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Research Article

Convergence Comparison of Several Iteration Algorithms for the Common Fixed Point Problems

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We discuss the following viscosity approximations with the weak contraction A for a non-expansive mapping sequence $\{T_n\}$, $y_n = \alpha_n A y_n + (1-\alpha_n)T_n y_n$, $x_{n+1} = \alpha_n A x_n + (1-\alpha_n)T_n x_n$. We prove that Browder's and Halpern's type convergence theorems imply Moudafi's viscosity approximations with the weak contraction, and give the estimate of convergence rate between Halpern's type iteration and Mouda's viscosity approximations with the weak contraction.

1. Introduction

The following famous theorem is referred to as the Banach Contraction Principle.

Theorem 1.1 (Banach [1]). *Let (E,d) be a complete metric space and let A be a contraction on X, that is, there exists $\beta \in (0,1)$ such that*

$$d(Ax, Ay) \le \beta d(x,y), \quad \forall x, y \in E. \tag{1.1}$$

Then A has a unique fixed point.

In 2001, Rhoades [2] proved the following very interesting fixed point theorem which is one of generalizations of Theorem 1.1 because the weakly contractions contains contractions as the special cases ($\varphi(t) = (1-\beta)t$).

Theorem 1.2 (Rhoades[2], Theorem 2). *Let (E,d) be a complete metric space, and let A be a weak contraction on E, that is,*

$$d(Ax, Ay) \le d(x,y) - \varphi(d(x,y)), \quad \forall x, y \in E, \tag{1.2}$$

for some $\varphi : [0,+\infty) \to [0,+\infty)$ *is a continuous and nondecreasing function such that* φ *is positive on* $(0,+\infty)$ *and* $\varphi(0) = 0$*. Then* A *has a unique fixed point.*

The concept of the weak contraction is defined by Alber and Guerre-Delabriere [3] in 1997. The natural generalization of the contraction as well as the weak contraction is nonexpansive. Let K be a nonempty subset of Banach space E, $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \qquad \forall x, y \in K. \tag{1.3}$$

One classical way to study nonexpansive mappings is to use a contraction to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a contraction $T_t : K \to K$ by $T_t x = tu + (1-t)Tx, x \in K$, where $u \in K$ is a fixed point. Banach Contraction Principle guarantees that T_t has a unique fixed point x_t in K, that is,

$$x_t = tu + (1-t)Tx_t. \tag{1.4}$$

Halpern [4] also firstly introduced the following explicit iteration scheme in Hilbert spaces: for $u, x_0 \in K$, $\alpha_n \in [0,1]$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \ge 0. \tag{1.5}$$

In the case of T having a fixed point, Browder [5] (resp. Halpern [4]) proved that if E is a Hilbert space, then $\{x_t\}$ (resp. $\{x_n\}$) converges strongly to the fixed point of T, that is, nearest to u. Reich [6] extended Halpern's and Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then $\{x_t\}$ and $\{x_n\}$ converge strongly to a same fixed point of T, respectively, and the limit of $\{x_t\}$ defines the (unique) sunny nonexpansive retraction from K onto $\operatorname{Fix}(T)$. In 1984, Takahashi and Ueda [7] obtained the same conclusion as Reich's in uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Recently, Xu [8] showed that the above result holds in a reflexive Banach space which has a weakly continuous duality mapping J_φ. In 1992, Wittmann [9] studied the iterative scheme (1.5) in Hilbert space, and obtained convergence of the iterations. In particular, he proved a strong convergence result [9, Theorem 2] under the control conditions

$$(C1) \lim_{n\to\infty} \alpha_n = 0, \quad (C2) \sum_{n=1}^{\infty} \alpha_n = \infty, \quad (C3) \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty. \tag{1.6}$$

In 2002, Xu [10, 11] extended wittmann's result to a uniformly smooth Banach space, and gained the strong convergence of $\{x_n\}$ under the control conditions $(C1)$, $(C2)$, and

$$(C4) \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1. \tag{1.7}$$

Actually, Xu [10, 11] and Wittmann [9] proved the following approximate fixed points theorem. Also see [12, 13].

