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Research Article

Existence of Solution and Positive Solution for a Nonlinear Third-Order *m*-Point BVP

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In this paper, we are concerned with the following nonlinear third-order m-point boundary value problem: $u'''(t) + f(t, u(t), u''(t)) = 0$, $t \in [0,1]$, $u(0) = A$, $u'(1) - \sum_{i=1}^{m-2} a_i u'(\xi_i) = B$, $u''(0) = C$. Some existence criteria of solution and positive solution are established by using the Schauder fixed point theorem. An example is also included to illustrate the importance of the results obtained.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves, or gravity-driven flows and so on [1].

Recently, third-order two-point or three-point boundary value problems (BVPs) have received much attention from many authors; see [2–10] and the references therein. In particular, Yao [10] employed the Leray-Schauder fixed point theorem to prove the existence of solution and positive solution for the BVP

$$\begin{aligned} &u'''(t) + f(t, u(t), u''(t)) = 0, \quad t \in [0,1], \\ &u(0) = A, \qquad u(1) = B, \qquad u''(0) = C. \end{aligned} \tag{1.1}$$

Although there are many excellent results on third-order two-point or three-point BVPs, few works have been done for more general third-order m-point BVPs [11–13]. It is worth mentioning that Jin and Lu [12] studied some third-order differential equation with the following m-point boundary conditions:

$$u(0) = 0, \qquad u'(1) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u''(0) = 0. \tag{1.2}$$

The main tool used was Mawhin's continuation theorem.

Motivated greatly by [10, 12], in this paper, we investigate the following nonlinear third-order m-point BVP:

$$\begin{gathered} u'''(t) + f\big(t, u(t), u''(t)\big) = 0, \quad t \in [0,1], \\ u(0) = A, \qquad u'(1) - \sum_{i=1}^{m-2} a_i u'(\xi_i) = B, \quad u''(0) = C. \end{gathered} \tag{1.3}$$

Throughout, we always assume that $a_i \geq 0$ $(i = 1, 2, \ldots, m-2)$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. The purpose of this paper is to consider the local properties of f on some bounded sets and establish some existence criteria of solution and positive solution for the BVP (1.3) by using the Schauder fixed point theorem. An example is also included to illustrate the importance of the results obtained.

2. Main Results

Lemma 2.1. *Let $\sum_{i=1}^{m-2} a_i \neq 1$. Then, for any $v \in C[0,1]$, the BVP*

$$\begin{gathered} u''(t) = v(t), \quad t \in [0,1], \\ u(0) = A, \qquad u'(1) - \sum_{i=1}^{m-2} a_i u'(\xi_i) = B \end{gathered} \tag{2.1}$$

has a unique solution

$$u(t) = \frac{B - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} v(s)ds}{1 - \sum_{i=1}^{m-2} a_i} t + A - \int_0^1 G(t,s)v(s)ds, \quad t \in [0,1], \tag{2.2}$$

where

$$G(t,s) = \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ t, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Proof. If u is a solution of the BVP (2.1), then we may suppose that

$$u(t) = Mt + N - \int_0^1 G(t,s)v(s)ds, \quad t \in [0,1]. \tag{2.4}$$

By the boundary conditions in (2.1), we know that

$$M = \frac{B - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} v(s)ds}{1 - \sum_{i=1}^{m-2} a_i}, \quad N = A. \tag{2.5}$$

Therefore, the unique solution of the BVP (2.1)

$$u(t) = \frac{B - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} v(s)ds}{1 - \sum_{i=1}^{m-2} a_i} t + A - \int_0^1 G(t,s)v(s)ds, \quad t \in [0,1]. \tag{2.6}$$

□

In the remainder of this paper, we always assume that $\sum_{i=1}^{m-2} a_i \neq 1$. For convenience, we denote

$$\begin{gathered} R = (-\infty, +\infty), \qquad R_+ = [0, +\infty), \qquad R_- = (-\infty, 0], \\ \sigma = \frac{2\left|1 - \sum_{i=1}^{m-2} a_i\right|}{2\sum_{i=1}^{m-2} a_i + \left|1 - \sum_{i=1}^{m-2} a_i\right|}, \qquad \eta = \max\left\{|A|, \left|\frac{B}{1 - \sum_{i=1}^{m-2} a_i}\right|, \frac{|C|}{\sigma}\right\}. \end{gathered} \tag{2.7}$$

The following theorem guarantees the existence of solution for the BVP (1.3).

