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Research Article

Strong Convergence Theorems for Infinitely Nonexpansive Mappings in Hilbert Space

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We introduce a modified Ishikawa iterative process for approximating a fixed point of two infinitely nonexpansive self-mappings by using the hybrid method in a Hilbert space and prove that the modified Ishikawa iterative sequence converges strongly to a common fixed point of two infinitely nonexpansive self-mappings.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H, T a self-mapping of C. Recall that T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$.

Construction of fixed points of nonexpansive mappings via Mann's iteration [1] has extensively been investigated in literature (see, e.g., [2–5] and reference therein). But the convergence about Mann's iteration and Ishikawa's iteration is in general not strong (see the counterexample in [6]). In order to get strong convergence, one must modify them. In 2003, Nakajo and Takahashi [7] proposed such a modification for a nonexpansive mapping T.

Consider the algorithm,

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarity}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{ v \in C : \|y_n - v\| \leq \|x_n - v\| \}, \\ Q_n = \{ v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{gathered} \tag{1.1}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H. They prove the sequence $\{x_n\}$ generated by that algorithm (1.1) converges strongly to a fixed point of T provided that the control sequence $\{\alpha_n\}$ is chosen so that $\sup_{n\geq 0}\alpha_n < 1$.

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings of C, $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of nonnegative numbers in $[0,1]$. For each $n \geq 1$, defined a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{1.2}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$; see [8].

In this paper, motivated by [9], for any given $x_i \in C$ ($i = 0,1,\ldots,q, q \in \mathbb{N}$ is a fixed number), we will propose the following iterative progress for two infinitely nonexpansive mappings $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ in a Hilbert space H:

$$\begin{gathered}
x_0, x_1, \ldots, x_q \in C \text{ chosen arbitrarity},\\
y_n = \alpha_n x_n + (1-\alpha_n) W_n^{(1)} z_{n-q},\\
z_n = \overline{\alpha}_n x_n + (1-\overline{\alpha}_n) W_n^{(2)} x_n,\\
C_n = \left\{ v \in K : \left\|y_n - v\right\|^2 \leq \|x_n - v\|^2 + (1-\alpha_n)\left(\left\|x_{n-q} - x^*\right\|^2 - \|x_n - x^*\|^2\right)\right\},\\
Q_n = \{ v \in K : \langle x_n - v, x_n - x_q \rangle \leq 0 \},\\
x_{n+1} = P_{C_n \cap Q_n}(x_q), n \geq q
\end{gathered} \tag{1.3}$$

and prove, $\{x_n\}$ converges strongly to a fixed point of $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$.

We will use the notation:

$\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.

$\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of x_n.

2. Preliminaries

In this paper, we need some facts and tools which are listed as lemmas below.

Lemma 2.1 (see [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and T a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.2 (see [11]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H. Given also a real number $a \in \mathbb{R}$ and $x, y, z \in H$. Then the set $D := \{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$ is closed and convex.*

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C, where C is a nonempty closed convex subset of a strictly convex Banach space E. Given a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $[0,1]$, *one defines a sequence $\{W_n\}_{n=1}^{\infty}$ of self-mappings on C by* (1.2). *Then one has the following results.*

Lemma 2.3 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E, $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in* $(0,b]$ *for some* $b \in (0,1)$. *Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.4. It can be known from Lemma 2.3 that if D is a nonempty bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that $\sup_{x\in D}\|U_{n,k}x - U_k x\| \leq \varepsilon$ for all $n > n_0$.

Remark 2.5. Using Lemma 2.3, we can define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \tag{2.1}$$

for all $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive mapping, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|. \tag{2.2}$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 2.4 that for $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$, $\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D}\|U_{n,1}x - U_1 x\| \leq \varepsilon$. This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0. \tag{2.3}$$

Lemma 2.6 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in* $(0,b]$ *for some* $b \in (0,1)$. *Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

3. Strong Convergence Theorem

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H and let $\{W_n^{(1)}\}$ and $\{W_n^{(2)}\}$ be defined as* (1.2)*. Assume that $\alpha_n \leq a$ for all n and for some $0 < a < 1$, and $\{\overline{\alpha}_n\} \in [b, c]$ for all n and $0 < b < c < 1$. If $F = \bigcap_{n=1}^{\infty}[F(T_n^{(1)}) \bigcap F(T_n^{(2)})] \neq \emptyset$, then $\{x_n\}$ generated by* (1.3) *converges strongly to $P_F(x_q)$.*

Proof. Firstly, we observe that C_n is convex by Lemma 2.2. Next, we show that $F \subset C_n$ for all n.