Theorem 1.3. *Let K be a nonempty closed convex subset of a Banach space E. provided that $T : K \to K$ is nonexpansive with* $\mathrm{Fix}(T) \neq \emptyset$*, and $\{x_n\}$ is given by* (1.5) *and $\alpha_n \in (0,1)$ satisfies the condition* (C1), (C2), *and* (C3) *(or* (C4)*). Then $\{x_n\}$ is bounded and* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

In 2000, for a nonexpansive selfmapping T with $\mathrm{Fix}(T) \neq \emptyset$ and a fixed contractive selfmapping f, Moudafi [14] introduced the following viscosity approximation method for T:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \tag{1.8}$$

and proved that $\{x_n\}$ converges to a fixed point p of T in a Hilbert space. They are very important because they are applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations. Xu [15] extended Moudafi's results to a uniformly smooth Banach space. Recently, Song and Chen [12, 13, 16–18] obtained a number of strong convergence results about viscosity approximations (1.8). Very recently, Petrusel and Yao [19], Wong, et al. [20] also studied the convergence of viscosity approximations, respectively.

In this paper, we naturally introduce viscosity approximations (1.9) and (1.10) with the weak contraction A for a nonexpansive mapping sequence $\{T_n\}$,

$$y_n = \alpha_n A y_n + (1-\alpha_n)T_n y_n, \tag{1.9}$$

$$x_{n+1} = \alpha_n A x_n + (1-\alpha_n)T_n x_n. \tag{1.10}$$

We will prove that Browder's and Halpern's type convergence theorems imply Moudafi's viscosity approximations with the weak contraction, and give the estimate of convergence rate between Halpern's type iteration and Moudafi's viscosity approximations with the weak contraction.

2. Preliminaries and Basic Results

Throughout this paper, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^*\rangle$ and the *normalized duality mapping* J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall x \in E. \tag{2.1}$$

Let $\mathrm{Fix}(T)$ denote the set of all fixed points for a mapping T, that is, $\mathrm{Fix}(T) = \{x \in E : Tx = x\}$, and let $\mathbb{N}$ denote the set of all positive integers. We write $x_n \rightharpoonup x$ (resp. $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence x_n weakly (resp. weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence.

In the proof of our main results, we need the following definitions and results. Let $S(E) := \{x \in E; \|x\| = 1\}$ denote the unit sphere of a Banach space E. E is said to have (i) *a Gâteaux differentiable norm* (we also say that E is *smooth*), if the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}, \tag{2.2}$$

exists for each $x, y \in S(E)$; (ii) *a uniformly Gâteaux differentiable norm*, if for each y in $S(E)$, the limit (2.2) is uniformly attained for $x \in S(E)$; (iii) *a Fréchet differentiable norm*, if for each $x \in S(E)$, the limit (2.2) is attained uniformly for $y \in S(E)$; (iv) *a uniformly Fréchet differentiable norm* (we also say that E is *uniformly smooth*), if the limit (2.2) is attained uniformly for $(x, y) \in S(E) \times S(E)$. A Banach space E is said to be (v) *strictly convex* if $\|x\| = \|y\| = 1, x \neq y$ implies $\|(x+y)/2\| < 1$; (vi) *uniformly convex* if for all $\varepsilon \in [0,2]$, $\exists \delta_\varepsilon > 0$ such that $\|x\| = \|y\| = 1$ with $\|x-y\| \geq \varepsilon$ implies $\|x+y\|/2 < 1 - \delta_\varepsilon$. For more details on geometry of Banach spaces, see [21, 22].

If C is a nonempty convex subset of a Banach space E, and D is a nonempty subset of C, then a mapping $P : C \to D$ is called a *retraction* if P is continuous with $\mathrm{Fix}(P) = D$. A mapping $P : C \to D$ is called *sunny* if $P(Px + t(x - Px)) = Px$, for all $x \in C$ whenever $Px + t(x - Px) \in C$, and $t > 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. We note that if K is closed and convex of a Hilbert space E, then the metric projection coincides with the sunny nonexpansive retraction from C onto D. The following lemma is well known which is given in [22, 23].

