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Research Article

Convergence Theorems on Asymptotically Pseudocontractive Mappings in the Intermediate Sense

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A new nonlinear mapping is introduced. The convergence of Ishikawa iterative processes for the class of asymptotically pseudocontractive mappings in the intermediate sense is studied. Weak convergence theorems are established. A strong convergence theorem is also established without any compact assumption by considering the so-called hybrid projection methods.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. The symbols $\to$ and $\rightharpoonup$ are denoted by strong convergence and weak convergence, respectively. $\omega_w(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$. Let C be a nonempty closed and convex subset of H and $T : C \to C$ a mapping. In this paper, we denote the fixed point set of T by $F(T)$.

Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall n \ge 1,\ \forall x, y \in C. \tag{1.2}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed convex and bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C, then T has a fixed point.

T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty}\sup_{x,y\in C}\left(\|T^n x - T^n y\| - \|x-y\|\right) \le 0. \tag{1.3}$$

Observe that if we define

$$\tau_n = \max\left\{0, \sup_{x,y\in C}\left(\|T^n x - T^n y\| - \|x-y\|\right)\right\}, \tag{1.4}$$

then $\tau_n \to 0$ as $n \to \infty$. It follows that (1.3) is reduced to

$$\|T^n x - T^n y\| \le \|x-y\| + \tau_n, \quad \forall n \ge 1,\ \forall x, y \in C. \tag{1.5}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2]. It is known [3] that if C is a nonempty close convex subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Recall that T is said to be *strictly pseudocontractive* if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\| \le \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.6}$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [4] in a real Hilbert space. Marino and Xu [5] proved that the fixed point set of strict pseudocontractions is closed convex, and they also obtained a weak convergence theorem for strictly pseudocontractive mappings by Mann iterative process; see [5] for more details.

Recall that T is said to be a *asymptotically strict pseudocontraction* if there exist a constant $k \in [0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \le k_n\|x-y\|^2 + k\|(I-T^n)x - (I-T^n)y\|^2, \quad \forall x, y \in C. \tag{1.7}$$

The class of asymptotically strict pseudocontractions was introduced by Qihou [6] in 1996 (see also [7]). Kim and Xu [8] proved that the fixed point set of asymptotically strict pseudocontractions is closed convex. They also obtained that the class of asymptotically strict pseudocontractions is demiclosed at the origin; see [8, 9] for more details.

Recently, Sahu et al. [10] introduced a class of new mappings: asymptotically strict pseudocontractive mappings in the intermediate sense. Recall that T is said to be an *asymptotically strict pseudocontraction in the intermediate sense* if

$$\limsup_{n\to\infty}\sup_{x,y\in C}\left(\|T^nx-T^ny\|^2-k_n\|x-y\|^2-k\|(I-T^n)x-(I-T^n)y\|^2\right)\leq 0, \tag{1.8}$$

where $k\in[0,1)$ and $\{k_n\}\subset[1,\infty)$ such that $k_n\to 1$ as $n\to\infty$. Put

$$\xi_n=\max\left\{0,\sup_{x,y\in C}\left(\|T^nx-T^ny\|^2-k_n\|x-y\|^2-k\|(I-T^n)x-(I-T^n)y\|^2\right)\right\}. \tag{1.9}$$

It follows that $\xi_n\to 0$ as $n\to\infty$. Then, (1.8) is reduced to the following:

$$\|T^nx-T^ny\|^2\leq k_n\|x-y\|^2+k\|(I-T^n)x-(I-T^n)y\|^2+\xi_n,\quad \forall x,y\in C. \tag{1.10}$$

They obtained a weak convergence theorem of modified Mann iterative processes for the class of mappings. Moreover, a strong convergence theorem was also established in a real Hilbert space by considering the so-called hybrid projection methods; see [10] for more details.

