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Research Article

On Series-Like Iterative Equation with a General Boundary Restriction

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By means of Schauder fixed point theorem and Banach contraction principle, we investigate the existence and uniqueness of Lipschitz solutions of the equation $\mathcal{P}(f) \circ f = F$. Moreover, we get that the solution f depends continuously on F. As a corollary, we investigate the existence and uniqueness of Lipschitz solutions of the series-like iterative equation $\sum_{n=1}^{\infty} a_n f^n(x) = F(x),\ x \in \mathbb{B}$ with a general boundary restriction, where $F : \mathbb{B} \to \mathbb{A}$ is a given Lipschitz function, and $\mathbb{B}, \mathbb{A}$ are compact convex subsets of $\mathbb{R}^N$ with nonempty interior.

1. Introduction

Let f be a self-mapping on a topological space X. For integer $n \geq 0$ define the nth iterate of f by $f^n = f \circ f^{n-1}$ and $f^0 = \mathrm{id}$, where id denotes the identity mapping on X, and $\circ$ denotes the composition of mappings. Let $C(X, X)$ be the set of all continuous self-mappings on X. An equation with iteration as its main operation is simply called an iterative equation. It is one of the most interesting classes of functional equations [1–4] because it concludes the problem of iterative roots [2, 5, 6], that is, finding $f \in C(X, X)$ such that f^n is identical to a given $F \in C(X, X)$ and the problem of invariant curves [7]. Iteration equations also appear in the study on transversal homoclinic intersection for diffeomorphisms [8], normal form of dynamical systems [9], and dynamics of a quadratic mapping [10]. The well-known Feigenbaum equation $f(x) = -(1/\lambda) f(f(\lambda x))$, arising in the discussion of period-doubling bifurcations [11, 12], is also an iterative equation.

As a natural generalization of the problem of iterative roots, a class of iterative equations which is called polynomial-like iterative equation:

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in I = [a, b], \tag{1.1}$$

always fascinates many scholars' attentions [3, 13]. It is more difficult than the analogous differential equation, where each f^j is replaced with the jth derivative $f^{(j)}$ of f because differentiation is a linear operator but iteration is not.

In 1986, Zhang [14] constructed an interesting operator called "structural operator" $L : f \to Lf$ for (1.1) and used the fixed point theory in Banach spaces to get the solutions of (1.1). By means of this method, Zhang and Si made a series of works concerning these qualitative problems such as [15–19]. Recently, Zhang et al. [20, 21] developed this method and made a series of works on (1.1). Furthermore, they have got the nonmonotonic and decreasing solutions of (1.1), and the convexity of solutions is also considered.

In 2002, Kulczycki and Tabor [22] investigated iterative functional equations in the class of Lipschitz functions. In 2004, Tabor and Żołdak [23] studied the iterative equations in Banach spaces. In the above references, the authors first gave theorems for the existence of solutions of

$$\mathcal{D}(f) \circ f = F. \tag{1.2}$$

By virtue of these theorems, in [22], the authors considered the existence of Lipschitz solutions of the iterative functional equation:

$$\sum_{n=1}^{\infty} a_n f^n(x) = F(x), \quad x \in \mathbb{B}, \tag{1.3}$$

where $\mathbb{B}$ is a compact convex subset of $\mathbb{R}^N$ with nonempty interior, and $F : \mathbb{B} \to \mathbb{B}$ is a given Lipschitz function. In [23], the existence of solutions of

$$\sum_i A_i f^i(x) = F(x), \quad \sum_i f(\phi_i(x)) = F(x), \quad x \in \mathbb{B} \tag{1.4}$$

is investigated, where $\mathbb{B}$ is a nonempty closed subset of a Banach Space X. But they all considered the case $F|_{\partial\mathbb{B}} = \mathrm{id}|_{\partial\mathbb{B}}$.

It is easy to see that (1.1) is the special case of (1.3) with $a_i = 0,\ i = n+1,\ n+2,\dots,$ and $\mathbb{B} = [a, b]$. Since the left-hand side of (1.3) is a functional series, in this paper we call it series-like iterative equation. In [14–20], the authors considered the solutions of (1.1) with $F : I \to I,\ F(a) = a,\ F(b) = b$. In [22], the authors considered the solutions of (1.3) with $F : \mathbb{B} \to \mathbb{B},\ F|_{\partial\mathbb{B}} = \mathrm{id}_{\partial\mathbb{B}}$. In fact, the above authors had studied the solutions of

$$\begin{gathered} \sum_{n=1}^{\infty} a_n f^n(x) = F(x), \quad x \in \mathbb{B}, \\ F : \mathbb{B} \longrightarrow \mathbb{B}, \quad F|_{\partial\mathbb{B}} = \mathrm{id}_{\partial\mathbb{B}}, \end{gathered} \tag{1.5}$$

where $\mathbb{B} \subset \mathbb{R}^N$ is a convex compact set with nonempty interior. In [21], the authors considered the solutions of (1.1) with $F : I \to J,\ F(a) = d,\ F(b) = c,\ I = [a,b],\ J = [c,d]$. Obviously, the more general case is

$$\begin{gathered}\sum_{n=1}^{\infty} a_n f^n(x) = F(x), \quad x \in \mathbb{B}, \\ F : \mathbb{B} \longrightarrow \mathbb{A}, \quad F|_{\partial\mathbb{B}} = g,\end{gathered} \tag{1.6}$$

where $\mathbb{B},\ \mathbb{A}$ are convex compact subsets of $\mathbb{R}^N$ with nonempty interior, and $g : \partial\mathbb{B} \to \partial\mathbb{A}$ is a continuous surjective map. Since g could be any map, in this paper we call (1.6) series-like iterative equation with a general boundary restriction. It is easy to see that [14–22] all considered one special case of (1.6).