Lemma 2.1 (see [22, Lemma 5.1.6]). *Let C be nonempty convex subset of a smooth Banach space E, $\emptyset \neq D \subset C$, $J : E \to E^*$ the normalized duality mapping of E, and $P : C \to D$ a retraction. Then P is both sunny and nonexpansive if and only if there holds the inequality:*

$$\langle x - Px, J(y - Px) \rangle \leq 0, \ \forall x \in C, \ y \in D. \tag{2.3}$$

Hence, there is at most one sunny nonexpansive retraction from C onto D.

In order to showing our main outcomes, we also need the following results. For completeness, we give a proof.

Proposition 2.2. *Let K be a convex subset of a smooth Banach space E. Let C be a subset of K and let P be the unique sunny nonexpansive retraction from K onto C. Suppose A is a weak contraction with a function φ on K, and T is a nonexpansive mapping. Then*

(i) the composite mapping TA is a weak contraction on K;

(ii) For each $t \in (0,1)$, a mapping $T_t = (1-t)T + tA$ is a weak contraction on K. Moreover, $\{x_t\}$ defined by (2.4) is well definition:

$$x_t = tAx_t + (1-t)Tx_t; \tag{2.4}$$

(iii) $z = P(Az)$ if and only if z is a unique solution of the following variational inequality:

$$\langle Az - z, J(y - z) \rangle \leq 0, \quad \forall y \in C. \tag{2.5}$$

Proof. For any $x, y \in K$, we have

$$\|T(Ax) - T(Ay)\| \leq \|Ax - Ay\| \leq \|x - y\| - \varphi(\|x - y\|). \tag{2.6}$$

So, TA is a weakly contractive mapping with a function φ. For each fixed $t \in (0,1)$, and $\psi(s) = t\varphi(s)$, we have

$$\begin{aligned}\|T_t x - T_t y\| &= \|(tAx + (1-t)Tx) - (tAy + (1-t)Ty)\| \\ &\leq (1-t)\|Tx - Ty\| + t\|Ax - Ay\| \\ &\leq (1-t)\|x-y\| + t\|x-y\| - t\varphi(\|x-y\|) \\ &= \|x-y\| - \psi(\|x-y\|).\end{aligned} \tag{2.7}$$

Namely, T_t is a weakly contractive mapping with a function ψ. Thus, Theorem 1.2 guarantees that T_t has a unique fixed point x_t in K, that is, $\{x_t\}$ satisfying (2.4) is uniquely defined for each $t \in (0,1)$. (i) and (ii) are proved. □

Subsequently, we show (iii). Indeed, by Theorem 1.2, there exists a unique element $z \in K$ such that $z = P(Az)$. Such a $z \in C$ fulfils (2.5) by Lemma 2.1. Next we show that the variational inequality (2.5) has a unique solution z. In fact, suppose $p \in C$ is another solution of (2.5). That is,

$$\langle Ap - p, J(z-p)\rangle \leq 0, \quad \langle Az - z, J(p-z)\rangle \leq 0. \tag{2.8}$$

Adding up gets

$$\varphi(\|p-z\|)\|p-z\| \leq \|p-z\|^2 - \|Ap - Az\|\|p-z\| \leq \langle (p-z) - (Ap - Az), J(p-z)\rangle \leq 0. \tag{2.9}$$

Hence $z = p$ by the property of φ. This completes the proof.

Let $\{T_n\}$ be a sequence of nonexpansive mappings with $F = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ on a closed convex subset K of a Banach space E and let $\{\alpha_n\}$ be a sequence in $(0,1]$ with (C1). $(E, K, \{T_n\}, \{\alpha_n\})$ is said to have *Browder's property* if for each $u \in K$, a sequence $\{y_n\}$ defined by

$$y_n = (1-\alpha_n)T_n y_n + \alpha_n u, \tag{2.10}$$

for $n \in \mathbb{N}$, converges strongly. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ with (C1) and (C2). Then $(E, K, \{T_n\}, \{\alpha_n\})$ is said to have *Halpern's property* if for each $u \in K$, a sequence $\{y_n\}$ defined by

$$y_{n+1} = (1-\alpha_n)T_n y_n + \alpha_n u, \tag{2.11}$$

for $n \in \mathbb{N}$, converges strongly.

