Research Article

The Theory of Reich's Fixed Point Theorem for Multivalued Operators

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The purpose of this paper is to present a theory of Reich's fixed point theorem for multivalued operators in terms of fixed points, strict fixed points, multivalued weakly Picard operators, multivalued Picard operators, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation, well-posedness of the fixed point problem, and the generated fractal operator.

1. Introduction

Let (X, d) be a metric space and consider the following family of subsets $P_{cl}(X) := \{Y \subseteq X \mid Y \text{ is nonempty and closed}\}$. We also consider the following (generalized) functionals:

$$D : P(X) \times P(X) \longrightarrow \mathbb{R}_+, \quad D(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}, \quad (1.1)$$

D is called the gap functional between A and B . In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$:

$$\rho : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}, \quad \rho(A, B) := \sup\{D(a, B) \mid a \in A\}, \quad (1.2)$$

ρ is called the (generalized) excess functional:

$$H : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}, \quad H(A, B) := \max\{\rho(A, B), \rho(B, A)\}, \quad (1.3)$$

H is the (generalized) Pompeiu-Hausdorff functional.

It is well known that if (X, d) is a complete metric space, then the pair $(P_{cl}(X), H)$ is a complete generalized metric space. (See [1, 2]).

Definition 1.1. If (X, d) is a metric space, then a multivalued operator $T : X \rightarrow P_{cl}(X)$ is said to be a Reich-type multivalued (a, b, c) -contraction if and only if there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$H(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y)), \quad \text{for each } x, y \in X. \quad (1.4)$$

Reich proved that any Reich-type multivalued (a, b, c) -contraction on a complete metric space has at least one fixed point (see [3]).

In a recent paper Petruşel and Rus introduced the concept of “theory of a metric fixed point theorem” and used this theory for the case of multivalued contraction (see [4]). For the singlevalued case, see [5].

The purpose of this paper is to extend this approach to the case of Reich-type multivalued (a, b, c) -contraction. We will discuss Reich’s fixed point theorem in terms of

- (i) fixed points and strict fixed points,
- (ii) multivalued weakly Picard operators,
- (iii) multivalued Picard operators,
- (iv) data dependence of the fixed point set,
- (v) sequence of multivalued operators and fixed points,
- (vi) Ulam-Hyers stability of a multivalued fixed point equation,
- (vii) well-posedness of the fixed point problem;
- (viii) fractal operators.

Notice also that the theory of fixed points and strict fixed points for multivalued operators is closely related to some important models in mathematical economics, such as optimal preferences, game theory, and equilibrium of an abstract economy. See [6] for a nice survey.

2. Notations and Basic Concepts

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used (see the papers by Kirk and Sims [7], Granas and Dugundji [8], Hu and Papageorgiou [2], Rus et al. [9], Petruşel [10], and Rus [11]).

Let X be a nonempty set. Then we denote.

$$\mathcal{P}(X) = \{Y \mid Y \text{ is a subset of } X\}, \quad P(X) = \{Y \in \mathcal{P}(X) \mid Y \text{ is nonempty}\}. \quad (2.1)$$

Let (X, d) be a metric space. Then $\delta(Y) = \sup\{d(a, b) \mid a, b \in Y\}$ and

$$P_b(X) = \{Y \in P(X) \mid \delta(Y) < +\infty\}, \quad P_{cp}(X) = \{Y \in P(X) \mid Y \text{ is compact}\}. \quad (2.2)$$

Let $T : X \rightarrow P(X)$ be a multivalued operator. Then the operator $\hat{T} : P(X) \rightarrow P(X)$, which is defined by

$$\hat{T}(Y) := \bigcup_{x \in Y} T(x), \quad \text{for } Y \in P(X), \quad (2.3)$$

is called the fractal operator generated by T . For a well-written introduction on the theory of fractals see the papers of Barnsley [12], Hutchinson [13], Yamaguti et al. [14].

It is known that if (X, d) is a metric space and $T : X \rightarrow P_{cp}(X)$, then the following statements hold:

- (a) if T is upper semicontinuous, then $T(Y) \in P_{cp}(X)$, for every $Y \in P_{cp}(X)$;
- (b) the continuity of T implies the continuity of $\hat{T} : P_{cp}(X) \rightarrow P_{cp}(X)$.

The set of all nonempty invariant subsets of T is denoted by $I(T)$, that is,

$$I(T) := \{Y \in P(X) \mid T(Y) \subset Y\}. \quad (2.4)$$

A sequence of successive approximations of T starting from $x \in X$ is a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in X with $x_0 = x$, $x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$.