The problem of differentiable solutions of iterative equations has also fascinated many scholars' attentions. In Zhang [16] and Si [19], the C^1 and C^2 solutions of (1.1) are considered. In Wang and Si [24], the differentiable solutions of

$$H\left(x, \phi^{n_1}(x), \ldots, \phi^{n_i}(x)\right) = F(x), \quad x \in I = [a,b] \tag{1.7}$$

are considered. Murugan and Subrahmanyam [25–27] offered theorems on the existence and uniqueness of differentiable solutions to the iterative equations involving iterated functional series:

$$\begin{gathered}\sum_{i=1}^{\infty} \lambda_i H_i(f^i(x)) = F(x), \quad x \in I = [a,b], \\ \sum_{i=1}^{\infty} \lambda_i H_i\left(x, \phi^{a_{i1}}(x), \ldots, \phi^{a_{in_i}}(x)\right) = F(x), \quad x \in I = [a,b].\end{gathered} \tag{1.8}$$

But the references above only considered the case that $F(a) = a,\ F(b) = b$.

The problem of differentiable solutions of higher dimensional iterative equations is also interesting. By constructing a new operator for the structure of (1.3), which simplifies the procedure of applying fixed point theorems in some sense, Li [28] studies the smoothness of solutions of (1.3). In [29], C^1 solutions of

$$\sum_{n=1}^{\infty} \lambda_n(x) f^n(x) = F(x), \quad x \in \mathbb{B} \tag{1.9}$$

are discussed, where $\mathbb{B}$ is a compact convex subset of $\mathbb{R}^N$ and for any $n \geq 1$, $\lambda_n(x) : \mathbb{B} \to \mathbb{R}$ is continuous. The boundary restrictions are not considered in the two references above because they only consider the case that $F(\mathbb{B}) \subseteq \mathbb{B}$.

It should be pointed out that Mai and Liu [30] made an important contribution to C^m solutions of iterative equations. Using the method of approximating fixed points by small shift of maps, choosing suitable metrics, and finding a relation between uniqueness and stability of fixed points of maps of general spaces, Mai and Liu proved the existence, uniqueness of C^m solutions of a relatively general kind of iterative equations:

$$G\left(x, f(x), \ldots, f^n(x)\right) = 0, \quad x \in J, \tag{1.10}$$

where J is a connected closed subset of $\mathbb{R}$ and $G \in C^m(J^{n+1}, \mathbb{R}),\ n \geq 2$. Here, $C^m(J^{n+1}, \mathbb{R})$ denotes the set of all C^m mappings from J^{n+1} to $\mathbb{R}$.

Inspired and motivated by the above work as well as [14–30], we will study (1.6) and investigate the existence and uniqueness of Lipschitz solution of this equation.

The rest of this paper is organized as follows. In Section 2, we will give some definitions and lemmas. In Section 3, we will give a main theorem concerning the existence and uniqueness of solution of

$$\mathcal{D}(f) \circ f = F. \tag{1.11}$$

In Sections 4 and 5, we will study some special cases of (1.6) by means of the above main theorem.

2. Preliminary

Let $\mathbb{B}$ be a compact convex subset of $\mathbb{R}^N$ with nonempty interior. Let $C(\mathbb{B}, \mathbb{R}^N) = \{f : \mathbb{B} \to R^N \mid f \text{ is continuous}\},\ N \in \mathbb{Z}^+$. In $C(\mathbb{B}, \mathbb{R}^N)$, we use the supremum norm

$$\|f\|_{\mathbb{B}} = \sup_{x \in \mathbb{B}} \|f(x)\|, \quad \text{for } f \in C\left(\mathbb{B}, \mathbb{R}^N\right), \tag{2.1}$$

where $\|\cdot\|$ denotes the usual metric of $\mathbb{R}^N$. Obviously, $C(\mathbb{B}, \mathbb{R}^N)$ is a complete metric space.

Definition 2.1. Let $\mathbb{A},\ \mathbb{B}$ be two convex compact subsets of $\mathbb{R}^N$ with nonempty interior. For $m \in [0,1],\ M \in [1, \infty]$, define

$$\begin{aligned} \operatorname{Lip}(\mathbb{B}, \mathbb{A}, m, M) := \{ & f : \mathbb{B} \longrightarrow \mathbb{A} \mid f \text{ is continuous, } f(\mathbb{B}) = \mathbb{A},\ \forall x, y \in \mathbb{B}, \\ & m\|x - y\| \leq \|f(x) - f(y)\| \leq M\|x - y\|\}. \end{aligned} \tag{2.2}$$

Let $g : \partial\mathbb{B} \to \partial\mathbb{A}$ be a continuous surjective map, and let $C(\mathbb{B}, \mathbb{A}, m, M, g)$ denote the subset of $\operatorname{Lip}(\mathbb{B}, \mathbb{A}, m, M)$ whose elements satisfy $f(\partial\mathbb{B}) = \partial\mathbb{A}$ and $f|_{\partial\mathbb{B}} = g$.