Indeed, for all $x^* \in F$,

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\left\|z_{n-q} - x^*\right\|^2 \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)\left(\left\|z_{n-q} - x^*\right\|^2 - \|x_n - x^*\|^2\right), \\
\left\|z_{n-q} - x^*\right\|^2 &= \left\|\overline{\alpha}_{n-q}x_{n-q} + (1-\overline{\alpha}_{n-q})W_{n-q}^{(2)}x_{n-q} - x^*\right\| \\
&= \overline{\alpha}_{n-q}\left\|x_{n-q} - x^*\right\|^2 + (1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x^*\right\|^2 \\
&\quad - \overline{\alpha}_{n-q}(1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x_{n-q}\right\|^2 \\
&\leq \overline{\alpha}_{n-q}\left\|x_{n-q} - x^*\right\|^2 + (1-\overline{\alpha}_{n-q})\left\|x_{n-q} - x^*\right\|^2 \\
&\quad - \overline{\alpha}_{n-q}(1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x_{n-q}\right\|^2 \\
&= \left\|x_{n-q} - x^*\right\|^2 - \overline{\alpha}_{n-q}(1-\overline{\alpha}_{n-q})\left\|W_{n-q}^{(2)}x_{n-q} - x_{n-q}\right\|^2 \\
&\leq \left\|x_{n-q} - x^*\right\|^2.
\end{aligned} \tag{3.1}$$

Therefore,

$$\left\|y_n - x^*\right\|^2 \leq \|x_n - x^*\|^2 + (1-\alpha_n)\left(\left\|x_{n-q} - x^*\right\|^2 - \|x_n - x^*\|^2\right). \tag{3.2}$$

That is $x^* \in C_n$ for all $n \geq q$. Next we show that $F \subset Q_n$ for all $n \geq q$.

We prove this by induction. For $n = q$, we have $F \subset C = Q_q$. Assume that $F \subset Q_n$ for all $n \geq q+1$, since x_{n+1} is the projection of x_q onto $C_n \bigcap Q_n$, so

$$\langle x_{n+1} - z, x_q - x_{n+1}\rangle \geq 0, \quad \forall z \in C_n \bigcap Q_n. \tag{3.3}$$

As $F \subset C_n \bigcap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $x^* \in F$. This together with definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence $F \subset C_n \bigcap Q_n$ for all $n \geq q$.

Notice that the definition of Q_n implies $x_n = P_{Q_n}x_q$. This together with the fact $F \subset Q_n$ further implies $\|x_n - x_q\| \leq \|x^* - x_q\|$ for all $x^* \in F$.

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_q \rangle \geq 0$ implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \left\|(x_{n+1} - x_q) - (x_n - x_q)\right\|^2 \\ &= \left\|x_{n+1} - x_q\right\|^2 - \left\|x_n - x_q\right\|^2 - 2\langle x_{n+1} - x_n, x_n - x_q\rangle \\ &\leq \left\|x_{n+1} - x_q\right\|^2 - \left\|x_n - x_q\right\|^2 \longrightarrow 0 \ (n \longrightarrow \infty). \end{aligned} \tag{3.4}$$

We now claim that $\|W^{(1)}x_n - x_n\| \to 0$ and $\|W^{(2)}x_n - x_n\| \to 0$. Indeed,

$$\begin{aligned} \left\|x_n - W_n^{(1)} z_{n-q}\right\| &= \frac{\|x_n - y_n\|}{1 - \alpha_n} \\ &\leq \frac{\|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\|}{1 - \alpha_n}, \end{aligned} \tag{3.5}$$

since $x_{n+1} \in C_n$, we have

$$\left\|y_n - x_{n+1}\right\|^2 \leq \|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\left(\left\|x_{n-q} - x^*\right\|^2 - \|x_n - x^*\|^2\right) \longrightarrow 0. \tag{3.6}$$

Thus

$$\left\|x_n - W_n^{(1)} z_{n-q}\right\| \longrightarrow 0. \tag{3.7}$$

We now show $\lim_{n\to\infty}\|W_n^{(2)}x_n - x_n\| = 0$. Let $\{\|W_{n_k}^{(2)}x_{n_k} - x_{n_k}\|\}$ be any subsequence of $\{\|W_n^{(2)}x_n - x_n\|\}$. Since C is a bounded subset of H, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\left\|x_{n_{k_j}} - x^*\right\| = \limsup_{k\to\infty}\|x_{n_k} - x^*\| := r. \tag{3.8}$$

Since

$$\begin{aligned} \left\|x_{n_{k_j}} - x^*\right\| &\leq \left\|x_{n_{k_j}} - W_{n_{k_j}}^{(1)} z_{n_{k_j}-q}\right\| + \left\|W_{n_{k_j}}^{(1)} z_{n_{k_j}-q} - x^*\right\| \\ &\leq \left\|x_{n_{k_j}} - W_{n_{k_j}}^{(1)} z_{n_{k_j}-q}\right\| + \left\|z_{n_{k_j}-q} - x^*\right\|, \end{aligned} \tag{3.9}$$

it follows that $r = \lim_{j\to\infty}\|x_{n_{k_j}} - x^*\| \leq \liminf_{j\to\infty}\|z_{n_{k_j}} - x^*\|$. By (3.1), we have

$$\left\|z_{n_{k_j}} - x^*\right\| \leq \left\|x_{n_{k_j}} - x^*\right\|^2. \tag{3.10}$$