If $T : Y \subseteq X \rightarrow P(X)$, then $F_T := \{x \in Y \mid x \in T(x)\}$ denotes the fixed point set of T and $(SF)_T := \{x \in Y \mid \{x\} = T(x)\}$ denotes the strict fixed point set of T . By

$$\text{Graph}(T) := \{(x, y) \in Y \times X : y \in T(x)\} \quad (2.5)$$

we denote the graph of the multivalued operator T .

If $T : X \rightarrow P(X)$, then $T^0 := 1_X$, $T^1 := T$, $\dots$, $T^{n+1} = T \circ T^n$, $n \in \mathbb{N}$, denote the iterate operators of T .

Definition 2.1 (see [15]). Let (X, d) be a metric space. Then, $T : X \rightarrow P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that

- (i) $x_0 = x$ and $x_1 = y$;
- (ii) $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$;
- (iii) the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of T .

For the following concepts see the papers by Rus et al. [15], Petruşel [10], Petruşel and Rus [16], and Rus et al. [9].

Definition 2.2. Let (X, d) be a metric space, and let $T : X \rightarrow P(X)$ be an MWP operator. The multivalued operator $T^\infty : \text{Graph}(T) \rightarrow P(F_T)$ is defined by the formula $T^\infty(x, y) = \{z \in F_T \mid \text{there exists a sequence of successive approximations of } T \text{ starting from } (x, y) \text{ that converges to } z\}$.

Definition 2.3. Let (X, d) be a metric space and $T : X \rightarrow P(X)$ an MWP operator. Then T is said to be a c -multivalued weakly Picard operator (briefly c -MWP operator) if and only if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x, y)) \leq cd(x, y)$ for all $(x, y) \in \text{Graph}(T)$.

We recall now the notion of multivalued Picard operator.

Definition 2.4. Let (X, d) be a metric space and $T : X \rightarrow P(X)$. By definition, T is called a multivalued Picard operator (briefly MP operator) if and only if

- (i) $(SF)_T = F_T = \{x^*\}$;
- (ii) $T^n(x) \xrightarrow{H} \{x^*\}$ as $n \rightarrow \infty$, for each $x \in X$.

In [10] other results on MWP operators are presented. For related concepts and results see, for example, [1, 17–23].

3. A Theory of Reich's Fixed Point Principle

We recall the fixed point theorem for a single-valued Reich-type operator, which is needed for the proof of our first main result.

Theorem 3.1 (see [3]). *Let (X, d) be a complete metric space, and let $f : X \rightarrow X$ be a Reich-type single-valued (a, b, c) -contraction, that is, there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)), \quad \text{for each } x, y \in X. \quad (3.1)$$

Then f is a Picard operator, that is, we have:

- (i) $F_f = \{x^*\}$;
- (ii) *for each $x \in X$ the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges in (X, d) to x^* .*

Our main result concerning Reich's fixed point theorem is the following.

Theorem 3.2. *Let (X, d) be a complete metric space, and let $T : X \rightarrow P_{cl}(X)$ be a Reich-type multivalued (a, b, c) -contraction. Let $\alpha := (a + b)/(1 - c)$. Then one has the following*

- (i) $F_T \neq \emptyset$;
- (ii) T is a $1/(1 - \alpha)$ -multivalued weakly Picard operator;
- (iii) *let $S : X \rightarrow P_{cl}(X)$ be a Reich-type multivalued (a, b, c) -contraction and $\eta > 0$ such that $H(S(x), T(x)) \leq \eta$ for each $x \in X$, then $H(F_S, F_T) \leq \eta/(1 - \alpha)$;*
- (iv) *let $T_n : X \rightarrow P_{cl}(X)$ ($n \in \mathbb{N}$) be a sequence of Reich-type multivalued (a, b, c) -contraction, such that $T_n(x) \xrightarrow{H} T(x)$ uniformly as $n \rightarrow +\infty$. Then, $F_{T_n} \xrightarrow{H} F_T$ as $n \rightarrow +\infty$.*

If, moreover $T(x) \in P_{cp}(X)$ for each $x \in X$, then one additionally has:

- (v) *(Ulam-Hyers stability of the inclusion $x \in T(x)$) Let $\epsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \epsilon$, then there exists $x^* \in F_T$ such that $d(x, x^*) \leq \epsilon/(1 - \alpha)$;*
- (vi) $\hat{T} : (P_{cp}(X), H) \rightarrow (P_{cp}(X), H)$, $\hat{T}(Y) := \bigcup_{x \in Y} T(x)$ is a set-to-set (a, b, c) -contraction and (thus) $F_{\hat{T}} = \{A_T^*\}$;
- (vii) $T^n(x) \xrightarrow{H} A_T^*$ as $n \rightarrow +\infty$, for each $x \in X$;
- (viii) $F_T \subset A_T^*$ and F_T are compact;
- (ix) $A_T^* = \bigcup_{n \in \mathbb{N} \setminus \{0\}} T^n(x)$ for each $x \in F_T$.