Lemmas 2.2–2.4 can be proved by a corresponding method which is contained in the proofs of Observation 2.2, Lemma 2.3, and Lemma 2.4 of [22].

Lemma 2.2. *Let $m \in (0,1],\ M \in [1, \infty]$, and $f \in \operatorname{Lip}(\mathbb{B}, \mathbb{A}, m, M)$ be arbitrary, then $f^{-1} \in \operatorname{Lip}(\mathbb{A}, \mathbb{B}, 1/M, 1/m)$.*

Lemma 2.3. *For every $m > 0$, the mapping*

$$\mathcal{L} : f \in \mathrm{Lip}(\mathbb{B}, \mathbb{A}, m, \infty) \longrightarrow f^{-1} \in \mathrm{Lip}\left(\mathbb{A}, \mathbb{B}, 0, \frac{1}{m}\right) \tag{2.3}$$

is well defined and Lipschitz with constant $1/m$.

Lemma 2.4. *For $1 \leq K,\ M < \infty$ and $F \in \mathrm{Lip}(\mathbb{B}, \mathbb{A}, 0, K)$, the mapping*

$$\mathcal{S}_F : f \in \mathrm{Lip}(\mathbb{A}, \mathbb{B}, 0, M) \longrightarrow f \circ F \in \mathrm{Lip}(\mathbb{B}, \mathbb{B}, 0, K, M) \tag{2.4}$$

is Lipschitz with constant 1.

Lemmas 2.5 and 2.6 can be proved by a method which is contained in the proof of Proposition 1 in [23].

Lemma 2.5. *If $H,\ G$ are homeomorphisms from $\mathbb{B}$ to $\mathbb{A}$ with Lipschitz constant L, then $\|H - G\|_{\mathbb{B}} \leq L\|H^{-1} - G^{-1}\|_{\mathbb{A}}$.*

Lemma 2.6. *If $f, g \in \mathrm{Lip}(\mathbb{B}, \mathbb{B}, m, M)$, then $\|f^k - g^k\|_{\mathbb{B}} \leq \sum_{j=0}^{k-1} M^j \|f - g\|_{\mathbb{B}}$.*

Lemma 2.7. *For any $M \in [1, \infty)$ and $g : \partial\mathbb{B} \to \partial\mathbb{B}$, which is a surjective map, $C(\mathbb{B}, \mathbb{B}, 0, M, g)$ is a compact subset of $C(\mathbb{B}, \mathbb{R}^N)$.*

Proof. It is easy to see that $C(\mathbb{B}, \mathbb{B}, 0, M, g)$ is uniformly bounded and equicontinuous. By Ascoli-Arzelá lemma for any sequence $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{B}, \mathbb{B}, 0, M, g)$, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ which converges to a continuous map $f \in C(\mathbb{B}, \mathbb{R}^N)$. Without any loss of generality, we suppose $\lim_{n\to\infty} f_n = f$. We can easily get $f|_{\partial\mathbb{B}} = g,\ f(\partial\mathbb{B}) = \partial\mathbb{B}$, and $\|f(x) - f(y)\| \leq M\|x - y\|,\ \forall x, y \in \mathbb{B}$. We only need to prove $f(\mathbb{B}) = \mathbb{B}$. Since $f_n(\mathbb{B}) = \mathbb{B}$, so for any $y \in \mathbb{B}$ there is a $x_n \in \mathbb{B}$ with $f_n(x_n) = y$. By the compactness of $\mathbb{B}$, we suppose $\lim_{n\to\infty} x_n = x$. Noticing that

$$\|y - f(x)\| = \|f_n(x_n) - f(x)\| \leq \|f_n(x_n) - f_n(x)\| + \|f_n(x) - f(x)\|, \tag{2.5}$$

we can get $f(x) = y$. Then, $C(\mathbb{B}, \mathbb{A}, 0, M, g)$ is compact. □

Let $D^N = \{x \mid x \in \mathbb{R}^N, \|x\| \leq 1\}$. Then, $\partial D^N = S^{N-1}$. Obviously, $\mathbb{B}$ is homeomorphic to D^N, and $\partial\mathbb{B}$ is homeomorphic to S^{N-1}.