Recall that T is said to be *asymptotically pseudocontractive* if there exists a sequence $k_n\subset[1,\infty)$ with $k_n\to 1$ as $n\to\infty$ such that

$$\langle T^nx-T^ny,x-y\rangle\leq k_n\|x-y\|^2,\quad \forall x,y\in C. \tag{1.11}$$

The class of asymptotically pseudocontractive mapping was introduced by Schu [11] (see also [12]). In [13], Rhoades gave an example to show that the class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings; see [13] for more details. In 1991, Schu [11] established the following classical results.

Theorem JS. *Let H be a Hilbert space: $\emptyset\neq A\subset H$ closed bounded and covnex; $L>0$; $T:A\to A$ completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\}\subset[1,\infty)$; $q_n=2k_n-1$ for all $n\geq 1$; $\sum_{n=1}^{\infty}(q_n-1)<\infty$; $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$; $\epsilon\leq\alpha_n\leq\beta_n\leq b$ for all $n\geq 1$, some $\epsilon>0$ and some $b\in(0,L^{-2}[\sqrt{1+L^2}-1])$; $x_1\in A$; for all $n\geq 1$, define*

$$\begin{aligned} z_n&=\beta_nT^nx_n+(1-\beta_n)x_n,\\ x_{n+1}&=\alpha_nT^nz_n+(1-\alpha_n)y_n,\quad \forall n\geq 1, \end{aligned} \tag{1.12}$$

then $\{x_n\}$ converges strongly to some fixed point of T.

Recently, Zhou [14] showed that every uniformly Lipschitz and asymptotically pseudocontractive mapping which is also uniformly asymptotically regular has a fixed point. Moreover, the fixed point set is closed and convex.

In this paper, we introduce and consider the following mapping.

Definition 1.1. A mapping $T : C \to C$ is said to be a *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2 \right) \le 0, \tag{1.13}$$

where $\{k_n\}$ is a sequence in $[1,\infty)$ such that $k_n \to 1$ as $n \to \infty$. Put

$$\nu_n = \max\left\{ 0, \sup_{x,y\in C} \left(\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2 \right) \right\}. \tag{1.14}$$

It follows that $\nu_n \to 0$ as $n \to \infty$. Then, (1.13) is reduced to the following:

$$\langle T^n x - T^n y, x - y \rangle \le k_n \|x - y\|^2 + \nu_n, \quad \forall n \ge 1,\ x, y \in C. \tag{1.15}$$

In real Hilbert spaces, we see that (1.15) is equivalent to

$$\|T^n x - T^n y\|^2 \le (2k_n - 1)\|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2 + 2\nu_n, \quad \forall n \ge 1,\ x, y \in C. \tag{1.16}$$

We remark that if $\nu_n = 0$ for each $n \ge 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically pseudocontractive mappings.

In this paper, we consider the problem of convergence of Ishikawa iterative processes for the class of mappings which are asymptotically pseudocontractive in the intermediate sense.

In order to prove our main results, we also need the following lemmas.

Lemma 1.2 (see [15]). *Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \le (1 + s_n) r_n + t_n, \quad \forall n \ge n_0, \tag{1.17}$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 1.3. *In a real Hilbert space, the following inequality holds:*

$$\|ax + (1 - a)y\|^2 = a\|x\|^2 + (1 - a)\|y\|^2 - a(1 - a)\|x - y\|^2, \quad \forall a \in [0,1],\ x, y \in C. \tag{1.18}$$

From now on, we always use M to denotes $(\text{diam } C)^2$.