Proof. (i) Let $x_0 \in X$ and $x_1 \in T(x_0)$ be arbitrarily chosen. Then, for each arbitrary $q > 1$ there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \leq qH(T(x_0), T(x_1))$. Hence

$$\begin{aligned} d(x_1, x_2) &\leq q[ad(x_0, x_1) + bD(x_0, T(x_0)) + cD(x_1, T(x_1))] \\ &\leq q[ad(x_0, x_1) + bd(x_0, x_1) + cd(x_1, x_2)]. \end{aligned} \quad (3.2)$$

Thus

$$d(x_1, x_2) \leq \frac{q(a+b)}{1-qc} d(x_0, x_1). \quad (3.3)$$

Denote $\beta := q(a+b)/(1-qc)$. By an inductive procedure, we obtain a sequence of successive approximations for T starting from $(x_0, x_1) \in \text{Graph}(T)$ such that, for each $n \in \mathbb{N}$, we have $d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1)$. Then

$$d(x_n, x_{n+p}) \leq \beta^n \frac{1-\beta^p}{1-\beta} d(x_0, x_1), \quad \text{for each } n, p \in \mathbb{N} \setminus \{0\}. \quad (3.4)$$

If we choose $1 < q < 1/(a+b+c)$, then by (3.4) we get that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence convergent in (X, d) to some $x^* \in X$.

Notice that, by $D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, T(x^*)) \leq d(x_{n+1}, x^*) + H(T(x_n), T(x^*)) \leq d(x_{n+1}, x^*) + ad(x_n, x^*) + bD(x_n, T(x_n)) + cD(x^*, T(x^*)) \leq d(x_{n+1}, x^*) + ad(x_n, x^*) + bd(x_n, x_{n+1}) + cD(x^*, T(x^*))$, we obtain that

$$D(x^*, T(x^*)) \leq \frac{1}{1-c} [d(x_{n+1}, x^*) + ad(x_n, x^*) + bd(x_n, x_{n+1})] \longrightarrow \quad \text{as } n \longrightarrow +\infty. \quad (3.5)$$

Hence $x^* \in F_T$.

(ii) Let $p \rightarrow +\infty$ in (3.4). Then we get that

$$d(x_n, x^*) \leq \beta^n \frac{1}{1-\beta} d(x_0, x_1) \quad \text{for each } n \in \mathbb{N} \setminus \{0\}. \quad (3.6)$$

For $n = 1$, we get

$$d(x_1, x^*) \leq \frac{\beta}{1-\beta} d(x_0, x_1). \quad (3.7)$$

Then

$$d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*) \leq \frac{1}{1-\beta} d(x_0, x_1). \quad (3.8)$$

Let $q \searrow 1$ in (3.8), then

$$d(x_0, x^*) \leq \frac{1}{1-\alpha} d(x_0, x_1). \quad (3.9)$$

Hence T is a $1/(1-\alpha)$ -multivalued weakly Picard operator.

(iii) Let $x_0 \in F_S$ be arbitrarily chosen. Then, by (ii), we have that

$$d(x_0, t^\infty(x_0, x_1)) \leq \frac{1}{1-\alpha} d(x_0, x_1), \quad \text{for each } x_1 \in T(x_0). \quad (3.10)$$

Let $q > 1$ be an arbitrary. Then, there exists $x_1 \in T(x_0)$ such that

$$d(x_0, t^\infty(x_0, x_1)) \leq \frac{1}{1-\alpha} qH(S(x_0), T(x_0)) \leq \frac{q\eta}{1-\alpha}. \quad (3.11)$$

In a similar way, we can prove that for each $y_0 \in F_T$ there exists $y_1 \in S(y_0)$ such that

$$d(y_0, s^\infty(y_0, y_1)) \leq \frac{q\eta}{1-\alpha}. \quad (3.12)$$

Thus, (3.11) and (3.12) together imply that $H(F_S, F_T) \leq q\eta/(1-\alpha)$ for every $q > 1$. Let $q \searrow 1$ and we get the desired conclusion.