Lemma 2.8. *If $f : D^N \to D^N$ is continuous and $f(S^{N-1}) \subset S^{N-1}$. Let f_0 denote $f|_{S^{N-1}}$. If $\deg(f_0) \neq 0$, then f is surjective, where $\deg(f_0)$ denotes the degree of f_0.*

Proof. Suppose that f is not surjective. Let $x_0 \in D^N \setminus f(D^N)$. If $x_0 \in S^{N-1}$, then f_0 is homotopic to a constant and $\deg(f_0) = 0$, a contradiction. So $x_0 \notin S^{N-1}$, then there exists a retraction mapping $r : D^N \setminus \{x_0\} \to S^{N-1}$. Thus, $r \circ f : D^N \to S^{N-1}$ is a continuous mapping, and $f_0 = (r \circ f)|_{S^{N-1}}$. This means that $(f_0)_{*(N-1)}$ is trivial, then $\deg(f_0) = 0$. So f is surjective. □

Lemma 2.9. *Let $M \in [1,\infty)$ and $C(\mathbb{B},\mathbb{B},0,M,g)$ be defined as above, where the surjective map $g : \partial\mathbb{B} \to \partial\mathbb{B}$ is the restriction of the elements of $C(\mathbb{B},\mathbb{B},0,M,g)$. If $\deg(g) \neq 0$, then $C(\mathbb{B},\mathbb{B},0,M,g)$ is a convex subset of $C(\mathbb{B},\mathbb{R}^N)$.*

Proof. For $\forall t \in [0,1]$ and $\forall f,h \in C(\mathbb{B},\mathbb{B},0,M,g)$, $tf+(1-t)h$ is continuous and

$$[tf+(1-t)h]|_{\partial\mathbb{B}} = tg+(1-t)g = g. \tag{2.6}$$

It is easy to see that

$$\|(tf+(1-t)h)(x)-(tf+(1-t)h)(y)\| \le M\|x-y\|, \quad \forall x,y \in \mathbb{B}. \tag{2.7}$$

By lemma 2.8, $tf+(1-t)h$ is surjective. Thus, $tf+(1-t)h \in C(\mathbb{B},\mathbb{B},0,M,g)$, that is, $C(\mathbb{B},\mathbb{B},0,M,g)$ is convex. □

3. Main Result

Theorem 3.1. *Give $M,K \in [1,\infty)$ and $\mathbb{A}$, $\mathbb{B}$ which are compact convex subsets of $\mathbb{R}^N$ with nonempty interior. Suppose that both $g : \partial\mathbb{B} \to \partial\mathbb{B}$ and $T : \partial\mathbb{B} \to \partial\mathbb{A}$ are continuous surjective maps and $\deg(g) \neq 0$. If there exist a decreasing function $\alpha : [1,\infty] \to [0,1]$ and a continuous map $\mathcal{D}$ defined on $C(\mathbb{B},\mathbb{B},0,M,g)$ such that*

$$\begin{gathered}\mathcal{D}(f) \in C(\mathbb{B},\mathbb{A},\alpha(M),\infty,T), \quad \forall f \in C(\mathbb{B},\mathbb{B},0,M,g)\\ M\cdot\alpha(M) \ge K.\end{gathered} \tag{3.1}$$

Then, for any $F \in C(\mathbb{B},\mathbb{A},0,K,T\circ g)$, there exists a $f \in C(\mathbb{B},\mathbb{B},0,M,g)$ such that

$$\mathcal{D}(f)\circ f = F. \tag{3.2}$$

Furthermore, if $\mathcal{D}$ is Lipschitz with a Lipschitz constant d which satisfies $d/\alpha(M) < 1$, then f is unique, and f depends continuously on F.

Proof. Firstly, we prove that $\mathcal{D}(f) : \mathbb{B} \to \mathbb{A}$ is a homeomorphism for all $f \in C(\mathbb{B},\mathbb{B},0,M,g)$. Since the interior of $\mathbb{A}$ is nonempty, $\alpha(M) > 0$ (otherwise we would get that F is a constant, while we suppose $F(\mathbb{B}) = \mathbb{A}$). Then, by Lemma 2.2, $\mathcal{D}(f)$ is a homeomorphism. We also get $M \ge K/\alpha(M)$.

By Lemmas 2.3 and 2.4, the mapping

$$\begin{gathered}\mathcal{L} : f \in \operatorname{Lip}(\mathbb{B},\mathbb{A},\alpha(M),\infty) \longrightarrow f^{-1} \in \operatorname{Lip}\left(\mathbb{A},\mathbb{B},0,\frac{1}{\alpha(M)}\right),\\ \mathcal{S}_F : f \in \operatorname{Lip}\left(\mathbb{A},\mathbb{B},0,\frac{1}{\alpha(M)}\right) \longrightarrow f\circ F \in \operatorname{Lip}\left(\mathbb{B},\mathbb{B},0,\frac{K}{\alpha(M)}\right)\end{gathered} \tag{3.3}$$

are both well defined and continuous.

From the above discussions, for $\forall f \in C(\mathbb{B}, \mathbb{B}, 0, M, g)$, we can get that

$$\begin{aligned}
&\mathcal{P}(f) \in C(\mathbb{B}, \mathbb{A}, \alpha(M), \infty, T), \\
&\mathcal{P}(f)^{-1} = \mathcal{L} \circ \mathcal{P}(f) \in C\left(\mathbb{A}, \mathbb{B}, 0, \frac{1}{\alpha(M)}, T^{-1}\right), \\
&S_F \circ \mathcal{L} \circ \mathcal{P}(f) \in C\left(\mathbb{B}, \mathbb{B}, 0, \frac{K}{\alpha(M)}, g\right) \subset C(\mathbb{B}, \mathbb{B}, 0, M, g).
\end{aligned} \tag{3.4}$$

These mean that $S_F \circ \mathcal{L} \circ \mathcal{P} : C(\mathbb{B}, \mathbb{B}, 0, M, g) \to C(\mathbb{B}, \mathbb{B}, 0, M, g)$ is well defined and continuous.

