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Research Article

Existence and Localization Results for $p(x)$-Laplacian via Topological Methods

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We show the existence of a week solution in $W_0^{1,p(x)}(\Omega)$ to a Dirichlet problem for $-\Delta_{p(x)}u = f(x,u)$ in Ω, and its localization. This approach is based on the nonlinear alternative of Leray-Schauder.

1. Introduction

In this work, we consider the boundary value problem

$$\begin{aligned} -\Delta_{p(x)}u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a nonempty bounded open set with smooth boundary $\partial\Omega$, $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the so-called $p(x)$-Laplacian operator, and (CAR): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function which satisfies the growth condition

$$\left|f(x,s)\right| \leq a(x) + C|s|^{q(x)/q'(x)} \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}, \tag{1.1}$$

with $C = \text{const.} > 0$, $1/q(x) + 1/q'(x) = 1$ for a.e. $x \in \Omega$, and $a \in L^{q'(x)}(\Omega)$, $a(x) \geq 0$ for a.e. $x \in \Omega$.

We recall in what follows some definitions and basic properties of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$. In that context, we refer to [1, 2] for the fundamental properties of these spaces.

Set

$$L_+^\infty(\Omega) = \left\{ p : p \in L^\infty(\Omega), \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1 \right\}. \tag{1.2}$$

For $p \in L_+^\infty(\Omega)$, let $p_1 := \text{ess inf}_{x\in\Omega} p(x) \le p(x) \le p_2 := \text{ess sup}_{x\in\Omega} p(x) < \infty$, for a.e. $x \in \Omega$.

Let us define by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω. For any $p \in L_+^\infty(\Omega)$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \in \mathcal{U}(\Omega) : \rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}. \tag{1.3}$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{p(x)} = \inf\left\{ \delta > 0 : \rho_{p(x)}\left(\frac{u}{\delta}\right) \le 1 \right\}, \tag{1.4}$$

and $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ becomes a Banach space.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega),\ i = 1, \dots, N \right\} \tag{1.5}$$

and we define on this space the norm

$$\|u\| = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \tag{1.6}$$

for all $u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 1.1 (see [1, 2]). *If $p \in L_+^\infty(\Omega)$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 1.2 (see [1, 2]). *If $u \in L^{p(x)}(\Omega)$ and $p_2 < \infty$, then we have*

(i) $\|u\|_{p(x)} < 1 (= 1;\ > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 (= 1;\ > 1)$,

(ii) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p_1} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p_2}$,

(iii) $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p_2} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p_1}$,

(iv) $\|u\|_{p(x)} = a > 0 \Leftrightarrow \rho_{p(x)}(u/a) = 1$.

Proposition 1.3 (see [3]). *Assume that Ω is bounded and smooth. Denote by $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h_1 > 1\}$.*

(i) *Let $p, q \in C_+(\overline{\Omega})$. If*

$$q(x) < p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \tag{1.7}$$

then $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is compactly imbedded in $L^{q(x)}(\Omega)$.

(ii) *(Poincaré inequality, see [1, Theorem 2.7]). If $p \in C_+(\overline{\Omega})$, then there is a constant $C > 0$ such that*

$$\|u\|_{p(x)} \leq C \||\nabla u|\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{1.8}$$

Consequently, $\|u\|_{1,p(x)} = \||\nabla u|\|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. In what follows, $W_0^{1,p(x)}(\Omega)$, with $p \in C_+(\overline{\Omega})$, will be considered as endowed with the norm $\|u\|_{1,p(x)}$.

