_Z u\rangle \le 0$. Indeed, we can take a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$\limsup_{k\to\infty}\langle u - P_S u, x_k - P_S u\rangle = \lim_{i\to\infty}\langle u - P_S u, x_{k_i} - P_S u\rangle. \tag{3.5}$$

We may assume that $x_{k_i} \rightharpoonup x^*$ by the reflexivity of H and the boundedness of $\{x_k\}$. Then $x^* \in Z = T^{-1}0 = F(J_r^T)$. In fact, since

$$\begin{aligned}
\left\|x_{k_i} - J_r^T x^*\right\|^2 &= \left\|x_{k_i} - x^* + x^* - J_r^T x^*\right\|^2 \\
&= \|x_{k_i} - x^*\|^2 + 2\left\langle x_{k_i} - x^*, x^* - J_r^T x^*\right\rangle + \left\|x^* - J_r^T x^*\right\|^2, \\
\left\|x_{k_i} - J_r^T x^*\right\| &= \left\|x_{k_i} - J_r^T x_{k_i} + J_r^T x_{k_i} - J_r^T x^*\right\| \\
&\le \left\|x_{k_i} - J_r^T x_{k_i}\right\| + \left\|J_r^T x_{k_i} - J_r^T x^*\right\| \\
&\le \left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|,
\end{aligned} \tag{3.6}$$

then, for some constant $L > 0$, we have

$$\begin{aligned}
&\|x_{k_i} - x^*\|^2 + 2\left\langle x_{k_i} - x^*, x^* - J_r^T x^*\right\rangle + \left\|x^* - J_r^T x^*\right\|^2 \\
&\quad = \left\|x_{k_i} - J_r^T x^*\right\|^2 \le \left(\left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|\right)^2 \\
&\quad = \left(\left\|x_{k_i} - J_r^T x_{k_i}\right\| + 2\|x_{k_i} - x^*\|\right)\left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|^2 \le L\left\|x_{k_i} - J_r^T x_{k_i}\right\| + \|x_{k_i} - x^*\|^2.
\end{aligned} \tag{3.7}$$

Thus,

$$2\langle x_{k_i} - x^*, x^* - J_r^T x^*\rangle + \left\|x^* - J_r^T x^*\right\|^2 \le L\left\|x_{k_i} - J_r^T x_{k_i}\right\|. \tag{3.8}$$

Take $i \to \infty$ on two sides of the above equation by means of (3.4), we must have $\|x^* - J_r^T x^*\|^2 = 0$. So, $x^* \in Z$. Hence, noting the projection inequality (2.3), we obtain

$$\limsup_{k\to\infty}\langle u - P_Z u, x_k - P_Z u\rangle = \lim_{i\to\infty}\langle u - P_Z u, x_{k_i} - P_Z u\rangle = \langle u - P_Z u, x^* - P_Z u\rangle \le 0. \tag{3.9}$$

Step 4. $x_k \to P_Z u$. Indeed,

$$\begin{aligned}\|x_{k+1}-P_Z u\|^2 &= \left\|J_{r_k}^T(t_k u+(1-t_k)x_k)-P_Z u\right\|^2 \\ &= \left\|J_{r_k}^T y_k - P_Z u\right\|^2 \le \left\|y_k - P_Z u\right\|^2 \\ &\le \|t_k(u-P_Z u)+(1-t_k)(x_k-P_Z u)\|^2 \\ &\le (1-t_k)^2\|x_k-P_Z u\|^2 + t_k^2\|u-P_Z u\|^2 + 2t_k(1-t_k)\langle u-P_Z u, x_k - P_Z u\rangle.\end{aligned} \tag{3.10}$$

Therefore,

$$\|x_{k+1}-P_Z u\|^2 \le (1-t_k)\|x_k-P_Z u\|^2 + t_k\beta_k, \tag{3.11}$$

where $\beta_k = t_k\|u-P_Z u\|^2 + 2(1-t_k)\langle u-P_Z u, x_k-P_Z u\rangle$. So, an application of Lemma 2.1 onto (3.11) yields the desired result. □