Hence

$$\limsup_{j\to\infty}\left\|z_{n_{k_j}} - x^*\right\| \leq \lim_{j\to\infty}\left\|x_{n_{k_j}} - x^*\right\| = r. \tag{3.11}$$

Thus,

$$\lim_{j\to\infty}\left\|z_{n_{k_j}} - x^*\right\| = r = \lim_{j\to\infty}\left\|x_{n_{k_j}} - x^*\right\|. \tag{3.12}$$

Using (3.1) again, we obtain that

$$\overline{\alpha}_{n_{k_j}-q}\left(1-\overline{\alpha}_{n_{k_j}-q}\right)\left\|W^{(2)}_{n_{k_j}-q}x_{n_{k_j}-q} - x_{n_{k_j}-q}\right\|^2 \le \left\|x_{n_{k_j}-q} - x^*\right\|^2 - \left\|z_{n_{k_j}-q} - x^*\right\|^2 \longrightarrow 0. \tag{3.13}$$

This imply that $\lim_{j\to\infty}\|W^{(2)}_{n_{k_j}}x_{n_{k_j}} - x_{n_{k_j}}\| = 0$. For the arbitrariness of $\{x_{n_k}\} \subset \{x_n\}$, we have $\lim_{n\to\infty}\|W^{(2)}_n x_n - x_n\| = 0$ and

$$\|z_n - x_n\| = (1-\overline{\alpha}_n)\left\|W^{(2)}_n x_n - x_n\right\| \longrightarrow 0. \tag{3.14}$$

Thus, by (3.4), (3.7) and (3.14), we have

$$\begin{aligned}
\left\|W^{(1)}_n x_n - x_n\right\| &\le \left\|W^{(1)}_n x_n - W^{(1)}_n z_{n-q}\right\| + \left\|W^{(1)}_n z_{n-q} - x_n\right\| \\
&\le \left\|z_{n-q} - x_n\right\| + \left\|W^{(1)}_n z_{n-q} - x_n\right\| \\
&\le \left\|W^{(1)}_n z_{n-q} - x_n\right\| + \left\|z_{n-q} - x_{n-q}\right\| + \left\|x_{n-q} - x_{n-q+1}\right\| \\
&\quad + \left\|x_{n-q+1} - x_{n-q+2}\right\| + \cdots + \|x_{n-1} - x_n\| \\
&\longrightarrow 0.
\end{aligned} \tag{3.15}$$

Since $\lim_{n\to\infty}\|W^{(1)}_n x_n - W^{(1)}x_n\| = 0$ and $\lim_{n\to\infty}\|W^{(2)}_n x_n - W^{(2)}x_n\| = 0$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left\|W^{(1)}x_n - x_n\right\| = 0, \\
&\lim_{n\to\infty}\left\|W^{(2)}x_n - x_n\right\| = 0.
\end{aligned} \tag{3.16}$$

Thus, using (3.16), Lemma 2.1, and the boundedness of $\{x_n\}$, we get that $\emptyset \ne \omega_w(x_n) \subset F$. Since $x_n = P_{Q_n}(x_q)$ and $F \subset Q_n$, we have $\|x_n - x_q\| \le \|x^* - x_q\|$ where $x^* := P_F(x_q)$. By the weak lower semicontinuity of the norm, we have $\|w - x_q\| \le \|x^* - x_q\|$ for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F$, we must have $w = x^*$ for all $w \in \omega_w(x_n)$. Hence $x_n \rightharpoonup x^* = P_F(x_q)$ and

$$\begin{aligned}
\|x_n - x^*\|^2 &= \left\|x_n - x_q\right\|^2 + 2\langle x_n - x_q, x_q - x^*\rangle + \left\|x_q - x^*\right\|^2 \\
&\le 2\left(\left\|x^* - x_q\right\|^2 + \langle x_n - x_q, x_q - x^*\rangle\right) \longrightarrow 0.
\end{aligned} \tag{3.17}$$

That is, $\{x_n\}$ converges to $P_F(x_q)$.

This completes the proof. □

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References

[1] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.
[2] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.
[3] L. Deng, "On Chidume's open questions," *Journal of Mathematical Analysis and Applications*, vol. 174, no. 2, pp. 441–449, 1993.
[4] C. E. Chidume, "An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces," *Journal of Mathematical Analysis and Applications*, vol. 151, no. 2, pp. 453–461, 1990.
[5] B. E. Rhoades, "Comments on two fixed point iteration methods," *Journal of Mathematical Analysis and Applications*, vol. 56, no. 3, pp. 741–750, 1976.
[6] A. Genel and J. Lindenstrauss, "An example concerning fixed points," *Israel Journal of Mathematics*, vol. 22, no. 1, pp. 81–86, 1975.
[7] K. Nakajo and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 372–379, 2003.
[8] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.
[9] L. Deng and Q. Liu, "Iterative scheme for nonself generalized asymptotically quasi-nonexpansive mappings," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 317–324, 2008.
[10] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.
[11] C. Martinez-Yanes and H.-K. Xu, "Strong convergence of the CQ method for fixed point iteration processes," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 11, pp. 2400–2411, 2006.