Lemma 1.4. *Assume that $r \in L_+^\infty(\Omega)$ and $p \in C_+(\overline{\Omega})$. If $|u|^{r(x)} \in L^{p(x)}(\Omega)$, then we have*

$$\min\left\{\|u\|_{r(x)p(x)}^{r_1}, \|u\|_{r(x)p(x)}^{r_2}\right\} \leq \left\| |u|^{r(x)} \right\|_{p(x)} \leq \max\left\{\|u\|_{r(x)p(x)}^{r_1}, \|u\|_{r(x)p(x)}^{r_2}\right\}. \tag{1.9}$$

Proof. By Proposition 1.2 (iv), we have

$$\begin{aligned} 1 &= \int_\Omega \left| \frac{|u|^{r(x)}}{\left\| |u|^{r(x)} \right\|_{p(x)}} \right|^{p(x)} dx \\ &= \int_\Omega \left| \frac{|u|}{\|u\|_{r(x)p(x)}} \right|^{r(x)p(x)} \frac{\|u\|_{r(x)p(x)}^{r(x)p(x)}}{\left\| |u|^{r(x)} \right\|_{p(x)}^{p(x)}} dx \\ &\leq \int_\Omega \left| \frac{|u|}{\|u\|_{r(x)p(x)}} \right|^{r(x)p(x)} \frac{\max\left\{\|u\|_{r(x)p(x)}^{r_1 p(x)}, \|u\|_{r(x)p(x)}^{r_2 p(x)}\right\}}{\left\| |u|^{r(x)} \right\|_{p(x)}^{p(x)}} dx. \end{aligned} \tag{1.10}$$

By the mean value theorem, there exists $\xi \in \overline{\Omega}$ such that

$$1 \leq \frac{\max\left\{\|u\|_{r(x)p(x)}^{r_1 p(\xi)}, \|u\|_{r(x)p(x)}^{r_2 p(\xi)}\right\}}{\left\| |u|^{r(x)} \right\|_{p(x)}^{p(\xi)}} \int_\Omega \left| \frac{|u|}{\|u\|_{r(x)p(x)}} \right|^{r(x)p(x)} dx \tag{1.11}$$

and we have

$$\left\| |u|^{r(x)} \right\|_{p(x)} \leq \max\left\{ \|u\|^{r_1}_{r(x)p(x)}, \|u\|^{r_2}_{r(x)p(x)} \right\}. \tag{1.12}$$

Similarly

$$\begin{gathered} 1 \geq \frac{\min\left\{ \|u\|^{r_1 p(\xi)}_{r(x)p(x)}, \|u\|^{r_2 p(\xi)}_{r(x)p(x)} \right\}}{\left\| |u|^{r(x)} \right\|^{p(\xi)}_{p(x)}} dx, \\ \left\| |u|^{r(x)} \right\|_{p(x)} \geq \min\left\{ \|u\|^{r_1}_{r(x)p(x)}, \|u\|^{r_2}_{r(x)p(x)} \right\}. \end{gathered} \tag{1.13}$$

□

Remark 1.5. If $r(x) = r = \text{const.}$, then

$$\left\| |u|^{r} \right\|_{p(x)} = \|u\|^{r}_{rp(x)}. \tag{1.14}$$

For simplicity of notation, we write

$$\begin{gathered} X = W_0^{1,p(x)}(\Omega), \qquad X^* = \left(W_0^{1,p(x)}(\Omega) \right)^*, \qquad Y = L^{q(x)}(\Omega), \qquad Y^* = L^{q'(x)}(\Omega), \\ \|\cdot\|_X = \|\cdot\|_{1,p(x)}, \qquad \|\cdot\|_Y = \|\cdot\|_{q(x)}. \end{gathered} \tag{1.15}$$

In [4], a topological method, based on the fundamental properties of the Leray-Schauder degree, is used in proving the existence of a week solution in X to the Dirichlet problem (P) that is an adaptation of that used by Dinca et al. for Dirichlet problems with classical p-Laplacian [5]. In this work, we use the nonlinear alternative of Leray-Schauder and give the existence of a solution and its localization. This method is used for finding solutions in Hölder spaces, while in [6], solutions are found in Sobolev spaces.