We know that if E is a uniformly smooth Banach space or a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, K is bounded, $\{T_n\}$ is a constant sequence T, then $(E, K, \{T_n\}, \{1/n\})$ has both Browder's and Halpern's property (see [7, 10, 11, 23], resp.).

Lemma 2.3 (see [24, Proposition 4]). *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Browder's property. For each $\in K$, put $Pu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by* (2.10). *Then P is a nonexpansive mapping on K.*

Lemma 2.4 (see [24, Proposition 5]). *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Halpern's property. For each $\in K$, put $Pu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by* (2.11). *Then the following hold: (i) Pu does not depend on the initial point y_1. (ii) P is a nonexpansive mapping on K.*

Proposition 2.5. *Let E be a smooth Banach space, and $(E, K, \{T_n\}, \{\alpha_n\})$ have Browder's property. Then F is a sunny nonexpansive retract of K, and moreover, $Pu = \lim_{n\to\infty} y_n$ define a sunny nonexpansive retraction from K to F.*

Proof. For each $p \in F$, it is easy to see from (2.10) that

$$\begin{aligned}\langle u - y_n, J(p - y_n)\rangle &= \frac{1-\alpha_n}{\alpha_n}\langle y_n - p + T_n p - T_n y_n, J(p - y_n)\rangle \\ &\le \frac{1-\alpha_n}{\alpha_n}\left(\|T_n p - T_n y_n\|\|p - y_n\| - \|p - y_n\|^2\right) \le 0,\end{aligned} \tag{2.12}$$

$$\langle u - y_n, J(p - y_n)\rangle = \langle u - Pu, J(p - y_n)\rangle + \langle Pu - y_n, J(p - y_n)\rangle. \tag{2.13}$$

This implies for any $p \in F$ and some $L \ge \|y_n - p\|$,

$$\langle u - Pu, J(p - y_n)\rangle \le \langle y_n - Pu, J(p - y_n)\rangle \le L\|y_n - Pu\| \longrightarrow 0. \tag{2.14}$$

The smoothness of E implies the norm weak* continuity of J [22, Theorems 4.3.1, 4.3.2], so

$$\lim_{n\to\infty}\langle u - Pu, J(p - y_n)\rangle = \langle u - Pu, J(p - Pu)\rangle. \tag{2.15}$$

Thus

$$\langle u - Pu, J(p - Pu)\rangle \le 0, \quad \forall p \in F. \tag{2.16}$$

By Lemma 2.1, $Pu = \lim_{n\to\infty} y_n$ is a sunny nonexpansive retraction from K to F. □

We will use the following facts concerning numerical recursive inequalities (see [25–27]).

Lemma 2.6. *Let $\{\lambda_n\}$, and $\{\beta_n\}$ be two sequences of nonnegative real numbers, and $\{\alpha_n\}$ a sequence of positive numbers satisfying the conditions $\sum_{n=0}^{\infty}\gamma_n = \infty$, and $\lim_{n\to\infty}\beta_n/\alpha_n = 0$. Let the recursive inequality*

$$\lambda_{n+1} \le \lambda_n - \alpha_n \psi(\lambda_n) + \beta_n, \quad n = 0, 1, 2, \ldots, \tag{2.17}$$

be given where $\psi(\lambda)$ is a continuous and strict increasing function for all $\lambda \geq 0$ with $\psi(0) = 0$. Then (1) $\{\lambda_n\}$ converges to zero, as $n \to \infty$; (2) there exists a subsequence $\{\lambda_{n_k}\} \subset \{\lambda_n\}, k = 1, 2, \ldots$, such that

$$\begin{gathered}
\lambda_{n_k} \leq \psi^{-1}\left(\frac{1}{\sum_{m=0}^{n_k} \alpha_m} + \frac{\beta_{n_k}}{\alpha_{n_k}}\right), \\
\lambda_{n_k+1} \leq \psi^{-1}\left(\frac{1}{\sum_{m=0}^{n_k} \alpha_m} + \frac{\beta_{n_k}}{\alpha_{n_k}}\right) + \beta_{n_k}, \\
\lambda_n \leq \lambda_{n_k+1} - \sum_{m=n_k+1}^{n-1} \frac{\alpha_m}{\theta_m}, \; n_k + 1 < n < n_{k+1}, \; \theta_m = \sum_{i=0}^{m} \alpha_i, \\
\lambda_{n+1} \leq \lambda_0 - \sum_{m=0}^{n} \frac{\alpha_m}{\theta_m} \leq \lambda_0, \quad 1 \leq n \leq n_k - 1, \\
1 \leq n_k \leq s_{max} = \max\left\{ s; \sum_{m=0}^{s} \frac{\alpha_m}{\theta_m} \leq \lambda_0 \right\}.
\end{gathered} \tag{2.18}$$