Lemma 1.4. *Let C be a nonempty close convex subset of a real Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\}$ and $\{\nu_n\}$ as defined in* (1.15). *Then $F(T)$ is a closed convex subset of C.*

Proof. To show that $F(T)$ is convex, let $f_1 \in F(T)$ and $f_2 \in F(T)$. Put $f = tf_1 + (1-t)f_2$, where $t \in (0,1)$. Next, we show that $f = Tf$. Choose $\alpha \in (0, 1/(1+L))$ and define $y_{\alpha,n} = (1-\alpha)f + \alpha T^n f$ for each $n \geq 1$. From the assumption that T is uniformly L-Lipschitz, we see that

$$\langle f - y_{\alpha,n}, (f - T^n f) - (y_{\alpha,n} - T^n y_{\alpha,n}) \rangle \leq (1+L)\|f - y_{\alpha,n}\|^2. \tag{1.19}$$

For any $g \in F(T)$, it follows that

$$\begin{aligned}
\|f - T^n f\|^2 &= \langle f - T^n f, f - T^n f \rangle \\
&= \frac{1}{\alpha}\langle f - y_{\alpha,n}, f - T^n f \rangle \\
&= \frac{1}{\alpha}\langle f - y_{\alpha,n}, (f - T^n f) - (y_{\alpha,n} - T^n y_{\alpha,n}) \rangle + \frac{1}{\alpha}\langle f - y_{\alpha,n}, y_{\alpha,n} - T^n y_{\alpha,n} \rangle \\
&= \frac{1}{\alpha}\langle f - y_{\alpha,n}, (f - T^n f) - (y_{\alpha,n} - T^n y_{\alpha,n}) \rangle \\
&\quad + \frac{1}{\alpha}\langle f - g, y_{\alpha,n} - T^n y_{\alpha,n} \rangle + \frac{1}{\alpha}\langle g - y_{\alpha,n}, y_{\alpha,n} - g \rangle + \frac{1}{\alpha}\langle g - y_{\alpha,n}, g - T^n y_{\alpha,n} \rangle \\
&\leq \alpha(1+L)\|f - T^n f\|^2 + \frac{1}{\alpha}\langle f - g, y_{\alpha,n} - T^n y_{\alpha,n} \rangle + \frac{(k_n - 1)\|g - y_{\alpha,n}\|^2 + \nu_n}{\alpha}.
\end{aligned} \tag{1.20}$$

This implies that

$$\alpha[1 - \alpha(1+L)]\|f - T^n f\|^2 \leq \langle f - g, y_{\alpha,n} - T^n y_{\alpha,n} \rangle + (k_n - 1)M + \nu_n, \quad \forall g \in F(T). \tag{1.21}$$

Letting $g = f_1$ and $g = f_2$ in (1.21), respectively, we see that

$$\begin{aligned}
\alpha[1 - \alpha(1+L)]\|f - T^n f\|^2 &\leq \langle f - f_1, y_{\alpha,n} - T^n y_{\alpha,n} \rangle + (k_n - 1)M + \nu_n, \\
\alpha[1 - \alpha(1+L)]\|f - T^n f\|^2 &\leq \langle f - f_2, y_{\alpha,n} - T^n y_{\alpha,n} \rangle + (k_n - 1)M + \nu_n.
\end{aligned} \tag{1.22}$$

It follows that

$$\alpha[1 - \alpha(1+L)]\|f - T^n f\|^2 \leq (k_n - 1)M + \nu_n. \tag{1.23}$$

Letting $n \to \infty$ in (1.23), we obtain that $T^n f \to f$. Since T is uniformly L-Lipschitz, we see that $f = Tf$. This completes the proof of the convexity of $F(T)$. From the continuity of T, we can also obtain the closedness of $F(T)$. The proof is completed. □

Lemma 1.5. *Let C be a nonempty close convex subset of a real Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and asymptotically pseudocontractive mapping in the intermediate sense such that $F(T)$ is nonempty. Then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup \overline{x}$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Next, we show that $\overline{x} \in C$ and $\overline{x} = T\overline{x}$. Since C is closed and convex, we see that $\overline{x} \in C$. It is sufficient to show that $\overline{x} = T\overline{x}$. Choose $\alpha \in (0, 1/(1+L))$ and define $y_{\alpha,m} = (1-\alpha)\overline{x} + \alpha T^m\overline{x}$ for arbitrary but fixed $m \geq 1$. From the assumption that T is uniformly L-Lipschitz, we see that

$$\begin{aligned}\|x_n - T^m x_n\| &\leq \|x_n - Tx_n\| + \left\|Tx_n - T^2x_n\right\| + \cdots + \left\|T^{m-1}x_n - T^m x_n\right\| \\ &\leq [1 + (m-1)L]\|x_n - Tx_n\|.\end{aligned} \tag{1.24}$$