Let us recall some results borrowed from Dinca [4] about $p(x)$-Laplacian and Nemytskii operator N_f. Firstly, since $q(x) < p(x) < p^*(x)$ for all $x \in \overline{\Omega}$, X is compactly embedded in Y. Denote by i the compact injection of X in $Y$and by $i^* : Y^* \to X^*$, $i^*v = v \circ i$ for all $v \in Y^*$, its adjoint.

Since the Caratheodory function f satisfies (CAR), the Nemytskii operator N_f generated by f, $(N_f u)(x) = f(x, u(x))$, is well defined from Y into Y^*, continuous, and bounded ([3, Proposition 2.2]). In order to prove that problem (P) has a weak solution in X it is sufficient to prove that the equation

$$-\Delta_{p(x)} u = \left(i^* N_f i\right) u \tag{1.16}$$

has a solution in X.

Indeed, if $u \in X$ satisfies (1.16) then, for all $v \in X$, one has

$$\left\langle -\Delta_{p(x)} u, v \right\rangle_{X,X^*} = \left\langle \left(i^* N_f i\right) u, v \right\rangle_{X,X^*} = \left\langle N_f(iu), iv \right\rangle_{Y,Y^*} \tag{1.17}$$

which rewrites as

$$\int_\Omega |\nabla u|^{p(x)} \nabla u \nabla v dx = \int_\Omega f v dx \tag{1.18}$$

and tells us that u is a weak solution in X to problem (P).

Since $-\Delta_{p(x)}$ is a homeomorphism of X onto X^*, (1.16) may be equivalently written as

$$u = \left(-\Delta_{p(x)}\right)^{-1} \left(i^* N_f i\right) u. \tag{1.19}$$

Thus, proving that problem (P) has a weak solution in X reduces to proving that the compact operator

$$K = \left(-\Delta_{p(x)}\right)^{-1} \left(i^* N_f i\right) \ : X \to X \tag{1.20}$$

has a fixed point.

Theorem 1.6 (Alternative of Leray-Schauder, [7]). *Let $B[0,R]$ denote the closed ball in a Banach space E, $\{u \in E : \|u\| \leq R\}$, and let $K : B[0,R] \to E$ be a compact operator. Then either*

(i) *the equation $\lambda K u = u$ has a solution in $B[0,R]$ for $\lambda = 1$ or*

(ii) *there exists an element $u \in E$ with $\|u\| = R$ satisfying $\lambda K u = u$ for some $\lambda \in (0,1)$.*

2. Main Results

In this work, we present new existence and localization results for X-solutions to problem (P), under (CAR) condition on f. Our approach is based on regularity results for the solutions of Dirichlet problems and again on the nonlinear alternative of Leray-Schauder.

We start with an existence and localization principle for problem (P).

Theorem 2.1. *Assume that there is a constant $R > 0$, independent of $\lambda > 0$, with $\|u\|_X \neq R$ for any solution $u \in X$ to*

$$\begin{gathered} -\Delta_{p(x)} u = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{P_λ}$$

and for each $\lambda \in (0,1)$. Then the Dirichlet problem (P) *has at least one solution $u \in X$ with $\|u\|_X \leq R$.*

Proof. By [3, Theorem 3.1], $-\Delta_{p(x)}$ is a homeomorphism of X onto X^*. We will apply Theorem 2.1 to $E = X$ and to operator $K : X \to X$,

$$Ku = \left(-\Delta_{p(x)}\right)^{-1} \left(i^* N_f i\right) u, \tag{2.1}$$

where $i^* N_f i : X \to X^*$ is given by $(N_f u)(x) = f(x, u(x))$. Notice that, according to a well-known regularity result [4], the operator $(-\Delta_{p(x)})^{-1}$ from X to X is well defined, continuous, and order preserving. Consequently, K is a compact operator. On the other hand, it is clear that the fixed points of K are the solutions of problem (P). Now the conclusion follows from Theorem 1.6 since condition (ii) is excluded by hypothesis. □