(iv) follows immediately from (iii).

(v) Let $\epsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \epsilon$. Then, since $T(x)$ is compact, there exists $y \in T(x)$ such that $d(x, y) \leq \epsilon$. From the proof of (i), we have that

$$d(x, t^\infty(x, y)) \leq \frac{1}{1-\alpha} d(x, y). \quad (3.13)$$

Since $x^* := t^\infty(x, y) \in F_T$, we get that $d(x, x^*) \leq \epsilon/(1-\alpha)$.

(vi) We will prove for any $A, B \in P_{cp}(X)$ that

$$H(T(A), T(B)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B)). \quad (3.14)$$

For this purpose, let $A, B \in P_{cp}(X)$ and let $u \in T(A)$. Then, there exists $x \in A$ such that $u \in T(x)$. Since the sets A, B are compact, there exists $y \in B$ such that

$$d(x, y) \leq H(A, B). \quad (3.15)$$

From (3.15) we get that $D(u, T(B)) \leq D(u, T(y)) \leq H(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y)) \leq ad(x, y) + b\rho(A, T(x)) + c\rho(B, T(y)) \leq aH(A, B) + b\rho(A, T(A)) + c\rho(B, T(B)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B))$. Hence

$$\rho(T(A), T(B)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B)). \quad (3.16)$$

In a similar way we obtain that

$$\rho(T(B), T(A)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B)). \quad (3.17)$$

Thus, (3.16) and (3.17) together imply that

$$H(T(A), T(B)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B)). \quad (3.18)$$

Hence, $\hat{T}$ is a Reich-type single-valued (a, b, c) -contraction on the complete metric space $(P_{cp}(X), H)$. From Theorem 3.1 we obtain that

(a) $F_{\hat{T}} = \{A_T^*\}$ and

(b) $\hat{T}^n(A) \xrightarrow{H} A_T^*$ as $n \rightarrow +\infty$, for each $A \in P_{cp}(X)$. □

(vii) From (vi)-(b) we get that $T^n(\{x\}) = \hat{T}^n(\{x\}) \xrightarrow{H} A_T^*$ as $n \rightarrow +\infty$, for each $x \in X$.

(viii)-(ix) Let $x \in F_T$ be an arbitrary. Then $x \in T(x) \subset T^2(x) \subset \dots \subset T^n(x) \subset \dots$. Hence $x \in T^n(x)$, for each $n \in \mathbb{N}^*$. Moreover, $\lim_{n \rightarrow +\infty} T^n(x) = \bigcup_{n \in \mathbb{N}^*} T^n(x)$. From (vii), we immediately get that $A_T^* = \bigcup_{n \in \mathbb{N}^*} T^n(x)$. Hence $x \in \bigcup_{n \in \mathbb{N}^*} T^n(x) = A_T^*$. The proof is complete.

A second result for Reich-type multivalued (a, b, c) -contractions formulates as follows.

Theorem 3.3. *Let (X, d) be a complete metric space and $T : X \rightarrow P_{cl}(X)$ a Reich-type multivalued (a, b, c) -contraction with $(SF)_T \neq \emptyset$. Then, the following assertions hold:*

(x) $F_T = (SF)_T = \{x^*\}$;

(xi) (Well-posedness of the fixed point problem with respect to D [24]) If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X such that $D(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow \infty$, then $x_n \xrightarrow{d} x^*$ as $n \rightarrow \infty$;

(xii) (Well-posedness of the fixed point problem with respect to H [24]) If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X such that $H(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow \infty$, then $x_n \xrightarrow{d} x^*$ as $n \rightarrow \infty$.

Proof. (x) Let $x^* \in (SF)_T$. Note that $(SF)_T = \{x^*\}$. Indeed, if $y \in (SF)_T$, then $d(x^*, y) = H(T(x^*), T(y)) \leq ad(x^*, y) + bD(x^*, T(x^*)) + cD(y, T(y)) = ad(x^*, y)$. Thus $y = x^*$.

Let us show now that $F_T = \{x^*\}$. Suppose that $y \in F_T$. Then, $d(x^*, y) = D(T(x^*), y) \leq H(T(x^*), T(y)) \leq ad(x^*, y) + bD(x^*, T(x^*)) + cD(y, T(y)) = ad(x^*, y)$. Thus $y = x^*$. Hence $F_T \subset (SF)_T = \{x^*\}$. Since $(SF)_T \subset F_T$, we get that $(SF)_T = F_T = \{x^*\}$.