It follows from the assumption that

$$\lim_{n\to\infty} \|x_n - T^m x_n\| = 0. \tag{1.25}$$

Note that

$$\begin{aligned}\langle \overline{x} - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle &= \langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + \langle x_n - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle \\ &= \langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + \langle x_n - y_{\alpha,m}, T^m x_n - T^m y_{\alpha,m}\rangle \\ &\quad - \langle x_n - y_{\alpha,m}, x_n - y_{\alpha,m}\rangle + \langle x_n - y_{\alpha,m}, x_n - T^m x_n\rangle \\ &\leq \langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + k_m\left\|x_n - y_{\alpha,m}\right\|^2 + \nu_m \\ &\quad - \left\|x_n - y_{\alpha,m}\right\|^2 + \left\|x_n - y_{\alpha,m}\right\|\|x_n - T^m x_n\| \\ &\leq \langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + (k_m - 1)M + \nu_m \\ &\quad + \left\|x_n - y_{\alpha,m}\right\|\|x_n - T^m x_n\|.\end{aligned} \tag{1.26}$$

Since $x_n \rightharpoonup \overline{x}$ and (1.25), we arrive at

$$\langle \overline{x} - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle \leq (k_m - 1)M + \nu_m. \tag{1.27}$$

On the other hand, we have

$$\langle \overline{x} - y_{\alpha,m}, (\overline{x} - T^m\overline{x}) - (y_{\alpha,m} - T^m y_{\alpha,m})\rangle \leq (1+L)\left\|\overline{x} - y_{\alpha,m}\right\|^2 = (1+L)\alpha^2\|\overline{x} - T^m\overline{x}\|^2. \tag{1.28}$$

Note that

$$\begin{aligned}\|\overline{x} - T^m\overline{x}\|^2 = \langle \overline{x} - T^m\overline{x}, \overline{x} - T^m\overline{x}\rangle &= \frac{1}{\alpha}\langle \overline{x} - y_{\alpha,m}, \overline{x} - T^m\overline{x}\rangle \\ &= \frac{1}{\alpha}\langle \overline{x} - y_{\alpha,m}, (\overline{x} - T^m\overline{x}) - (y_{\alpha,m} - T^m y_{\alpha,m})\rangle \\ &\quad + \frac{1}{\alpha}\langle \overline{x}-, y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle.\end{aligned} \tag{1.29}$$

Substituting (1.27) and (1.28) into (1.29), we arrive at

$$\|\overline{x} - T^m\overline{x}\|^2 \le (1+L)\alpha\|\overline{x} - T^m\overline{x}\|^2 + \frac{(k_m-1)M + \nu_m}{\alpha}. \tag{1.30}$$

This implies that

$$\alpha[1-(1+L)\alpha]\|\overline{x} - T^m\overline{x}\|^2 \le (k_m-1)M + \nu_m, \quad \forall m \ge 1. \tag{1.31}$$

Letting $m \to \infty$ in (1.31), we see that $T^m\overline{x} \to \overline{x}$. Since T is uniformly L-Lipschitz, we can obtain that $\overline{x} = T\overline{x}$. This completes the proof. □

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex bounded subset of a real Hilbert space H and $T: C \to C$ a uniformly L-Lipschitz and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1,\infty)$ and $\{\nu_n\} \subset [0,\infty)$ defined as in (1.15). Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ y_n = \beta_n T^n x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad \forall n \ge 1, \end{gathered} \tag{$*$}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Assume that the following restrictions are satisfied:

(a) $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, *where* $q_n = 2k_n - 1$ *for each* $n \ge 1$;

(b) $a \le \alpha_n \le \beta_n \le b$ *for some* $a > 0$ *and some* $b \in (0, L^{-2}[\sqrt{1+L^2} - 1])$,

then the sequence $\{x_n\}$ generated by $(*)$ converges weakly to fixed point of T.