By Lemmas 2.7, 2.9 and Schauder's fixed points theorem, $S_F \circ \mathcal{L} \circ \mathcal{P}$ has a fixed point f in $C(\mathbb{B}, \mathbb{B}, 0, M, g)$. Then,

$$S_F \circ \mathcal{L} \circ \mathcal{P}(f)(x) = f(x), \quad \forall x \in \mathbb{B}, \tag{3.5}$$

which implies

$$\mathcal{P}(f)^{-1} \circ F = f. \tag{3.6}$$

This means that f satisfies the assertion of the theorem.

For $f_1, f_2 \in C(\mathbb{B}, \mathbb{B}, 0, M, g)$, by Lemma 2.5,

$$\begin{aligned}
&\left\| S_F \circ \mathcal{L} \circ \mathcal{P}(f_1) - S_F \circ \mathcal{L} \circ \mathcal{P}(f_2) \right\|_{\mathbb{B}} \\
&\quad = \left\| \mathcal{P}(f_1)^{-1} \circ F - \mathcal{P}(f_2)^{-1} \circ F \right\|_{\mathbb{B}} \\
&\quad \le \left\| \mathcal{P}(f_1)^{-1} - \mathcal{P}(f_2)^{-1} \right\|_{\mathbb{A}} \\
&\quad \le \left(\frac{1}{\alpha(M)} \right) \left\| \mathcal{P}(f_1) - \mathcal{P}(f_2) \right\|_{\mathbb{B}} \\
&\quad \le \left(\frac{d}{\alpha(M)} \right) \left\| f_1 - f_2 \right\|_{\mathbb{B}}.
\end{aligned} \tag{3.7}$$

Since $d/\alpha(M) < 1$, the mapping $S_F \circ \mathcal{L} \circ \mathcal{P}$ is a contraction on $C(\mathbb{B}, \mathbb{B}, 0, M, g)$. By Banach contraction principle, f is unique.

Suppose $F_1, F_2 \in C(\mathbb{B}, \mathbb{A}, 0, K, T \circ g)$ and $f_1, f_2 \in C(\mathbb{B}, \mathbb{B}, 0, M, g)$ such that $\mathcal{P}(f_1) \circ f_1 = F_1$ and $\mathcal{P}(f_2) \circ f_2 = F_2$, then,

$$\begin{aligned}
\left\| f_1 - f_2 \right\|_{\mathbb{B}} &= \left\| S_{F_1} \circ \mathcal{L} \circ \mathcal{P}(f_1) - S_{F_2} \circ \mathcal{L} \circ \mathcal{P}(f_2) \right\|_{\mathbb{B}} \\
&= \left\| \mathcal{P}(f_1)^{-1} \circ F_1 - \mathcal{P}(f_2)^{-1} \circ F_2 \right\|_{\mathbb{B}} \\
&\le \left\| \mathcal{P}(f_1)^{-1} \circ F_1 - \mathcal{P}(f_2)^{-1} \circ F_1 \right\|_{\mathbb{B}} + \left\| \mathcal{P}(f_2)^{-1} \circ F_1 - \mathcal{P}(f_2)^{-1} \circ F_2 \right\|_{\mathbb{B}} \\
&\le \left(\frac{d}{\alpha(M)} \right) \left\| f_1 - f_2 \right\|_{\mathbb{B}} + \left(\frac{1}{\alpha(M)} \right) \left\| F_1 - F_2 \right\|_{\mathbb{B}}.
\end{aligned} \tag{3.8}$$

This means that

$$\|f_1 - f_2\|_{\mathbb{B}} \le \left(\frac{1}{\alpha(M)}\right)\left(1 - \frac{d}{\alpha(M)}\right)^{-1} \|F_1 - F_2\|_{\mathbb{B}}, \tag{3.9}$$

then f depends continuously on F. □

Theorem 3.2. *Let the sequence* $\{a_k\}_{k=0}^{\infty} \subset R$ *satisfy that* $\sum_{k=0}^{\infty} a_k$ *is absolutely convergent, then for all* $M \in [1, \infty)$ *the mapping*

$$\wp : f \in \operatorname{Lip}(\mathbb{B}, \mathbb{B}, 0, M) \longrightarrow \sum_{k=0}^{\infty} a_k f^k \in C(\mathbb{B}, \mathbb{R}^N) \tag{3.10}$$

is well defined and continuous.

Proof. Since $\sum_{k=0}^{\infty} a_k$ is absolutely convergent and $\{f^k\}_{k=0}^{\infty}$ is a uniformly bounded sequence of continuous maps on the compact space $\mathbb{B}$ to itself, by Weierstrass M-test, $\wp(f)$ is well defined and continuous.