(xi) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $D(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow \infty$. Then, $d(x_n, x^*) \leq D(x_n, T(x_n)) + H(T(x_n), T(x^*)) \leq D(x_n, T(x_n)) + ad(x_n, x^*) + bD(x_n, T(x_n)) + cD(x^*, T(x^*)) = (1 + b)D(x_n, T(x_n)) + ad(x_n, x^*)$. Then $d(x_n, x^*) \leq ((1 + b)/(1 - a))D(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow +\infty$.

(xii) follows by (xi) since $D(x_n, T(x_n)) \leq H(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow +\infty$. □

A third result for the case of (a, b, c) -contraction is the following.

Theorem 3.4. *Let (X, d) be a complete metric space, and let $T : X \rightarrow P_{cp}(X)$ be a Reich-type multivalued (a, b, c) -contraction such that $T(F_T) = F_T$. Then one has*

(xiii) $T^n(x) \xrightarrow{H} F_T$ as $n \rightarrow +\infty$, for each $x \in X$;

(xiv) $T(x) = F_T$ for each $x \in F_T$;

(xv) If $(x_n)_{n \in \mathbb{N}} \subset X$ is a sequence such that $x_n \xrightarrow{d} x^* \in F_T$ as $n \rightarrow \infty$ and T is H -continuous, then $T(x_n) \xrightarrow{H} F_T$ as $n \rightarrow +\infty$.

Proof. (xiii) From the fact that $T(F_T) = F_T$ and Theorem 3.2 (vi) we have that $F_T = A_T^*$. The conclusion follows by Theorem 3.2 (vii).

(xiv) Let $x \in F_T$ be an arbitrary. Then $x \in T(x)$, and thus $F_T \subset T(x)$. On the other hand $T(x) \subset T(F_T) \subset F_T$. Thus $T(x) = F_T$, for each $x \in F_T$.

(xv) Let $(x_n)_{n \in \mathbb{N}} \subset X$ be a sequence such that $x_n \xrightarrow{d} x^* \in F_T$ as $n \rightarrow \infty$. Then, we have $T(x_n) \xrightarrow{H} T(x^*) = F_T$ as $n \rightarrow \infty$. The proof is complete. $\square$

For compact metric spaces we have the following result.

Theorem 3.5. *Let (X, d) be a compact metric space, and let $T : X \rightarrow P_{cl}(X)$ be a H -continuous Reich-type multivalued (a, b, c) -contraction. Then*

(xvi) *if $(x_n)_{n \in \mathbb{N}}$ is such that $D(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow \infty$, then there exists a subsequence $(x_{n_i})_{i \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $x_{n_i} \xrightarrow{d} x^* \in F_T$ as $i \rightarrow \infty$ (generalized well-posedness of the fixed point problem with respect to D [24, 25]).*

Proof. (xvi) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $D(x_n, T(x_n)) \rightarrow 0$ as $n \rightarrow \infty$. Let $(x_{n_i})_{i \in \mathbb{N}}$ be a subsequence of $(x_n)_{n \in \mathbb{N}}$ such that $x_{n_i} \xrightarrow{d} x^*$ as $i \rightarrow \infty$. Then, there exists $y_{n_i} \in T(x_{n_i})$, such that $y_{n_i} \xrightarrow{d} x^*$ as $i \rightarrow \infty$. Then $D(x^*, T(x^*)) \leq d(x^*, y_{n_i}) + D(y_{n_i}, T(x_{n_i})) + H(T(x_{n_i}), T(x^*)) \leq d(x^*, y_{n_i}) + ad(x^*, x_{n_i}) + bD(x_{n_i}, T(x_{n_i})) + cD(x^*, T(x^*))$. Hence

$$D(x^*, T(x^*)) \leq \frac{1}{1-c} [d(x^*, y_{n_i}) + ad(x^*, x_{n_i}) + bD(x_{n_i}, T(x_{n_i}))] \rightarrow 0 \quad (3.19)$$

as $n \rightarrow +\infty$. Hence $x^* \in F_T$. $\square$

Remark 3.6. For $b = c = 0$ we obtain the results given in [4]. On the other hand, our results unify and generalize some results given in [12, 13, 17, 26–34]. Notice that, if the operator T is singlevalued, then we obtain the well-posedness concept introduced in [35].

Remark 3.7. An open question is to present a theory of the Ćirić-type multivalued contraction theorem (see [36]). For some problems for other classes of generalized contractions, see for example, [17, 21, 27, 34, 37].

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