3. Main Results

We first discuss Browder's type convergence.

Theorem 3.1. *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Browder's property. For each $u \in K$, put $Pu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by* (2.10)*. Let $A : K \to K$ be a weak contraction with a function φ. Define a sequence $\{x_n\}$ in K by*

$$x_n = \alpha_n A x_n + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}. \tag{3.1}$$

Then $\{x_n\}$ converges strongly to the unique point $z \in K$ satisfying $P(Az) = z$.

Proof. We note that Proposition 2.2(ii) assures the existence and uniqueness of $\{x_n\}$. It follows from Proposition 2.2(i) and Lemma 2.3 that PA is a weak contraction on K, then by Theorem 1.2, there exists the unique element $z \in K$ such that $P(Az) = z$. Define a sequence $\{y_n\}$ in K by

$$y_n = \alpha_n A z + (1 - \alpha_n) T_n y_n, \quad \text{for any} \quad n \in \mathbb{N}. \tag{3.2}$$

Then by the assumption, $\{y_n\}$ converges strongly to $P(Az)$. For every n, we have

$$\begin{aligned}
\|x_n - y_n\| &\leq (1 - \alpha_n)\|T_n x_n - T_n y_n\| + \alpha_n \|A x_n - Az\| \\
&\leq (1 - \alpha_n)\|x_n - y_n\| + \alpha_n \|A x_n - A y_n\| + \alpha_n \|A y_n - Az\| \\
&\leq \|x_n - y_n\| - \alpha_n \varphi(\|x_n - y_n\|) + \alpha_n (\|y_n - z\| - \varphi(\|x_n - z\|)),
\end{aligned} \tag{3.3}$$

then

$$\varphi(\|x_n - y_n\|) \le \|y_n - z\|. \tag{3.4}$$

Therefore,

$$\lim_{n\to\infty} \varphi(\|x_n - y_n\|) \le 0, \quad \text{i.e.,} \quad \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.5}$$

Hence,

$$\lim_{n\to\infty} \|x_n - z\| \le \lim_{n\to\infty} (\|x_n - y_n\| + \|y_n - z\|) = 0. \tag{3.6}$$

Consequently, $\{x_n\}$ converges strongly to z. This completes the proof. □

We next discuss Halpern's type convergence.

Theorem 3.2. *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Halpern's property. For each $u \in K$, put $Pu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by (2.11). Let $A : K \to K$ be a weak contraction with a function φ. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and*

$$x_{n+1} = \alpha_n A x_n + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}. \tag{3.7}$$

Then $\{x_n\}$ converges strongly to the unique point $z \in K$ satisfying $P(Az) = z$. Moreover, there exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$, $k = 1, 2, \ldots$, and $\exists\{\varepsilon_n\} \subset (0, +\infty)$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$\begin{gathered}
\|y_{n_k} - x_{n_k}\| \le \varphi^{-1}\left(\frac{1}{\sum_{m=0}^{n_k} \alpha_m} + \varepsilon_{n_k}\right), \\
\|x_{n_k+1} - y_{n_k+1}\| \le \varphi^{-1}\left(\frac{1}{\sum_{m=0}^{n_k} \alpha_m} + \varepsilon_{n_k}\right) + \alpha_{n_k}\varepsilon_{n_k}, \\
\|x_n - y_n\| \le \|x_{n_k+1} - y_{n_k+1}\| - \sum_{m=n_k+1}^{n-1} \frac{\alpha_m}{\theta_m}, \; n_k + 1 < n < n_{k+1}, \; \theta_m = \sum_{i=0}^{m} \alpha_i, \\
\|y_{n+1} - x_{n+1}\| \le \|x_0 - y_0\| - \sum_{m=0}^{n} \frac{\alpha_m}{\theta_m} \le \|y_0 - x_0\|, \; 1 \le n \le n_k - 1, \\
1 \le n_k \le s_{\max} = \max\left\{ s; \sum_{m=0}^{s} \frac{\alpha_m}{\theta_m} \le \|y_0 - x_0\| \right\}.
\end{gathered} \tag{3.8}$$