Theorem 2.2 immediately yields the following existence and localization result.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain and let $p, q \in C_+(\overline{\Omega})$ be such that $q(x) < p(x)$ for all $x \in \overline{\Omega}$. Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function which satisfies the growth condition (CAR).*

Suppose, in addition, that

$$C\|i^*\|_{Y^* \to X^*} \max\left\{\|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1}\right\} < 1, \tag{2.2}$$

where C is the constant appearing in condition (CAR). Let $R \geq 1$ be a constant such that

$$R \geq \left(\frac{\|i^*\|_{Y^* \to X^*}\|a\|_{Y^*}}{1 - C\|i^*\|_{Y^* \to X^*} \max\left\{\|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1}\right\}}\right)^{1/(p_1 - 1)}. \tag{2.3}$$

Then the Dirichlet problem (P) *has at least a solution in X with $\|u\|_X \leq R$.*

Proof. Let $u \in X$ be a solution of problem (P_λ) with $\|u\|_X = R \geq 1$, corresponding to some $\lambda \in (0,1)$. Then by Propositions 1.2, 1.3, and Lemma 1.4, we obtain

$$\begin{aligned}
\|u\|_X^{p_1} &\leq \int_\Omega |\nabla u|^{p(x)} dx = \lambda\left\langle (i^* N_f i)u, u\right\rangle_{X,X^*} = \lambda\left\langle N_f(iu), iu\right\rangle_{Y,Y^*} \\
&\leq \lambda\|i^*\|_{Y^* \to X^*}\left\|N_f(iu)\right\|_{Y^*}\|u\|_X \\
&\leq \lambda\|i^*\|_{Y^* \to X^*}\|u\|_X\left(\|a\|_{Y^*} + C\max\left\{\|iu\|_Y^{q_1 - 1}, \|iu\|_Y^{q_2 - 1}\right\}\right) \\
&\leq \lambda\|i^*\|_{Y^* \to X^*}\|u\|_X\left(\|a\|_{Y^*} + C\|u\|_X^{q_2 - 1}\max\left\{\|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1}\right\}\right) \\
&\leq \lambda\|i^*\|_{Y^* \to X^*}\|u\|_X\left(\|a\|_{Y^*} + C\|u\|_X^{p_1 - 1}\max\left\{\|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1}\right\}\right).
\end{aligned} \tag{2.4}$$

Therefore, we have

$$\|u\|_X^{p_1 - 1} \leq \frac{\lambda\|i^*\|_{Y^* \to X^*}\|a\|_{Y^*}}{1 - \lambda C\|i^*\|_{Y^* \to X^*}\max\left\{\|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1}\right\}}. \tag{2.5}$$

Substituting $\|u\|_X = R$ in the above inequality, we obtain

$$R \leq \left(\frac{\lambda \|i^*\|_{Y^* \to X^*} \|a\|_{Y^*}}{1 - \lambda C \|i^*\|_{Y^* \to X^*} \max\left\{ \|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1} \right\}} \right)^{1/(p_1 - 1)}, \tag{2.6}$$

which, taking into account (2.3) and $\lambda \in (0,1)$, gives

$$\begin{aligned} R &\leq \lambda^{(1/p_1 - 1)} \left(\frac{\|i^*\|_{Y^* \to X^*} \|a\|_{Y^*}}{1 - C\lambda \|i^*\|_{Y^* \to X^*} \max\left\{ \|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1} \right\}} \right)^{1/(p_1 - 1)} \\ &\leq \lambda^{(1/p_1 - 1)} \left(\frac{\|i^*\|_{Y^* \to X^*} \|a\|_{Y^*}}{1 - C \|i^*\|_{Y^* \to X^*} \max\left\{ \|i\|_{X \to Y}^{q_1 - 1}, \|i\|_{X \to Y}^{q_2 - 1} \right\}} \right)^{1/(p_1 - 1)} \\ &\leq \lambda^{(1/p_1 - 1)} R < R, \end{aligned} \tag{2.7}$$

a contradiction. Theorem 2.1 applies. □

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