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Research Article

Some Weak Convergence Theorems for a Family of Asymptotically Nonexpansive Nonself Mappings

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A one-step iteration with errors is considered for a family of asymptotically nonexpansive nonself mappings. Weak convergence of the purposed iteration is obtained in a Banach space.

1. Introduction and Preliminaries

Let E be a real Banach space and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E. \tag{1.1}$$

Let $U_E = \{x \in E : \|x\| = 1\}$, where E is said to be smooth or said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for each $x, y \in U_E$, where E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$, where E is said to be uniformly smooth or said to have a uniformly Fréchet differentiable norm if the limit is attained uniformly for all $x, y \in U_E$, where E is said to be uniformly convex if for any $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$:

$$\|x-y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1-\delta. \tag{1.3}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. Recall that a Banach space E is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$ for more details on Kadec-Klee property, the reader is referred to [1, 2] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Recall that a Banach space E is said to satisfy the Opial condition [3] if, for each sequence $\{x_n\}$ in E, the convergence $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \ (y \neq x). \tag{1.4}$$

Let C be a nonempty closed and convex subset of E and T a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x, y \in C. \tag{1.5}$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x-y\|, \quad \forall x, y \in C, \ \forall n \geq 1. \tag{1.6}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed convex and bounded subset of a real uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point; see [4] for more details. Some classical results on asymptotically nonexpansive mappings and other important nonlinear mappings have been established by Kirk et al.; see [5–13].

However, T is said to be uniformly L-lipschitz if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x-y\|, \quad \forall x, y \in C, \ \forall n \geq 1. \tag{1.7}$$

Recall that the Mann iteration was introduced by Mann [14] in 1953. The Mann iteration sequence $\{x_n\}$ is defined in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$ and $T : C \to C$ is a mapping.

In 1979, Reich [15] obtained the following celebrated weak convergence theorem.

Theorem R-1. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differential norm, $T : C \to C$ a nonexpansive mapping with a fixed point, and $\{\alpha_n\}$ a real sequence such that $0 \le \alpha_n \le 1$ and $\sum_{n=1} \alpha_n(1-\alpha_n) = \infty$. Let $\{x_n\}$ be a sequence generated in* (1.8)*. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Note that the dual of reflexive Banach spaces with a Fréchet differentiable norm have the Kadec-Klee property. In 2001, García Falset et al. [16] obtained a new weak convergence theorem without the restriction E enjoys the Fréchet differential norm. To be more precise, they obtained the following results.

Theorem FKKR. *Let C be a closed convex subset of a uniformly convex Banach space E such that E^* has the Kadec-Klee property, $T : C \to C$ a nonexpansive mapping with a fixed point, and $\{\alpha_n\}$ a real sequence such that $0 \le \alpha_n \le 1$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Let $\{x_n\}$ be a sequence generated in* (1.8)*. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Recall that the modified Mann iteration which was introduced by Schu [17] generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \ge 1, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$ and $T : C \to C$ is an asymptotically nonexpansvie mapping.

In 1991, Schu [17] obtained the following weak convergence results for asymptotically nonexpansive mappings in a uniformly convex Banach space. To be more precise, they obtained the following results.

Theorem S. *Let E be a uniformly convex Banach space satisfying the Opial condition, $\emptyset \ne C \subset E$ closed bounded and convex and $S : C \to C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1,\infty)$ for which $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\} \in [0,1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in* (1.9)*. Then the sequence $\{x_n\}$ converges weakly to some fixed point of T.*

Note that each l^p $(1 \le p < \infty)$ satisfies the Opial condition, while all L^p do not have the property unless $p = 2$. In 1994, Tan and Xu [18] obtained the following results.

Theorem TX. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable, C a nonempty closed and convex subset of E, and $T : K \to K$ an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ such that $F(T)$ is nonempty. Let $\{x_n\}$ be sequence generated in* (1.9)*, where $\{\alpha_n\}$ is a real sequence bounded away from 0 and 1. Then the sequence $\{x_n\}$ converges weakly to some point in $F(T)$.*

Let E be a Banach space, K a nonempty subset of E, and $T : K \to E$ a mapping. For all $x \in K$, define a set $I_K(x)$ by

$$I_K(x) = \{x + \lambda(y - x) : \lambda > 0,\ y \in K\}, \tag{1.10}$$

where T is said to be inward if $Tx \in I_K(x)$ for all $x \in K$ and T is said to be weakly inward if $Tx \in \overline{I_K(x)}$ for all $x \in K$. Recall that the subset K of E is said to be retract if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. It is well known that every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. Let C and D be subsets of E. Then a mapping $P : C \to D$ is said to be sunny if $P(Px + t(x - Px)) = Px$, whenever $Px + t(x - Px) \in C$ for all $x \in C$ and $t \geq 0$.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space. See Reich [19].