Theorem 3.2. *Let $T, H, Z, K, \{x_k\}, \{t_k\}$ be as Theorem 3.1, the condition (iii) $\lim_{k\to\infty} r_k = \infty$ is replaced by the following condition:*

$$\sum_{k=0}^{+\infty}|t_{k+1}-t_k| < \infty; \qquad 0 < \liminf_{k\to\infty} r_k, \qquad \sum_{k=0}^{\infty}\left|1-\frac{r_k}{r_{k+1}}\right| < +\infty. \tag{3.12}$$

Then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.

Proof. From the proof of Theorem 3.1, we can observe that Steps 1, 3 and 4 still hold. So we only need show to Step 2: $\lim_{k\to\infty}\|x_k - J_r^T x_k\| = 0$ for each $r > 0$.

We first estimate $\|x_{k+1}-x_k\|$. From the resolvent identity (2.2), we have

$$J_{r_k}^T y_k = J_{r_{k-1}}^T\left(\frac{r_{k-1}}{r_k}y_k + \left(1-\frac{r_{k-1}}{r_k}\right)J_{r_k}^T y_k\right). \tag{3.13}$$

Therefore, for a constant $M > 0$ with $M \ge \max\{\|u\|, \|x_k\|, \|J_{r_k}^T y_k\|, \|y_k\|\}$,

$$\begin{aligned}\|x_{k+1}-x_k\| = \left\|J_{r_k}^T y_k - J_{r_{k-1}}^T y_{k-1}\right\| &\le \left\|\frac{r_{k-1}}{r_k}y_k + \left(1-\frac{r_{k-1}}{r_k}\right)J_{r_k}^T y_k - y_{k-1}\right\| \\ &\le \left\|\frac{r_{k-1}}{r_k}(y_k - y_{k-1}) + \left(1-\frac{r_{k-1}}{r_k}\right)\left(J_{r_k}^T y_k - y_{k-1}\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&\le \left\|y_k - y_{k-1}\right\| + \left|1 - \frac{r_{k-1}}{r_k}\right| \left\|J_{r_k}^T y_k - y_k\right\| \\
&\le |t_k - t_{k-1}|(\|u\| + \|x_{k-1}\|) + (1 - t_k)\|x_k - x_{k-1}\| + 2M\left|1 - \frac{r_{k-1}}{r_k}\right| \\
&\le (1 - t_k)\|x_k - x_{k-1}\| + 2M\left(|t_k - t_{k-1}| + \left|1 - \frac{r_{k-1}}{r_k}\right|\right).
\end{aligned}
\tag{3.14}
$$

It follows from Lemma 2.1 that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.15}$$

As $\|y_k - J_{r_k}^T y_k\| = \|y_k - x_{k+1}\| \le t_k\|u - x_{k+1}\| + (1 - t_k)\|x_k - x_{k+1}\|$, then

$$\lim_{k\to\infty} \left\|y_k - J_{r_k}^T y_k\right\| = 0. \tag{3.16}$$

Since $0 < \liminf_{k\to\infty} r_k$, then there exists $\varepsilon > 0$ and a positive integer $N > 0$ such that for all $k > N$, $r_k \ge \varepsilon$. Thus for each $r > 0$, we also have

$$
\begin{aligned}
\left\|x_{k+1} - J_r^T x_{k+1}\right\| &= \left\|J_{r_k}^T y_k - J_r^T J_{r_k}^T y_k\right\| = \left\|\left(I - J_r^T\right) J_{r_k}^T y_k\right\| \\
&= r\left\|A_r J_{r_k}^T y_k\right\| \le r\left|T J_{r_k}^T y_k\right| \le r\left\|A_{r_k} y_k\right\| \\
&= r\frac{\left\|y_k - J_{r_k}^T y_k\right\|}{r_k} \le \frac{r}{\varepsilon}\left\|y_k - J_{r_k}^T y_k\right\| \longrightarrow 0 \quad (k \longrightarrow \infty);
\end{aligned}
\tag{3.17}
$$

we have $\lim_{k\to\infty}\|x_k - J_r^T x_k\| = 0$. □

Corollary 3.3. *Let $H, \{t_k\}, \{r_k\}, Z$ be as Theorem 3.1 or 3.2. Suppose that T is a maximal monotone operator on H and for $x_0, u \in H$, $\{x_k\}$ is defined by* (3.1)*. Then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.*