Proof. Fix $x^* \in F(T)$. From (1.16) and Lemma 1.3, we see that

$$\begin{aligned} \left\|y_n - x^*\right\|^2 &= \left\|\beta_n(T^n x_n - x^*) + (1-\beta_n)(x_n - x^*)\right\|^2 \\ &= \beta_n\|T^n x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\le \beta_n\left(q_n\|x_n - x^*\|^2 + \|x_n - T^n x_n\| + 2\nu_n\right) + (1-\beta_n)\|x_n - x^*\|^2 \\ &\quad - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\le q_n\|x_n - x^*\|^2 + \beta_n^2\|T^n x_n - x_n\|^2 + 2\nu_n, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \left\|y_n - T^n y_n\right\|^2 &= \left\|\beta_n(T^n x_n - T^n y_n) + (1-\beta_n)(x_n - T^n y_n)\right\|^2 \\ &= \beta_n\left\|T^n x_n - T^n y_n\right\|^2 + (1-\beta_n)\left\|x_n - T^n y_n\right\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\le \beta_n^3 L^2\|x_n - T^n x_n\|^2 + (1-\beta_n)\left\|x_n - T^n y_n\right\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we arrive at

$$\begin{aligned}\|T^n y_n - x^*\|^2 &\le q_n\|y_n - x^*\|^2 + \|y_n - T^n y_n\|^2 + 2\nu_n \\ &\le q_n^2\|x_n - x^*\|^2 - \beta_n\left(1 - q_n\beta_n - \beta_n^2 L^2 - \beta_n\right)\|T^n x_n - x_n\|^2 \\ &\quad + 2(q_n+1)\nu_n + (1-\beta_n)\|x_n - T^n y_n\|^2.\end{aligned} \tag{2.3}$$

It follows that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(T^n y_n - x^*) + (1-\alpha_n)(x_n - x^*)\|^2 \\ &= \alpha_n\|T^n y_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\ &\le \alpha_n q_n^2\|x_n - x^*\|^2 - \alpha_n\beta_n\left(1 - q_n\beta_n - \beta_n^2L^2 - \beta_n\right)\|T^n x_n - x_n\|^2 + 2(q_n+1)\nu_n \\ &\quad + \alpha_n(1-\beta_n)\|x_n - T^n y_n\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\ &\le q_n^2\|x_n - x^*\|^2 - \alpha_n\beta_n\left(1 - q_n\beta_n - \beta_n^2L^2 - \beta_n\right)\|T^n x_n - x_n\|^2 + 2(q_n+1)\nu_n.\end{aligned} \tag{2.4}$$

From condition (b), we see that there exists n_0 such that

$$1 - q_n\beta_n - \beta_n^2L^2 - \beta_n \ge \frac{1 - 2b - L^2b^2}{2} > 0, \quad \forall n \ge n_0. \tag{2.5}$$

Note that

$$\|x_{n+1} - x^*\|^2 \le \left[1 + \left(q_n^2 - 1\right)\right]\|x_n - x^*\|^2 + 2(q_n+1)\nu_n, \quad \forall n \ge n_0. \tag{2.6}$$

In view of Lemma 1.2, we see that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. For any $n \ge n_0$, we see that

$$\frac{a^2(1 - 2b - L^2b^2)}{2}\|T^n x_n - x_n\|^2 \le \left(q_n^2 - 1\right)\|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2(q_n+1)\nu_n, \tag{2.7}$$

from which it follows that

$$\lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{2.8}$$

Note that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \alpha_n\left\|T^n y_n - x_n\right\| \le \alpha_n\left(\left\|T^n y_n - T^n x_n\right\| + \|T^n x_n - x_n\|\right)\\ &\le \alpha_n\left(L\left\|y_n - x_n\right\| + \|T^n x_n - x_n\|\right) \le \alpha_n(1+\beta_n L)\|T^n x_n - x_n\|.\end{aligned} \tag{2.9}$$