Proof. It follows from Proposition 2.2(i) and Lemma 2.4 that PA is a weak contraction on K, then by Theorem 1.2, there exists a unique element $z \in K$ such that $z = P(Az)$. Thus we may define a sequence $\{y_n\}$ in K by

$$y_{n+1} = \alpha_n Az + (1 - \alpha_n) T_n y_n, \quad n = 0, 1, 2, \ldots. \tag{3.9}$$

Then by the assumption, $y_n \to P(Az)$ as $n \to \infty$. For every n, we have

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &\le \alpha_n \|Ax_n - Az\| + (1-\alpha_n)\|T_n x_n - T_n y_n\| \\ &\le \alpha_n \left(\|Ax_n - Ay_n\| + \|Ay_n - Az\|\right) + (1-\alpha_n)\|x_n - y_n\| \\ &\le \|x_n - y_n\| - \alpha_n \varphi(\|x_n - y_n\|) + \alpha_n \left(\|y_n - z\| - \varphi(\|y_n - z\|)\right).\end{aligned} \tag{3.10}$$

Thus, we get for $\lambda_n = \|x_n - y_n\|$ the following recursive inequality:

$$\lambda_{n+1} \le \lambda_n - \alpha_n \varphi(\lambda_n) + \beta_n, \tag{3.11}$$

where $\beta_n = \alpha_n \varepsilon_n$, and $\varepsilon_n = \|y_n - z\|$. Thus by Lemma 2.6,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.12}$$

Hence,

$$\lim_{n\to\infty} \|x_n - z\| \le \lim_{n\to\infty} \left(\|x_n - y_n\| + \|y_n - z\|\right) = 0. \tag{3.13}$$

Consequently, we obtain the strong convergence of $\{x_n\}$ to $z = P(Az)$, and the remainder estimates now follow from Lemma 2.6. □

Theorem 3.3. *Let E be a Banach space E whose norm is uniformly Gâteaux differentiable, and $\{\alpha_n\}$ satisfies the condition (C2). Assume that $(E, K, \{T_n\}, \{\alpha_n\})$ have Browder's property and* $\lim_{n\to\infty}\|y_n - T_m y_n\| = 0$ *for every $m \in \mathbb{N}$, where $\{y_n\}$ is a bounded sequence in K defined by* (2.10). *then $(E, K, \{T_n\}, \{\alpha_n\})$ have Halpern's property.*

Proof. Define a sequence $\{z_m\}$ in K by $u \in K$ and

$$z_m = \alpha_m u + (1-\alpha_m) T_m z_m, \quad m \in \mathbb{N}. \tag{3.14}$$

It follows from Proposition 2.5 and the assumption that $Pu = \lim_{m\to\infty} z_m$ is the unique sunny nonexpansive retraction from K to F. Subsequently, we approved that

$$\forall \varepsilon > 0, \quad \limsup_{n\to\infty} \left\langle u - Pu, J(y_n - Pu)\right\rangle \le \varepsilon. \tag{3.15}$$

In fact, since $Pu \in F$, then we have

$$\begin{aligned}
\|z_m - y_n\|^2 &= (1-\alpha_m)\langle T_m z_m - y_n, J(z_m - y_n)\rangle + \alpha_m \langle u - y_n, J(z_m - y_n)\rangle \\
&= (1-\alpha_m)\left(\langle T_m z_m - T_m y_n, J(z_m - y_n)\rangle + \langle T_m y_n - y_n, J(z_m - y_n)\rangle\right) \\
&\quad + \alpha_m \langle u - Pu, J(z_m - y_n)\rangle + \alpha_m \langle Pu - z_m, J(z_m - y_n)\rangle \\
&\quad + \alpha_m \langle z_m - y_n, J(z_m - y_n)\rangle \\
&\le \|y_n - z_m\|^2 + \|T_m y_n - y_n\| M + \alpha_m \langle u - Pu, J(z_m - y_n)\rangle \\
&\quad + \alpha_m \|z_m - Pu\| M,
\end{aligned} \tag{3.16}$$