Theorem R-2. *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(1) *P is sunny and nonexpansive;*
(2) $\|Px - Py\|^2 \leq \langle x - y, J(Px - Py)\rangle$, $\forall x, y \in E$;
(3) $\langle x - Px, J(y - Px)\rangle \leq 0$, $\forall x \in E$, $y \in C$.

Recently, fixed point problems of nonself mappings have been studied by a number of authors; see, for example, [20–30]. Next, we draw our attention to nonself mappings. Let K be a nonempty subset of a Banach space E, $T : K \to E$ be a mapping and P a sunny nonexpansive retraction from E onto K.

The mapping T is said to be asymptotically nonexpansive with respect to P if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|(PT)^n x - (PT)^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in K, \ \forall n \geq 1. \tag{1.11}$$

The mapping T is said to be uniformly L-lipschitz with respect to P if there exists a positive constant L such that

$$\|(PT)^n x - (PT)^n y\| \leq L\|x - y\|, \quad \forall x, y \in K, \ \forall n \geq 1. \tag{1.12}$$

We remark that if T is a self mapping, then P is reduced to the identity mapping. It follows that (1.11) is reduced to (1.6).

In this paper, we consider a one-step iteration for a finite family of asymptotically nonexpansive nonself mappings. Weak convergence theorems are established in a real smooth and uniformly convex Banach space.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 (see [16, 31]). *Let E be a uniformly convex Banach space such that its dual has the Kadec-Klee property. Suppose that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty}\|ax_n + (1-a)f_1 - f_2\|$ exists for all $a \in [0, 1]$ and $f_1, f_2 \in \omega_w(x_n)$. Then $\omega_w(x_n)$ is a singleton.*

Lemma 1.2 (see [2, 25]). *Let E be a real smooth Banach space, K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and $T : K \to E$ a mapping which enjoys the weakly inward condition. Then $F(PT) = F(T)$.*

Lemma 1.3 (see [32]). *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1. \tag{1.13}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.4 (see [33]). *Let $p > 1$ and $s > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|) \tag{1.14}$$

for all $x, y \in B_s(0) = \{x \in E : \|x\| \le s\}$ and $\lambda \in [0,1]$, where $w_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

The following lemma is an immediate result of Lemma 1.4. See also Zhang [34].

Lemma 1.5. *Let E be a uniformly convex Banach space, $s > 0$ a positive number, and $B_s(0)$ a closed ball of E. There exits a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^{N}(\alpha_i x_i)\right\|^2 \le \sum_{i=1}^{N}\left(\alpha_i\|x_i\|^2\right) - \alpha_1\alpha_2 g(\|x_1 - x_2\|) \tag{1.15}$$

for all $x_1, x_2, \dots, x_N \in B_s(0) = \{x \in E : \|x\| \le s\}$ and $\alpha_1, \alpha_2, \dots, \alpha_N \in [0,1]$ such that $\sum_{i=1}^{N}\alpha_i = 1$.

Proof. We prove it by inductions. For $N = 2$, we from Lemma 1.4 see that (1.15) holds. For $N = j$, where $j \ge 3$ is some positive integer, suppose that (1.15) holds. We see that (1.15) still holds for $N = j+1$. Indeed, from Lemma 1.4, we see that