Thanks to (2.8), we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{2.10}$$

Note that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| + \left\|T^{n+1}x_n - Tx_n\right\|\\ &\le (1+L)\|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\|T^n x_n - x_n\|.\end{aligned} \tag{2.11}$$

From (2.8) and (2.10), we obtain that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{2.12}$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \overline{x}$. From Lemma 1.5, we see that $\overline{x} \in F(T)$.

Next we prove that $\{x_n\}$ converges weakly to $\overline{x}$. Suppose the contrary. Then we see that there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\widehat{x} \in C$ and $\widehat{x} \neq \overline{x}$. From Lemma 1.5, we can also prove that $\widehat{x} \in F(T)$. Put $d = \lim_{n\to\infty}\|x_n - \overline{x}\|$. Since H satisfies Opial property, we see that

$$\begin{aligned}d &= \liminf_{n_i\to\infty}\|x_{n_i} - \overline{x}\| < \liminf_{n_i\to\infty}\|x_{n_i} - \widehat{x}\|\\ &= \liminf_{n_j\to\infty}\left\|x_{n_j} - \widehat{x}\right\| < \liminf_{n_j\to\infty}\left\|x_{n_j} - \overline{x}\right\|\\ &= \liminf_{n_i\to\infty}\|x_{n_i} - \overline{x}\| = d.\end{aligned} \tag{2.13}$$

This derives a contradiction. It follows that $\widehat{x} = \overline{x}$. This completes the proof. □

Next, we modify Ishikawa iterative processes to obtain a strong convergence theorem without any compact assumption.

Theorem 2.2. *Let C be a nonempty closed convex bounded subset of a real Hilbert space H, P_C the metric projection from H onto C, and $T : C \to C$ a uniformly L-Lipschitz and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\nu_n\} \subset [0, \infty)$ as defined in (1.15). Let $q_n = 2k_n - 1$ for each $n \geq 1$. Assume that $F(T)$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$
\begin{gathered}
x_1 \in C, \quad \text{chosen arbitrarily}, \\
z_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\
y_n = (1 - \alpha_n)x_n + \alpha_n T^n z_n, \\
C_n = \left\{u \in C : \left\|y_n - u\right\|^2 \leq \|x_n - u\|^2 + \alpha_n\theta_n + \alpha_n\beta_n\left(q_n\beta_n + \beta_n^2L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2\right\} \\
Q_n = \{u \in C : \langle x_1 - x_n, x_n - u\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_1,
\end{gathered}
\tag{$**$}
$$

where $\theta_n = q_n([1 + \beta_n(q_n - 1)] - 1)M + 2(q_n + 1)\nu_n$ for each $n \geq 1$. Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen such that $a \leq \alpha_n \leq \beta_n \leq b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$. Then the sequence $\{x_n\}$ generated in $(**)$ converges strongly to a fixed point of T.

Proof. The proof is divided into seven steps.

Step 1. Show that $C_n \cap Q_n$ is closed and convex for each $n \geq 1$.

It is obvious that Q_n is closed and convex and C_n is closed for each $n \geq 1$. We, therefore, only need to prove that C_n is convex for each $n \geq 1$. Note that

$$
C_n = \left\{u \in C : \left\|y_n - u\right\|^2 \leq \|x_n - u\|^2 + \alpha_n\theta_n + \alpha_n\beta_n\left(q_n\beta_n + \beta_n^2L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2\right\}
\tag{2.14}
$$

is equivalent to

$$
C_n' = \left\{u \in C : 2\langle x_n - y_n, u\rangle \leq \|x_n\|^2 - \left\|y_n\right\|^2 + \alpha_n\theta_n + \alpha_n\beta_n\left(q_n\beta_n + \beta_n^2L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2\right\}.
\tag{2.15}
$$

It is easy to see that C_n' is convex for each $n \geq 1$. Hence, we obtain that $C_n \cap Q_n$ is closed and convex for each $n \geq 1$. This completes Step 1.