then

$$\langle u - Pu, J(y_n - z_m)\rangle \le \frac{\|y_n - T_m y_n\|}{\alpha_m} M + M\|z_m - Pu\|, \tag{3.17}$$

where M is a constant such that $M \ge \|y_n - z_m\|$ by the boundedness of $\{y_n\}$, and $\{z_m\}$. Therefore, using $\lim_{n\to\infty}\|y_n - T_m y_n\| = 0$, and $z_m \to Pu$, we get

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle u - Pu, J(y_n - z_m)\rangle \le 0. \tag{3.18}$$

On the other hand, since the duality map J is norm topology to weak* topology uniformly continuous in a Banach space E with uniformly Gâteaux differentiable norm, we get that as $m \to \infty$,

$$\left|\langle u - Pu, J(y_n - Pu) - J(y_n - z_m)\rangle\right| \to 0, \quad \forall n . \tag{3.19}$$

Therefore for any $\varepsilon > 0$, $\exists N > 0$ such that for all $m > N$ and all $n \ge 0$, we have that

$$\langle u - Pu, J(y_n - Pu)\rangle < \langle u - Pu, J(y_n - z_m)\rangle + \varepsilon. \tag{3.20}$$

Hence noting (3.18), we get that

$$\limsup_{n\to\infty} \langle u - Pu, J(y_n - Pu)\rangle \le \limsup_{m\to\infty} \limsup_{n\to\infty} \left(\langle u - Pu, J(y_n - z_m)\rangle + \varepsilon\right) \le \varepsilon. \tag{3.21}$$

(3.15) is proved. From (2.10) and $Pu \in F$, we have for all $n \ge 0$,

$$\begin{aligned}
\|y_{n+1} - Pu\|^2 &= \alpha_n \langle u - Pu, J(y_{n+1} - Pu)\rangle + (1-\alpha_n)\langle T_n y_n - Pu, J(y_{n+1} - Pu)\rangle \\
&\le (1-\alpha_n)\frac{\|T_n y_n - Pu\|^2 + \|J(y_{n+1} - Pu)\|^2}{2} + \alpha_n \langle u - Pu, J(y_{n+1} - Pu)\rangle \\
&\le (1-\alpha_n)\frac{\|y_n - Pu\|^2}{2} + \frac{\|y_{n+1} - Pu\|^2}{2} + \alpha_n \langle u - Pu, J(y_{n+1} - Pu)\rangle.
\end{aligned} \tag{3.22}$$

Thus,

$$\left\|y_{n+1}-Pu\right\|^2 \leq \left\|y_n-Pu\right\|^2-\alpha_n\left\|y_n-Pu\right\|^2+2\alpha_n\left\langle u-Pu, J\left(y_{n+1}-Pu\right)\right\rangle. \tag{3.23}$$

Consequently, we get for $\lambda_n = \|y_n - Pu\|^2$ the following recursive inequality:

$$\lambda_{n+1} \leq \lambda_n - \alpha_n\psi(\lambda_n) + \beta_n, \tag{3.24}$$

where $\psi(t) = t$, and $\beta_n = 2\alpha_n\varepsilon$. The strong convergence of $\{y_n\}$ to Pu follows from Lemma 2.6. Namely, $(E, K, \{T_n\}, \{\alpha_n\})$ have Halpern's property. □

4. Deduced Theorems

Using Theorems 3.1, 3.2, and 3.3, we can obtain many convergence theorems. We state some of them.