Proof. Since T is a maximal monotone, then T is monotone and satisfies the condition $\overline{D(T)} \subset H = R(I + rT)$ for all $r > 0$. Putting $K = H$, the desired result is reached. □

Corollary 3.4. *Let $H, \{t_k\}, \{r_k\}, Z$ be as Theorem 3.1 or 3.2. Suppose that T is a monotone operator on H satisfying the condition $\overline{D(T)} \subset R(I + rT)$ for all $r > 0$ and for $x_0, u \in \overline{D(T)}$, $\{x_k\}$ is defined by* (3.1)*. If $D(T)$ is convex, then the sequence $\{x_k\}$ converges strongly to $P_Z u$, where P_Z is the metric projection from H onto Z.*

Proof. Taking $K = \overline{D(T)}$, following Theorem 3.1 or 3.2, we easily obtain the desired result. □

4. Weakly Convergence Theorems

For a monotone operator T, if $\overline{D(T)} \subset R(I + rT)$ for all $r > 0$ and $x_0 \in \overline{D(T)}$, then the iteration $x_{k+1} = J_{r_k}^T x_k$ $(k \in \mathbb{N})$ is well defined. Next we will show weak convergence of $\{x_k\}$ under some assumptions.

Theorem 4.1. *Let T be a monotone operator on a Hilbert space H with $Z = T^{-1}0 \neq \emptyset$. Assume that $\overline{D(T)} \subset R(I + rT)$ for all $r > 0$ and for an initial value $x_0 \in \overline{D(T)}$, iteratively define*

$$x_{k+1} = J_{r_k}^T x_k. \tag{4.1}$$

If $\{r_k\} \subset (0, +\infty)$ satisfies

$$\lim_{k \to \infty} r_k = \infty, \tag{4.2}$$

then the sequence $\{x_k\}$ converges weakly to some $x^* \in Z$.

Proof. Take $z \in Z = T^{-1}0 = F(J_r^T)$, we have

$$\|x_{k+1} - z\| = \left\| J_{r_k}^T x_k - z \right\| \leq \|x_k - z\|. \tag{4.3}$$

Therefore, $\{\|x_k - z\|\}$ is nonincreasing and bounded below, and hence the limit $\lim_{k \to \infty} \|x_k - z\|$ exists for each $z \in Z$. Further, $\{x_k\}$ is bounded. So we have

$$\begin{aligned}
\left\| x_{k+1} - J_r^T x_{k+1} \right\| &= \left\| J_{r_k}^T x_k - J_r^T J_{r_k}^T x_k \right\| = \left\| \left(I - J_r^T \right) J_{r_k}^T x_k \right\| \\
&= r \left\| A_r J_{r_k}^T x_k \right\| \leq r \left| T J_{r_k}^T x_k \right| \leq r \|A_{r_k} x_k\| \\
&= r \frac{\left\| x_k - J_{r_k}^T x_k \right\|}{r_k} = \frac{r \|x_k - x_{k+1}\|}{r_k} \longrightarrow 0 \quad (k \longrightarrow \infty).
\end{aligned} \tag{4.4}$$

Hence,

$$\lim_{k \to \infty} \left\| x_k - J_r^T x_k \right\| = 0. \tag{4.5}$$

As $\{x_k\}$ is weakly sequentially compact by the reflexivity of H, and hence we may assume that there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup x^*$. Using the proof technique of Step 3 in Theorem 3.1, we must have that $x^* \in Z = T^{-1}0$.