By Lemma 2.6 and the absolute convergence of $\sum_{k=0}^{\infty} a_k$, the continuity of the mapping $\wp$ can be easily got. □

4. Iterative Equation in $\mathbb{R}$

Let $I = [a, b]$ and $J = [c, d]$ be two compact intervals. Let $h_1 = \mathrm{id}|_{\partial I}$ and $h_2 = r_1$ be the antipodal maps on ∂I. Let $g_1, g_2 : \partial I \to \partial J$ satisfy $g_1(a) = c,\ g_1(b) = d$ and $g_2(a) = d,\ g_2(b) = c$. Obviously, $g_1 = g_1 \circ h_1$ and $g_2 = g_1 \circ h_2$. Obviously, $\deg(h_1) = 1$ and $\deg(h_2) = -1$.

Theorem 4.1. *Suppose that the sequence* $\{a_k\}_{k=1}^{\infty} \subset R$ *satisfy* $a_1 > 0$ *and* $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent and* $M, K \ge 1$, *if*

$$1 \ge a_1 - \sum_{n=2}^{\infty} |a_n| M^{n-1} \ge \frac{K}{M}, \tag{4.1}$$

$$\sum_{i=1}^{\infty} a_i h_k^{i-1}(a) < \sum_{i=1}^{\infty} a_i h_k^{i-1}(b), \quad k = 1, 2. \tag{4.2}$$

Then, for any $F \in C(I, J, 0, K, g_k)$, (1.6) *has a solution in* $C(I, I, 0, K, h_k)$, *where* $I = [a, b]$ *and* $J = [\sum_{i=1}^{\infty} a_i h_k^{i-1}(a), \sum_{i=1}^{\infty} a_i h_k^{i-1}(b)],\ k = 1, 2$. *Moreover, if*

$$\frac{\left(\sum_{k=2}^{\infty} |a_k| \sum_{j=0}^{k-2} M^j\right)}{\alpha(M)} < 1, \tag{4.3}$$

f *is unique and depends continuously on* F.

Proof. For $t \in [1,\infty]$ define $\alpha(t)$ by $\alpha(t) = \min\{\max[a_1 - \sum_{i=2}^{\infty}|a_i|t^{i-1}, 0], 1\}$. Since $0 \leq \alpha(t) \leq 1$, we obtain that $\alpha : [1,\infty] \to [0,1]$. From (4.1), we have $M \cdot \alpha(M) \geq K$. By Theorem 3.2, we can define $\mathcal{D} : C(I, I, 0, M, h_k) \to C(I, \mathbb{R}),\ k = 1, 2$ by

$$\mathcal{D}(f)(x) = \sum_{i=1}^{\infty} a_i f^{i-1}(x), \quad \forall x \in I, \tag{4.4}$$

where $f \in C(I, I, 0, M, h_k)$, $k = 1, 2$. It is easy to see that $\mathcal{D}(f)(a) = \sum_{i=1}^{\infty} a_i h_k^{i-1}(a)$ and $\mathcal{D}(f)(b) = \sum_{i=1}^{\infty} a_i h_k^{i-1}(b)$. For $x, y \in I$ with $y > x$, one can check that

$$0 < \alpha(M)(y-x) \leq \mathcal{D}(f)(y) - \mathcal{D}(f)(x) \leq 2a_1(y-x). \tag{4.5}$$

This means that $\mathcal{D}(f)(I) = J$, and $\mathcal{D}(f)$ is an orientation preserving homeomorphism of I. The above discussions imply that

$$\mathcal{D}(f) \in C(I, J, \alpha(M), \infty, g_1). \tag{4.6}$$

For any $f, g \in C(I, I, 0, M, h_k)$ by Lemma 2.6, we get that

$$\begin{aligned}
\|\mathcal{D}(f) - \mathcal{D}(g)\|_I &= \sup_{x\in I}\left|\sum_{i=1}^{\infty} a_i f^{i-1}(x) - \sum_{i=1}^{\infty} a_i g^{i-1}(x)\right| \\
&\leq \sup_{x\in I}\sum_{i=1}^{\infty}|a_i| \cdot \left|f^{i-1}(x) - g^{i-1}(x)\right| \\
&\leq \sum_{i=2}^{\infty}|a_i| \cdot \|f^{i-1} - g^{i-1}\|_I \\
&\leq \sum_{i=2}^{\infty}|a_i| \cdot \sum_{j=0}^{i-2} M^j \|f - g\|_I.
\end{aligned} \tag{4.7}$$

By Theorem 3.1, the assertion is true. □

Example 4.2.

$$\begin{gathered}
\frac{54}{55} f(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-2}}{54^{i-1}} f^i(x) = x^2, \quad x \in I = [0,1], \\
F_1(x) = x^2 : I \longrightarrow I, \quad F_1\big|_{\partial I} = g_1.
\end{gathered} \tag{4.8}$$