$$\begin{aligned}
&\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_j x_j + \alpha_{j+1}x_{j+1}\|^2 \\
&\quad = \left\|(1-\alpha_{j+1})\left(\frac{\alpha_1}{1-\alpha_{j+1}}x_1 + \frac{\alpha_2}{1-\alpha_{j+1}}x_2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}x_j\right) + \alpha_{j+1}x_{j+1}\right\|^2 \\
&\quad \le (1-\alpha_{j+1})\left\|\frac{\alpha_1}{1-\alpha_{j+1}}x_1 + \frac{\alpha_2}{1-\alpha_{j+1}}x_2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}x_j\right\|^2 + \alpha_{j+1}\|x_{j+1}\|^2 \\
&\qquad - \alpha_j(1-\alpha_{j+1})g\left(\left\|\left(\frac{\alpha_1}{1-\alpha_{j+1}}x_1 + \frac{\alpha_2}{1-\alpha_{j+1}}x_2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}x_j\right) - x_{j+1}\right\|\right) \\
&\quad \le (1-\alpha_{j+1})\left(\frac{\alpha_1}{1-\alpha_{j+1}}\|x_1\|^2 + \frac{\alpha_2}{1-\alpha_{j+1}}\|x_2\|^2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}\|x_j\|^2\right. \\
&\qquad\qquad \left. - \frac{\alpha_1\alpha_2}{(1-\alpha_{j+1})(1-\alpha_{j+1})}g(\|x_1 - x_2\|)\right) + \alpha_{j+1}\|x_{j+1}\|^2 \\
&\quad = \alpha_1\|x_1\|^2 + \alpha_2\|x_2\|^2 + \cdots + \alpha_j\|x_j\|^2 + \alpha_{j+1}\|x_{j+1}\|^2 - \frac{\alpha_1\alpha_2}{1-\alpha_{j+1}}g(\|x_1 - x_2\|) \\
&\quad \le \alpha_1\|x_1\|^2 + \alpha_2\|x_2\|^2 + \cdots + \alpha_j\|x_j\|^2 + \alpha_{j+1}\|x_{j+1}\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|).
\end{aligned} \tag{1.16}$$

This completes the proof. □

Lemma 1.6 (see [35]). *Let E be a real uniformly convex Banach space, K a nonempty closed, and convex subset of E and $T : K \to K$ an asymptotically nonexpansive mapping. Then $I-T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply that $x = Tx$.*

2. Main Results

Lemma 2.1. *Let E be a real uniformly convex Banach space, K a nonempty closed and convex subset of E, and P a sunny nonexpansive retraction from E onto K. Let $T_i : K \to E$ be an asymptotically nonexpansive mapping with respect to P with a sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $i \in \{1,2,\ldots,N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be sequence generated in the following manner: $x_1 \in K$ and*

$$x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{N}\alpha_{n,i}(PT_i)^n x_n + \alpha_{n,N+1}u_n, \quad \forall n \geq 1, \tag{HCQ}$$

where $\{\alpha_{n,i}\}$ is a real sequence in $(0,1)$ and $\{u_n\}$ is a bounded sequence in K. Assume that

(a) $\sum_{i=0}^{N+1} \alpha_{n,i} = 1$;

(b) $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$ *for each* $i \in \{1,2,\ldots,N\}$;

(c) $\sum_{n=1}^{\infty} \alpha_{n,N+1} < \infty$.

Then $\lim_{n\to\infty}\|x_n - (PT_i)x_n\| = 0$ *for each* $i \in \{1,2,\ldots,N\}$.

Proof. Fix $q \in \mathcal{F}$ and $k_n = \max\{k_{n,1}, k_{n,2}, \ldots, k_{n,N}\}$. It follows that $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Since $\{u_n\}$ is a bounded sequence in K, we set $M = \sup\{\|u_n - q\| : n \geq 1\}$. It follows that

$$\begin{aligned}
\|x_{n+1} - q\| &= \left\|\alpha_{n,0}x_n + \sum_{i=1}^{N}\alpha_{n,i}(PT_i)^n x_n + \alpha_{n,N+1}u_n - q\right\| \\
&\leq \alpha_{n,0}\|x_n - q\| + \sum_{i=1}^{N}\alpha_{n,i}\|(PT_i)^n x_n - q\| + \alpha_{n,N+1}\|u_n - q\| \\
&\leq \alpha_{n,0}\|x_n - q\| + \sum_{i=1}^{N}\alpha_{n,i}k_{n,i}\|x_n - q\| + \alpha_{n,N+1}\|u_n - q\| \\
&\leq [1 + (k_n - 1)]\|x_n - q\| + \alpha_{n,N+1}M.
\end{aligned} \tag{2.1}$$

In view of the condition (c), we obtain from Lemma 1.3 that $\lim_{n\to\infty}\|x_n - q\|$ exists for any $q \in F(T)$. This in turn shows that the sequence $\{x_n\}$ is bounded.