Step 2. Show that $F(T) \subset C_n \cap Q_n$ for each $n \geq 1$.

Let $p \in F(T)$. From Lemma 1.3 and the algorithm $(**)$, we see that

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n z_n - p)\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n z_n - p\|^2 - \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\left(q_n\|z_n - p\|^2 + \|z_n - T^n z_n\|^2 + 2\nu_n\right) \\
&\quad - \alpha_n(1-\alpha_n)\|T^n z_n - x_n\|^2,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\|z_n - T^n z_n\|^2 &= \|(1-\beta_n)(x_n - T^n z_n) + \beta_n(T^n x_n - T^n z_n)\|^2 \\
&= (1-\beta_n)\|x_n - T^n z_n\|^2 + \beta_n\|T^n x_n - T^n z_n\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\
&\leq (1-\beta_n)\|x_n - T^n z_n\|^2 + \beta_n L^2\|x_n - z_n\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\
&\leq (1-\beta_n)\|x_n - T^n z_n\|^2 + \beta_n\left(\beta_n^2 L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n x_n - p\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\
&\leq [1 + \beta_n(q_n - 1)]\|x_n - p\|^2 + \beta_n^2\|x_n - T^n x_n\|^2 + 2\beta_n\nu_n.
\end{aligned} \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16), we arrive at

$$\begin{aligned}
\|y_n - p\|^2 &\leq \|x_n - p\|^2 + \alpha_n(q_n[1 + \beta_n(q_n - 1)] - 1)\|x_n - p\|^2 + 2\alpha_n(q_n + 1)\nu_n \\
&\quad + \alpha_n\beta_n\left(q_n\beta_n + \beta_n^2 L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n\beta_n\left(q_n\beta_n + \beta_n^2 L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2 + \alpha_n\theta_n,
\end{aligned} \tag{2.19}$$

where $\theta_n = q_n([1 + \beta_n(q_n - 1)] - 1)M + 2(q_n + 1)\nu_n$ for each $n \geq 1$. This implies that $p \in C_n$ for each $n \geq 1$. That is, $F(T) \subset C_n$ for each $n \geq 1$.

Next, we show that $F(T) \subset Q_n$ for each $n \geq 1$. We prove this by inductions. It is obvious that $F(T) \subset Q_1 = C$. Suppose that $F(T) \subset Q_k$ for some $k > 1$. Since x_{k+1} is the projection of x_1 onto $C_k \cap Q_k$, we see that

$$\langle x_1 - x_{k+1}, x_{k+1} - x \rangle \geq 0, \quad \forall x \in C_k \cap Q_k. \tag{2.20}$$

By the induction assumption, we know that $F(T) \subset C_k \cap Q_k$. In particular, for any $y \in F(T) \subset C$, we have

$$\langle x_1 - x_{k+1}, x_{k+1} - y \rangle \geq 0, \tag{2.21}$$

which implies that $y \in Q_{k+1}$. That is, $F(T) \subset C_{k+1}$. This proves that $F(T) \subset Q_n$ for each $n \geq 1$. Hence, $F(T) \subset C_n \cap Q_n$ for each $n \geq 1$. This completes Step 2.

Step 3. Show that $\lim_{n\to\infty}\|x_n - x_1\|$ exists.

In view of the algorithm $(**)$, we see that $x_n = P_{Q_n}x_1$ and $x_{n+1} \in Q_n$ which give that

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|. \tag{2.22}$$

This shows that the sequence $\|x_n - x_1\|$ is nondecreasing. Note that C is bounded. It follows that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. This completes Step 3.

Step 4. Show that $x_{n+1} - x_n \to 0$ as $n \to \infty$.