Obviously, $F_1(x) = x^2 \in C(I, I, 0, 2, g_1)$. Let $M = 4$ since

$$\alpha(M) = a_1 - \sum_{n=2}^{\infty} \left|a_n\right| M^{n-1} = \frac{54}{55} - \sum_{i=2}^{\infty} \frac{4^{i-1}}{54^{i-1}} = \frac{248}{275} > \frac{2}{4},$$
$$\frac{\sum_{k=2}^{\infty} \left|a_k\right| \sum_{j=0}^{k-2} M^j}{\alpha(M)} = \frac{\sum_{k=2}^{\infty} \left(1/54^{k-1}\right) \sum_{j=0}^{k-2} 4^j}{\alpha(M)} \le \frac{\sum_{k=2}^{\infty} \left(\left(2^{k-1} \cdot 4^{k-2}\right)/54^{k-1}\right)}{\alpha(M)} = \frac{275}{23 \times 248}. \tag{4.9}$$

Then, by Theorem 4.1, the equation has an unique strictly increasing solution in $C(I, I, 0, 4, h_1)$. For

$$F_2(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{4}\right], \\ \frac{1}{2}, & x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 2x - 1, & x \in \left[\frac{3}{4}, 1\right], \end{cases} \tag{4.10}$$

it is easy to see that $F_2 \in C(I, I, 0, 2, g_1)$. Then,

$$\begin{gathered} \frac{54}{55} f(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-2}}{54^{i-1}} f^i(x) = F_2(x), \quad x \in I = [0, 1], \\ F_2 : I \longrightarrow I, \quad F_2\big|_{\partial I} = g_1 \end{gathered} \tag{4.11}$$

has an unique increasing solution in $C(I, I, 0, 4, h_1)$.

For a nonmonotonic example, we consider $F_3(x) = x + (1/2)\sin(2\pi x) \in C(I, I, 0, 5, g_1)$. As mentioned in [20], F_3 has a local maximum at a point x_1 and a local minimum at a point x_2 in $(0, 1)$. The equation

$$\begin{gathered} \frac{210}{211} f(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-2}}{210^{i-1}} f^i(x) = F_3(x), \quad x \in I = [0, 1], \\ F_3 : I \longrightarrow I, \quad F_3\big|_{\partial I} = g_1 \end{gathered} \tag{4.12}$$

has an unique nonmonotonic solution in $C(I, I, 0, 10, h_1)$.

Example 4.3. For convenience, we only consider $\{a_k\}_{k=1}^{\infty}$ with $a_{2i} = 0,\ i = 1, 2, \ldots$. Obviously, for $I = [0, 1],\ F_1(x) = 1 - x^2 \in C(I, I, 0, 2, g_2)$. By Theorem 4.1,

$$\begin{gathered} \frac{254}{255} f(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-2}}{256^{i-1}} f^{2i-1}(x) = F_1(x), \quad x \in [0, 1], \\ F_1 : I \longrightarrow I, \quad F_1\big|_{\partial I} = g_2 \end{gathered} \tag{4.13}$$

has an unique strictly decreasing solution in $C(I, I, 0, 4, h_2)$. For

$$F_2(x) = \begin{cases} 1-2x, & x \in \left[0, \dfrac{1}{4}\right], \\ \dfrac{1}{2}, & x \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right], \\ 2-2x, & x \in \left[\dfrac{3}{4}, 1\right], \end{cases} \tag{4.14}$$

it is easy to see that $F_2 \in C(I, I, 0, 2, g_2)$. Then,

$$\begin{gathered} \frac{254}{255} f(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-2}}{256^{i-1}} f^{2i-1}(x) = F_2(x), \quad x \in [0,1], \\ F_2 : I \longrightarrow I, \quad F_2|_{\partial I} = g_2 \end{gathered} \tag{4.15}$$

has an unique decreasing solution in $C(I, I, 0, 4, h_2)$.

For a nonmonotonic example, we consider $F_3(x) = 1 - x - (1/2)\sin(2\pi x) \in C(I, I, 0, 5, g_2)$. F_3 has a local maximum at a point x_1 and a local minimum at a point x_2 in $(0,1)$. The equation

$$\begin{gathered} \frac{1000}{1001} f(x) + \sum_{i=2}^{\infty} \frac{(-1)^{i-2}}{1000^{i-1}} f^{2i-1}(x) = F_3(x), \quad x \in [0,1], \\ F_3 : I \longrightarrow I, \quad F_3|_{\partial I} = g_2 \end{gathered} \tag{4.16}$$

has an unique nonmonotonic solution in $C(I, I, 0, 10, h_2)$.

5. Iterative Equation in $\mathbb{R}^N$ $(N \geq 2)$

In [22], Kulczycki and Tabor got the existence of solutions of the iterative (1.6) on compact convex subsets of $\mathbb{R}^N$, but they only discussed the case $F|_{\partial\mathbb{B}} = \mathrm{id}_{\partial\mathbb{B}}$. In this section, we will continue the work of [22] and discuss the solutions for a special case of (1.6) on unit closed ball of $\mathbb{R}^N$.

Let $\xi : S^{N-1} \to S^{N-1}$ be a homeomorphism which satisfies $\xi \circ \xi = \mathrm{id}_{S^{N-1}}$. Obviously, $\deg(\xi) = \pm 1$.