We now discuss convergence theorems for families of nonexpansive mappings. Let K be a nonempty closed convex subset of a Banach space E. A (one parameter) nonexpansive semigroups is a family $\mathcal{F} = \{T(t) : t > 0\}$ of selfmappings of K such that

(i) $T(0)x = x$ for $x \in K$;

(ii) $T(t+s)x = T(t)T(s)x$ for $t, s > 0$, and $x \in K$;

(iii) $\lim_{t\to 0} T(t)x = x$ for $x \in K$;

(iv) for each $t > 0, T(t)$ is nonexpansive, that is,

$$\left\|T(t)x - T(t)y\right\| \leq \left\|x-y\right\|, \quad \forall x, y \in K. \tag{4.1}$$

We will denote by F the common fixed point set of $\mathcal{F}$, that is,

$$F := \mathrm{Fix}(\mathcal{F}) = \{x \in K : T(t)x = x, t > 0\} = \bigcap_{t>0} \mathrm{Fix}(T(t)). \tag{4.2}$$

A continuous operator semigroup $\mathcal{F}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) (see [28–31]) on K if for all $h \geq 0$ and any bounded subset C of K,

$$\lim_{t\to\infty} \sup_{x\in C} \|T(h)(T(t)x) - T(t)x\| = 0. \tag{4.3}$$

Recently, Song and Xu [31] showed that $(E, K, \{T(t_n)\}, \{\alpha_n\})$ have both Browder's and Halpern's property in a reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm whenever $t_n \to \infty$ $(n \to \infty)$. As a direct consequence of Theorems 3.1, 3.2, and 3.3, we obtain the following.

Theorem 4.1. *Let E be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and K a nonempty closed convex subset of E, and $\{T(t)\}$ a u.a.r. nonexpansive semigroup from K into itself such that $F := \mathrm{Fix}(\mathcal{F}) \neq \emptyset$, and $A : K \to K$ a weak contraction. Suppose* *that* $\lim_{n\to\infty} t_n = \infty$, *and* $\beta_n \in (0,1)$ *satisfies the condition* (C1), *and* $\alpha_n \in (0,1)$ *satisfies the conditions* (C1) *and* (C2). *If* $\{y_n\}$ *and* $\{x_n\}$ *defined by*

$$\begin{aligned} y_n &= \beta_n A y_n + (1-\beta_n) T(t_n) y_n, \quad n \in \mathbb{N}, \\ x_{n+1} &= \alpha_n A x_n + (1-\alpha_n) T(t_n) x_n, \quad n \geq 1. \end{aligned} \tag{4.4}$$

Then as $n \to \infty$, *both* $\{y_n\}$, *and*$\{x_n\}$ *strongly converge to* $z = P(Az)$, *where* P *is a sunny nonexpansive retraction from* K *to* F.

Let $\{t_n\}$ a sequence of positive real numbers divergent to ∞, and for each $t > 0$ and $x \in K$, $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s) x ds. \tag{4.5}$$

Recently, Chen and Song [32] showed that $(E, K, \{\sigma_{t_n}\}, \{\alpha_n\})$ have both Browder's and Halpern's property in a uniformly convex Banach space with a uniformly Gâeaux differentiable norm whenever $t_n \to \infty$ $(n \to \infty)$. Then we also have the following.

Theorem 4.2. *Let* E *be a uniformly convex Banach space with uniformly Gâteaux differentiable norm, and let* K, A *be as in Theorem 4.1. Suppose that* $\{T(t)\}$ *a nonexpansive semigroups from* K *into itself such that* $F := \mathrm{Fix}(\mathcal{F}) = \bigcap_{t>0} \mathrm{Fix}(T(t)) \neq \emptyset$, $\{y_n\}$, *and* $\{x_n\}$ *defined by*

$$\begin{aligned} y_n &= \beta_n A y_n + (1-\beta_n) \sigma_{t_n}(y_n), \quad n \in \mathbb{N}, \\ x_{n+1} &= \alpha_n A x_n + (1-\alpha_n) \sigma_{t_n}(x_n), \quad n \in \mathbb{N}, \end{aligned} \tag{4.6}$$

where $t_n \to \infty$, *and* $\beta_n \in (0,1)$ *satisfies the condition* (C1), *and* $\alpha_n \in (0,1)$ *satisfies the conditions* (C1) *and* (C2). *Then as* $n \to \infty$, *both* $\{y_n\}$, *and* $\{x_n\}$ *strongly converge to* $z = P(Az)$, *where* P *is a sunny nonexpansive retraction from* K *to* F.

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