Theorem 2.2. *Assume that $f : [0,1] \times R \times R \to R$ is continuous and there exist $c > 0$ and $1/4 \le k \le 1/2$ such that*

$$\max\{|f(t,u,v)| : t \in [0,1],\ |u| \le 4\eta + c,\ |v| \le \sigma k(4\eta + c)\} \le \sigma[(4k-1)\eta + kc]. \tag{2.8}$$

Then the BVP (1.3) *has one solution u_0 satisfying*

$$\begin{gathered} |u_0(t)| \le 4\eta + c, \quad t \in [0,1], \\ |u_0''(t)| \le \sigma k(4\eta + c), \quad t \in [0,1]. \end{gathered} \tag{2.9}$$

Proof. Let $E = C[0,1] \times C[0,1]$ be equipped with the norm $\|(u,v)\| = \max\{\|u\|_\infty, \|v\|_\infty/\sigma k\}$, where $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$. Then E is a Banach space.

Let $v(t) = u''(t)$, $t \in [0,1]$. Then the BVP (1.3) is equivalent to the following system:

$$\begin{gathered} u''(t) = v(t), \quad t \in [0,1], \\ v'(t) = -f(t, u(t), v(t)), \quad t \in [0,1], \\ u(0) = A, \qquad u'(1) - \sum_{i=1}^{m-2} a_i u'(\xi_i) = B, \qquad v(0) = C. \end{gathered} \tag{2.10}$$

Furthermore, it is easy to know that the system (2.10) is equivalent to the following system:

$$u(t) = \frac{B - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} v(s)ds}{1 - \sum_{i=1}^{m-2} a_i} t + A - \int_0^1 G(t,s)v(s)ds, \quad t \in [0,1],$$
$$v(t) = C - \int_0^t f(s,u(s),v(s))ds, \quad t \in [0,1]. \tag{2.11}$$

Now, if we define an operator $F : E \to E$ by

$$F(u,v) = (F_1(u,v), F_2(u,v)), \tag{2.12}$$

where

$$F_1(u,v)(t) = \frac{B - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} v(s)ds}{1 - \sum_{i=1}^{m-2} a_i} t + A - \int_0^1 G(t,s)v(s)ds, \quad t \in [0,1],$$
$$F_2(u,v)(t) = C - \int_0^t f(s,u(s),v(s))ds, \quad t \in [0,1], \tag{2.13}$$

then it is easy to see that $F : E \to E$ is completely continuous and the system (2.11) and so the BVP (1.3) is equivalent to the fixed point equation

$$F(u,v) = (u,v), \quad (u,v) \in E. \tag{2.14}$$

Let $V = \{(u,v) \in E : \|(u,v)\| \le 4\eta + c\}$. Then V is a closed convex subset of E. Suppose that $(u,v) \in V$. Then $\|u\|_\infty \le 4\eta + c$ and $\|v\|_\infty \le \sigma k(4\eta + c)$. So,