Now we prove that $\{x_n\}$ converges weakly to x^*. Supposed that there exists another subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which weakly converges to some $y \in K$. We also have $y \in Z = T^{-1}0$. Because $\lim_{k \to \infty} \|x_k - z\|$ exists for each $z \in Z = T^{-1}0$ and

$$\begin{aligned}
\left\| x_{k_j} - y \right\|^2 &= \left\| x_{k_j} - x^* \right\|^2 + 2\left\langle x_{k_j} - x^*, x^* - y \right\rangle + \|x^* - y\|^2, \\
\|x_{k_i} - x^*\|^2 &= \left\| x_{k_i} - y \right\|^2 + 2\langle x_{k_i} - y, y - x^* \rangle + \|y - x^*\|^2,
\end{aligned} \tag{4.6}$$

thus,

$$\begin{aligned}\lim_{k\to\infty}\|x_k - y\|^2 &= \limsup_{j\to\infty}\left\|x_{k_j} - y\right\|^2 \\ &= \limsup_{j\to\infty}\left(\left\|x_{k_j} - x^*\right\|^2 + 2\left\langle x_{k_j} - x^*, x^* - y\right\rangle + \|x^* - y\|^2\right) \\ &\leq \lim_{k\to\infty}\|x_k - x^*\|^2 - \|x^* - y\|^2.\end{aligned} \tag{4.7}$$

Similarly, we also have

$$\lim_{k\to\infty}\|x_k - x^*\|^2 \leq \lim_{k\to\infty}\|x_k - y\|^2 - \|x^* - y\|^2. \tag{4.8}$$

Adding up the above two equations, we must have $-\|x^* - y\|^2 \geq 0$. So, $x^* = y$.

In a summary, we have proved that the set $\{x_k\}$ is weakly sequentially compact and each cluster point in the weak topology equals to $x^* \in Z$. Hence, $\{x_k\}$ converges weakly to $x^* \in T^{-1}0$. The proof is complete. □

Theorem 4.2. *Let T be a maximal monotone operator on a Hilbert space H with $Z = T^{-1}0 \neq \emptyset$. For an initial value $x_0 \in H$, iteratively define*

$$x_{k+1} = J_{r_k}^T(x_k + e_k). \tag{4.9}$$

If $\{r_k\} \subset (0, +\infty)$ and $e_k \in H$ satisfy

$$\lim_{k\to\infty} r_k = \infty, \quad \sum_{k=0}^{+\infty}\|e_k\| < +\infty, \tag{4.10}$$

then the sequence $\{x_k\}$ converges weakly to some $x^* \in Z$.

Proof. Take $z \in Z = T^{-1}0 = F(J_r^T)$ and $y_k = x_k + e_k$, we have

$$\|x_{k+1} - z\| = \left\|J_{r_k}^T y_k - z\right\| \leq \|x_k - z\| + \|e_k\|. \tag{4.11}$$

It follows from Liu [21, Lemma 2] that the limit $\lim_{k\to\infty}\|x_k - z\|$ exists for each $z \in Z$ and hence both $\{x_k\}$ and $\{y_k\}$ are bounded. So we have

$$\begin{aligned}\left\|x_{k+1} - J_r^T x_{k+1}\right\| &= \left\|J_{r_k}^T y_k - J_r^T J_{r_k}^T y_k\right\| = \left\|\left(I - J_r^T\right)J_{r_k}^T y_k\right\| \\ &= r\left\|A_r J_{r_k}^T y_k\right\| \leq r\left|T J_{r_k}^T y_k\right| \leq r\|A_{r_k} y_k\| \\ &= r\frac{\|y_k - J_{r_k}^T y_k\|}{r_k} = \frac{r\|y_k - x_{k+1}\|}{r_k} \longrightarrow 0 \quad (k \longrightarrow \infty).\end{aligned} \tag{4.12}$$

Hence,

$$\lim_{k\to\infty}\left\|x_k - J_r^T x_k\right\| = 0. \tag{4.13}$$

The remainder of the proof is the same as Theorem 4.1; we omit it. □

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