On the other hand, we conclude from Lemma 1.4 that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \left\|\alpha_{n,0}x_n+\sum_{i=1}^{N}\alpha_{n,i}(PT_i)^n x_n+\alpha_{n,N+1}u_n-q\right\|^2\\
&\le \alpha_{n,0}\|x_n-q\|^2+\sum_{i=1}^{N}\alpha_{n,i}\|(PT_i)^n x_n-q\|^2+\alpha_{n,N+1}\|u_n-q\|^2\\
&\quad-\alpha_{n,0}\alpha_{n,1}g(\|x_n-(PT_1)^n x_n\|)\\
&\le \alpha_{n,0}\|x_n-q\|^2+\sum_{i=1}^{N}\alpha_{n,i}k_{n,i}^2\|x_n-q\|^2+\alpha_{n,N+1}\|u_n-q\|^2\\
&\quad-\alpha_{n,0}\alpha_{n,1}g(\|x_n-(PT_1)^n x_n\|)\\
&\le \left[1+\left(k_n^2-1\right)\right]\|x_n-q\|^2+\alpha_{n,N+1}\|u_n-q\|^2-\alpha_{n,0}\alpha_{n,1}g(\|x_n-(PT_1)^n x_n\|).
\end{aligned} \tag{2.2}$$

This shows that

$$\begin{aligned}
&\alpha_{n,0}\alpha_{n,1}g(\|x_n-(PT_1)^n x_n\|)\\
&\quad\le \|x_n-q\|^2-\|x_{n+1}-q\|^2+\left(k_n^2-1\right)\|x_n-q\|^2+\alpha_{n,N+1}\|u_n-q\|^2\\
&\quad\le (\|x_n-q\|-\|x_{n+1}-q\|)R_1+\left(k_n^2-1\right)R_2+\alpha_{n,N+1}\|u_n-q\|^2.
\end{aligned} \tag{2.3}$$

where $R_1=\sup\{\|x_n-q\|+\|x_{n+1}-q\| : n\ge 1\}$ and $R_2=\sup\{\|x_n-q\|^2 : n\ge 1\}$. In view of the conditions (b) and (c), we arrive at $\lim_{n\to\infty}g(\|x_n-(PT_1)^n x_n\|)=0$. In view of the property of the function g, we conclude that

$$\lim_{n\to\infty}\|x_n-(PT_1)^n x_n\|=0. \tag{2.4}$$

By repeating (2.2) and (2.3), we can conclude that

$$\lim_{n\to\infty}\|x_n-(PT_i)^n x_n\|=0,\quad \forall i\in\{1,2,\ldots,N\}. \tag{2.5}$$

Note that

$$\|x_{n+1}-x_n\|\le\sum_{i=1}^{N}\alpha_{n,i}\|(PT_i)^n x_n-x_n\|+\alpha_{n,N+1}\|u_n-x_n\|. \tag{2.6}$$

From (2.5) and condition (c), we see that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.7}$$

On the other hand, we have

$$\|x_n - (PT_i)x_n\| \le \|x_n - x_{n+1}\| + \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + \left\|(PT_i)^{n+1}x_{n+1} - (PT_i)^{n+1}x_n\right\| + \left\|(PT_i)^{n+1}x_n - (PT_i)x_n\right\|. \tag{2.8}$$

Since T_i is Lipschitz with respective to P for each $i \in \{1, 2, \ldots, N\}$, we obtain that

$$\lim_{n\to\infty} \|x_n - (PT_i)x_n\| = 0, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{2.9}$$

This completes the proof. □

Next, we give some weak convergence theorems.

Theorem 2.2. *Let E be a real smooth and uniformly convex Banach space which enjoys the Opial condition, K a nonempty closed and convex subset of E, and P a sunny nonexpansive retraction from E on K. Let $T_i : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with a sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i} - 1) < \infty$ for each $i \in \{1, 2, \ldots, N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be sequence generated in (HCQ), where $\{\alpha_{n,i}\}$ is a real sequence in $(0, 1)$ and $\{u_n\}$ is a bounded sequence in K. Assume that*

(a) $\sum_{i=0}^{N+1} \alpha_{n,i} = 1$;

(b) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for each* $i \in \{1, 2, \ldots, N\}$;

(c) $\sum_{n=1}^{\infty} \alpha_{n,N+1} < \infty$.

Then the sequence $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

Proof. Since E is reflexive and $\{x_n\}$ is bounded, we from Lemmas 1.2 and 1.6 conclude that $\omega_w(x_n) \subset F(PT_i) = F(T_i)$ for each $i \in \{1, 2, \ldots, N\}$. On the other hand, since the space E enjoys the Opial condition, we see that $\omega_w(x_n)$ is singleton. This completes the proof. □

If $T = T_i$ for each $i \in \{1, 2, \ldots, N\}$ and $\alpha_{n,N+1} = 0$ for each $n \ge 1$, then we have from Theorem 2.2 the following results.

Corollary 2.3. *Let E be a real smooth and uniformly convex Banach space which enjoys the Opial condition, K a nonempty closed and convex subset of E, and P a sunny nonexpansive retraction from E onto K. Let $T : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be sequence generated in the following manner: $x_1 \in K$ and*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(PT)^n x_n, \quad \forall n \ge 1, \tag{2.10}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to some point in $F(T)$.