Theorem 5.1. *Let $\{a_i\}_{i=1}^{\infty} \subset [0,1]$ with $\sum_{i=1}^{\infty} a_i = 1$, $a_1 > 0$ and there exist two constants $M, K \geq 1$ with*

$$a_1 - \sum_{i=2}^{\infty} a_i M^{2i-2} \geq \frac{K}{M}. \tag{5.1}$$

Then, for any $F \in C(D^N, D^N, 0, K, \xi)$,

$$\sum_{i=1}^{\infty} a_i f^{2i-1}(x) = F(x), \quad x \in D^N,$$
$$F : D^N \longrightarrow D^N, \quad F\big|_{S^{N-1}} = \xi \tag{5.2}$$

has a solution $f \in C(D^N, D^N, 0, M, \xi)$. *Moreover, if* $(\sum_{k=2}^{\infty} a_k \sum_{j=0}^{2k-3} M^j)/\alpha(M) < 1$, *then* f *is unique and depends continuously on* F.

Proof. For $t \in [1, \infty]$, define $\alpha(t) = \max\{a_0 - \sum_{i=1}^{\infty} a_i t^{2i-2}, 0\}$, then $0 \le \alpha(t) \le a_0 \le 1$. Define $\mathcal{P} : C(D^N, D^N, 0, M, \xi) \to C(D^N, \mathbb{R}^N)$ by

$$\mathcal{P}(f)(x) = \sum_{i=1}^{\infty} a_i f^{2i-2}(x). \tag{5.3}$$

Then, we get

$$\begin{aligned}
\|\mathcal{P}(f)(x) - \mathcal{P}(f)(y)\| &= \left\| \sum_{i=1}^{\infty} a_i f^{2i-2}(x) - \sum_{i=1}^{\infty} a_i f^{2i-2}(y) \right\| \\
&\ge a_1 \|x - y\| - \sum_{i=2}^{\infty} a_i \left\| f^{2i-2}(x) - f^{2i-2}(y) \right\| \\
&\ge \left(a_1 - \sum_{i=2}^{\infty} a_i M^{2i-2} \right) \|x - y\| \\
&= \alpha(M) \|x - y\|, \\
\|\mathcal{P}(f)(x) - \mathcal{P}(f)(y)\| &\le \left(a_1 + \sum_{i=2}^{\infty} a_i M^{2i-2} \right) \|x - y\|.
\end{aligned} \tag{5.4}$$

For all $x \in D^N$, we have

$$\sum_{i=1}^{\infty} a_i f^{2i-2}(x) \in \overline{\operatorname{conv}}\{x, f^2(x), f^4(x), \dots\} \subset D^N. \tag{5.5}$$

Since $\mathcal{P}(f)|_{S^{N-1}} = \sum_{i=1}^{\infty} a_i \xi^{2i-2}|_{S^{N-1}} = \mathrm{id}|_{S^{N-1}}$ and $\deg(\mathrm{id}|_{S^{N-1}}) = 1$, then $\mathcal{P}(f)(D^N) = D^N$ and $\mathcal{P}(f)$ is a homeomorphism from D^N to D^N. By the above discussion, we get that $\mathcal{P}(f) \in C(D^N, D^N, \alpha(M), \infty, \mathrm{id}|_{S^{N-1}})$. But $[\mathcal{P}(f) \circ f]|_{S^{N-1}} = \sum_{i=1}^{\infty} a_i \xi^{2i-1} = \xi$. For $f, g \in C(D^N, D^N, 0, M, \xi)$, by Lemma 2.6, we get that

$$\|\mathcal{P}(f) - \mathcal{P}(g)\|_{D^N} \le \sum_{k=2}^{\infty} a_k \sum_{j=0}^{2k-3} M^j \|f - g\|_{D^N}. \tag{5.6}$$

Obviously, the maps $\mathrm{id}|_{S^{N-1}}$ and ξ are the concrete forms of the maps T and g in Theorem 3.1. By Theorem 3.1, the assertion is true. □

Example 5.2. For $F(x) = -(x_1, x_2) + \operatorname{dist}(x, S^1)(1,0) = (1 - x_1 - \sqrt{x_1^2 + x_2^2}, -x_2)$, where $x = (x_1, x_2) \in D^2$, and $\operatorname{dist}(x, S^1)$ denotes the distance of the point x from S^1. Obviously, $F|_{S^1} = r_1$, where r_1 denotes the antipodal map on S^1. By simple calculation, we get that for any $x, y \in D^2$,

$$\begin{gathered} \|F(x) - F(y)\| \leq 2\|x - y\|, \\ \|F(x)\| \leq \|x\| + \operatorname{dist}\left(x, S^1\right) = 1 \end{gathered} \tag{5.7}$$

hold. By the above discussion, we get that $F \in C(D^2, D^2, 0, 2, r_1)$. Then, by Theorem 5.1,

$$\begin{gathered} \frac{254}{255} f(x) + \sum_{i=2}^{\infty} \frac{1}{256^{i-1}} f^{2i-1}(x) = F(x), \quad x \in D^2, \\ F : D^2 \longrightarrow D^2, \quad F|_{S^1} = r_1 \end{gathered} \tag{5.8}$$

has a unique solution in $C(D^2, D^2, 0, 4, r_1)$.

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