$$|u(t)| \le 4\eta + c, \quad t \in [0,1], \tag{2.15}$$

$$|v(t)| \le \sigma k(4\eta + c), \quad t \in [0,1], \tag{2.16}$$

which implies that

$$|f(t,u(t),v(t))| \le \sigma[(4k-1)\eta + kc], \quad t \in [0,1]. \tag{2.17}$$

From (2.16) and $1/4 \le k \le 1/2$, we have

$$\begin{aligned}
\|F_1(u,v)\|_\infty &\le \max_{t\in[0,1]}\left(\left|\frac{B}{1-\sum_{i=1}^{m-2}a_i}\right| + \left|\frac{\sum_{i=1}^{m-2}a_i\int_{\xi_i}^1 v(s)ds}{1-\sum_{i=1}^{m-2}a_i}\right|\right)t + |A| + \max_{t\in[0,1]}\int_0^1 G(t,s)|v(s)|ds \\
&\le \left|\frac{B}{1-\sum_{i=1}^{m-2}a_i}\right| + \frac{\sigma k(4\eta+c)\sum_{i=1}^{m-2}a_i}{\left|1-\sum_{i=1}^{m-2}a_i\right|} + |A| + \frac{\sigma k(4\eta+c)}{2} \\
&\le 2\eta + \frac{2\sum_{i=1}^{m-2}a_i + \left|1-\sum_{i=1}^{m-2}a_i\right|}{2\left|1-\sum_{i=1}^{m-2}a_i\right|}\sigma k(4\eta+c) \\
&= 4\eta\left(k+\frac{1}{2}\right) + kc.
\end{aligned} \tag{2.18}$$

On the other hand, it follows from (2.17) that

$$\begin{aligned}
\|F_2(u,v)\|_\infty &= \max_{t\in[0,1]}\left|C - \int_0^t f(s,u(s),v(s))ds\right| \\
&\le |C| + \int_0^1 \left|f(s,u(s),v(s))\right|ds \\
&\le \sigma\eta + \sigma\left[(4k-1)\eta + kc\right] \\
&= \sigma k(4\eta+c).
\end{aligned} \tag{2.19}$$

In view of (2.18) and (2.19), we know that

$$\begin{aligned}
\|(F_1(u,v),F_2(u,v))\| &= \max\left\{\|F_1(u,v)\|_\infty, \frac{\|F_2(u,v)\|_\infty}{\sigma k}\right\} \\
&\le 4\eta + c,
\end{aligned} \tag{2.20}$$

which shows that $F : V \to V$. Then it follows from the Schauder fixed point theorem that F has a fixed point $(u_0,v_0) \in V$. In other words, the BVP (1.3) has one solution u_0, which satisfies

$$\begin{aligned}
|u_0(t)| &\le 4\eta + c, \quad t\in[0,1], \\
\left|u_0''(t)\right| &\le \sigma k(4\eta+c), \quad t\in[0,1].
\end{aligned} \tag{2.21}$$

□

On the basis of Theorem 2.2, we now give some existence results of nonnegative solution and positive solution for the BVP (1.3).

Theorem 2.3. *Assume that $A \geq 0$, $B \geq 0$, $C \leq 0$, $\sum_{i=1}^{m-2} a_i < 1$, $f : [0,1] \times R_+ \times R_- \to R_+$ is continuous, and there exist $c > 0$ and $1/4 \leq k \leq 1/2$ such that*

$$\max\left\{f(t,u,v) : t \in [0,1],\ 0 \leq u \leq 4\eta + c,\ -\sigma k(4\eta + c) \leq v \leq 0\right\} \leq \sigma\left[(4k-1)\eta + kc\right]. \tag{2.22}$$

Then the BVP (1.3) *has one solution u_0 satisfying*

$$\begin{gathered} 0 \leq u_0(t) \leq 4\eta + c, \quad t \in [0,1], \\ -\sigma k(4\eta + c) \leq u_0''(t) \leq 0, \quad t \in [0,1]. \end{gathered} \tag{2.23}$$

Proof. Let

$$\begin{gathered} f_1(t,u,v) = \begin{cases} f(t,u,v), & (t,u,v) \in [0,1] \times R_+ \times R_-, \\ f(t,u,0), & (t,u,v) \in [0,1] \times R_+ \times R_+, \end{cases} \\ f_2(t,u,v) = \begin{cases} f_1(t,u,v) & (t,u,v) \in [0,1] \times R_+ \times R, \\ f_1(t,0,v) & (t,u,v) \in [0,1] \times R_- \times R. \end{cases} \end{gathered} \tag{2.24}$$

Then $f_2 : [0,1] \times R \times R \to R_+$ is continuous and

$$\begin{aligned} &\max\left\{\left|f_2(t,u,v)\right| : t \in [0,1],\ |u| \leq 4\eta + c,\ |v| \leq \sigma k(4\eta + c)\right\} \\ &\quad = \max\left\{f(t,u,v) : t \in [0,1],\ 0 \leq u \leq 4\eta + c,\ -\sigma k(4\eta + c) \leq v \leq 0\right\} \\ &\quad \leq \sigma\left[(4k-1)\eta + kc\right]. \end{aligned} \tag{2.25}$$