Theorem 2.4. *Let E be a real smooth and uniformly convex Banach space whose norm is Fréchet differentiable, K a nonempty closed and convex subset of E, and P a sunny nonexpansive retraction from E onto K. Let $T_i : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with a sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $i \in \{1,2,\ldots,N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be sequence generated in (HCQ), where $\{\alpha_{n,i}\}$ is a real sequence in $(0,1)$ and $\{u_n\}$ is a bounded sequence in K. Assume that*

(a) $\sum_{i=0}^{N+1} \alpha_{n,i} = 1$;

(b) $\liminf_{n\to\infty}\alpha_{n,0}\ \alpha_{n,i} > 0$ *for each* $i \in \{1,2,\ldots,N\}$;

(c) $\sum_{n=1}^{\infty} \alpha_{n,N+1} < \infty$.

Then the sequence $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

Proof. Since E is reflexive and $\{x_n\}$ is bounded, we from Lemma 1.2 and 1.6 conclude that $\omega_w(x_n) \subset F(PT_i) = F(T_i)$ for each $i \in \{1,2,\ldots,N\}$. From the proof of Tan and Xu [18, Lemma 2.2] (see also Cho et al. [35, Lemma 1.8]), we can show that, for every $f_1, f_2 \in \mathcal{F}$,

$$\langle p-q, J(f_1 - f_2)\rangle = 0, \quad \forall p,q \in \omega_w(x_n). \tag{2.11}$$

Let $p,q \in \omega_w(x_n)$. It follows that $p,q \in \mathcal{F}$; that is,

$$\|p-q\| = \langle p-q, J(p-q)\rangle = 0. \tag{2.12}$$

Therefore, $p = q$. This completes the proof. □

If $T = T_i$ for each $i \in \{1,2,\ldots,N\}$ and $\alpha_{n,N+1} = 0$ for each $n \geq 1$, then we from Theorem 2.4 have the following results.

Corollary 2.5. *Let E be a real smooth and uniformly convex Banach space whose norm is Fréchet differentiable, K a nonempty closed and convex subset of E, and P a sunny nonexpansive retraction from E onto K. Let $T : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be sequence generated in (2.10), where $\{\alpha_n\}$ is a real sequence in $(0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to some point in $F(T)$.*

Theorem 2.6. *Let E be a real smooth and uniformly convex Banach space such that its dual E^* has the Kadec-Klee property, K a nonempty closed and convex subset of E, and P a sunny nonexpansive retraction from E onto K. Let $T_i : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with a sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for each $i \in \{1,2,\ldots,N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be sequence generated in (HCQ), where $\{\alpha_{n,i}\}$ is a real sequence in $(0,1)$ and $\{u_n\}$ is a bounded sequence in K. Assume that*

(a) $\sum_{i=0}^{N+1} \alpha_{n,i} = 1$;

(b) $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$ *for each* $i \in \{1,2,\ldots,N\}$;

(c) $\sum_{n=1}^{\infty} \alpha_{n,N+1} < \infty$.

Then the sequence $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

Proof. Since E is reflexive and $\{x_n\}$ is bounded, we from Lemma 1.2 and Lemma 1.6 conclude that $\omega_w(x_n) \subset F(PT_i) = F(T_i)$ for each $i \in \{1, 2, \ldots, N\}$. From the proof of Lemma 2.2 of Tan and Xu [18] (see also of Cho et al. [35, Lemma 1.8]), we can show that $\lim_{n\to\infty}\|ax_n + (1 - a)f_1 - f_2\|$ exists for all $a \in [0,1]$ and $f_1, f_2 \in \omega_w(x_n)$. In view of Lemma 1.1, we see that $\omega_w(x_n)$ is singleton. This completes the proof. □

If $T = T_i$ for each $i \in \{1, 2, \ldots, N\}$ and $\alpha_{n,N+1} = 0$ for each $n \geq 1$, then we from Theorem 2.6 have the following results.

Corollary 2.7. *Let E be a real smooth and uniformly convex Banach space such that its dual E^* has the Kadec-Klee property, K a nonempty closed and convex subset of E and P a sunny nonexpansive retraction from E onto K. Let $T : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be sequence generated in* (2.10)*, where $\{\alpha_n\}$ is a real sequence in $(0, 1)$ such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. *Then the sequence $\{x_n\}$ converges weakly to some point in $F(T)$.*

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