Note that $x_n = P_{Q_n}x_1$ and $x_{n+1} = P_{C_n\cap Q_n}x_1 \in Q_n$. This implies that

$$\langle x_{n+1} - x_n, x_1 - x_n \rangle \leq 0, \tag{2.23}$$

from which it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_1) + (x_1 - x_n)\|^2 \\
&= \|x_{n+1} - x_1\|^2 + \|x_1 - x_n\|^2 + 2\langle x_{n+1} - x_1, x_1 - x_n \rangle \\
&= \|x_{n+1} - x_1\|^2 - \|x_1 - x_n\|^2 + 2\langle x_{n+1} - x_n, x_1 - x_n \rangle \\
&\leq \|x_{n+1} - x_1\|^2 - \|x_1 - x_n\|^2.
\end{aligned} \tag{2.24}$$

Hence, we have $x_{n+1} - x_n \to 0$ as $n \to \infty$. This completes Step 4.

Step 5. Show that $T^n x_n - x_n \to 0$ as $n \to \infty$.

In view of $x_{n+1} \in C_n$, we see that

$$\left\|y_n - x_{n+1}\right\|^2 \leq \|x_n - x_{n+1}\|^2 + \alpha_n\theta_n + \alpha_n\beta_n\left(q_n\beta_n + \beta_n^2L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2. \tag{2.25}$$

On the other hand, we have

$$\left\|y_n - x_{n+1}\right\|^2 = \left\|y_n - x_n + x_n - x_{n+1}\right\|^2 = \left\|y_n - x_n\right\|^2 + \|x_n - x_{n+1}\|^2 + 2\langle y_n - x_n, x_n - x_{n+1} \rangle. \tag{2.26}$$

Combining (2.25) and (2.26) and noting $y_n = (1-\alpha_n)x_n + \alpha_n T^n z_n$, we get that

$$\alpha_n\|T^n z_n - x_n\|^2 + 2\langle T^n z_n - x_n, x_n - x_{n+1} \rangle \leq \theta_n + \beta_n\left(q_n\beta_n + \beta_n^2L^2 + \beta_n - 1\right)\|T^n x_n - x_n\|^2. \tag{2.27}$$

From the assumption, we see that there exists n_0 such that

$$1 - q_n\beta_n - \beta_n^2L^2 - \beta_n \geq \frac{1 - 2b - L^2b^2}{2} > 0, \quad \forall n \geq n_0. \tag{2.28}$$

For any $n \geq n_0$, it follows from (2.27) that

$$\frac{a(1-2b-L^2b^2)}{2}\|T^n x_n - x_n\|^2 \leq \theta_n + 2\|T^n z_n - x_n\|\|x_n - x_{n+1}\|. \tag{2.29}$$

Note that $\theta_n \to 0$ as $n \to \infty$. Thanks to Step 4, we obtain that

$$\lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{2.30}$$

This completes Step 5.

Step 6. Show that $Tx_n - x_n \to 0$ as $n \to \infty$.

Note that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| + \left\|T^{n+1}x_n - Tx_n\right\| \\ &\leq (1+L)\|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\|T^n x_n - x_n\|.\end{aligned} \tag{2.31}$$

From Step 5, we can conclude the desired conclusion. This completes Step 6.

Step 7. Show that $x_n \to q$, where $q = P_{F(T)}x_1$ as $n \to \infty$.

Note that Lemma 1.5 ensures that $\omega_w(x_n) \subset F(T)$. From $x_n = P_{Q_n}x_1$ and $F(T) \subset Q_n$, we see that

$$\|x_1 - x_n\| \leq \left\|x_1 - q\right\|. \tag{2.32}$$

From Lemma 1.5 of Yanes and Xu [16], we can obtain Step 7. This completes the proof. □

Remark 2.3. The results of Theorem 2.2 are more general which includes the corresponding results of Kim and Xu [17], Marino and Xu [5], Qin et al. [18], Sahu et al. [10], Zhou [14, 19] as special cases.

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