Consider the BVP

$$\begin{gathered} u'''(t) + f_2\left(t, u(t), u''(t)\right) = 0, \quad t \in [0,1], \\ u(0) = A, \qquad u'(1) - \sum_{i=1}^{m-2} a_i u'(\xi_i) = B, \qquad u''(0) = C. \end{gathered} \tag{2.26}$$

By Theorem 2.2, we know that the BVP (2.26) has one solution u_0 satisfying

$$\begin{gathered} |u_0(t)| \leq 4\eta + c, \quad t \in [0,1], \\ \left|u_0''(t)\right| \leq \sigma k(4\eta + c), \quad t \in [0,1]. \end{gathered} \tag{2.27}$$

Since $C \leq 0$, we get

$$u_0''(t) = C - \int_0^t f_2\left(s, u_0(s), u_0''(s)\right) ds \leq 0, \quad t \in [0,1]. \tag{2.28}$$

In view of (2.28) and $u_0'(1) - \sum_{i=1}^{m-2} a_i u_0'(\xi_i) = B$, we have

$$u_0'(t) \geq u_0'(1) \geq \frac{B}{1 - \sum_{i=1}^{m-2} a_i} \geq 0, \quad t \in [0,1], \tag{2.29}$$

which implies that

$$u_0(t) \geq u_0(0) = A \geq 0, \quad t \in [0,1]. \tag{2.30}$$

It follows from (2.28), (2.30), and the definition of f_2 that

$$f_2\left(t, u_0(t), u_0''(t)\right) = f\left(t, u_0(t), u_0''(t)\right), \quad t \in [0,1]. \tag{2.31}$$

Therefore, u_0 is a solution of the BVP (1.3) and satisfies

$$\begin{gathered} 0 \leq u_0(t) \leq 4\eta + c, \quad t \in [0,1], \\ -\sigma k(4\eta + c) \leq u_0''(t) \leq 0, \quad t \in [0,1]. \end{gathered} \tag{2.32}$$

□

Corollary 2.4. *Assume that all the conditions of Theorem 2.3 are fulfilled. Then the BVP* (1.3) *has one positive solution if one of the following conditions is satisfied:*

(i) $A + B > 0$;

(ii) $C < 0$;

(iii) $f(t,0,0) \not\equiv 0$, $t \in [0,1]$.

Proof. Since it is easy to prove Cases (ii) and (iii), we only prove Case (i). It follows from Theorem 2.3 that the BVP (1.3) has a solution u_0, which satisfies

$$\begin{gathered} 0 \leq u_0(t) \leq 4\eta + c, \quad t \in [0,1], \\ -\sigma k(4\eta + c) \leq u_0''(t) \leq 0, \quad t \in [0,1]. \end{gathered} \tag{2.33}$$

Suppose that $A + B > 0$. Then for any $t \in (0,1)$, we have

$$\begin{aligned} u_0(t) &= \frac{B - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 u_0''(s)ds}{1 - \sum_{i=1}^{m-2} a_i} t + A - \int_0^1 G(t,s) u_0''(s)ds \\ &\geq \frac{Bt}{1 - \sum_{i=1}^{m-2} a_i} + A \\ &\geq Bt + A \\ &\geq (B + A)t \\ &> 0, \end{aligned} \tag{2.34}$$

which shows that u_0 is a positive solution of the BVP (1.3). □

Example 2.5. Consider the BVP

$$u'''(t) + f\big(t, u(t), u''(t)\big) = 0, \quad t \in [0,1],$$
$$u(0) = 1, \qquad u'(1) - \frac{1}{2}u'\left(\frac{1}{2}\right) = 0, \qquad u''(0) = -1, \tag{2.35}$$

where $f(t,u,v) = u^2/189 + (1-t)v^2/14 + 1/9$, $(t,u,v) \in [0,1] \times R_+ \times R_-$.

A simple calculation shows that $\sigma = 2/3$ and $\eta = 3/2$. Thus, if we choose $k = 1/3$ and $c = 1$, then all the conditions of Theorem 2.3 and (i) of Corollary 2.4 are fulfilled. It follows from Corollary 2.4 that the BVP (2.35) has a positive solution.

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