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Research Article

An Iterative Algorithm Combining Viscosity Method with Parallel Method for a Generalized Equilibrium Problem and Strict Pseudocontractions

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We introduce a new approximation scheme combining the viscosity method with parallel method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a family of finitely strict pseudocontractions. We obtain a strong convergence theorem for the sequences generated by these processes in Hilbert spaces. Based on this result, we also get some new and interesting results. The results in this paper extend and improve some well-known results in the literature.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and let C be a nonempty-closed convex subset of H. Let $\varphi : H \to R \cup \{+\infty\}$ be a function and let F be a bifunction from $C \times C$ to R such that $C \cap \operatorname{dom}\varphi \neq \varnothing$, where R is the set of real numbers and $\operatorname{dom}\varphi = \{x \in H : \varphi(x) < +\infty\}$. Flores-Bazán [1] introduced the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{GEP}(F, \varphi)$. Flores-Bazán [1] provided some characterizations of the nonemptiness of the solution set for problem (1.1) in reflexive Banach spaces in the quasiconvex case. Bigi et al. [2] studied a dual problem associated with the problem (1.1) with $C = H = R^n$.

Let $\varphi(x) = \delta_C(x)$, $\forall x \in H$. Here δ_C denotes the indicator function of the set C; that is, $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise. Then the problem (1.1) becomes the following equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(F)$. The problem (1.2) includes, as special cases, the optimization problem, the variational inequality problem, the fixed point problem, the nonlinear complementarity problem, the Nash equilibrium problem in noncooperative games, and the vector optimization problem. For more detail, please see [3–5] and the references therein.

If $F(x, y) = g(y) - g(x)$ for all $x, y \in C$, where $g : C \to R$ is a function, then the problem (1.1) becomes a problem of finding $x \in C$ which is a solution of the following minimization problem:

$$\min_{y \in C} \{\varphi(y) + g(y)\}. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{Argmin}(g, \varphi)$.

If $\varphi : H \to R \cup \{+\infty\}$ is replaced by a real-valued function $\phi : C \to R$, the problem (1.1) reduces to the following mixed equilibrium problem introduced by Ceng and Yao [6]:

$$\text{Find } x \in C \text{ such that } F(x, y) + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

Recall that a mapping $T : C \to C$ is said to be a κ-strict pseudocontraction [7] if there exists $0 \leq \kappa < 1$, such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C, \tag{1.5}$$

where I denotes the identity operator on C. When $\kappa = 0$, T is said to be nonexpansive. Note that the class of strict pseudocontraction mappings strictly includes the class of nonexpansive mappings. We denote the set of fixed points of S by $\mathrm{Fix}(S)$.

Ceng and Yao [6], Yao et al. [8], and Peng and Yao [9, 10] introduced some iterative schemes for finding a common element of the set of solutions of the mixed equilibrium problem (1.4) and the set of common fixed points of a family of finitely (infinitely) nonexpansive mappings (strict pseudocontractions) in a Hilbert space and obtained some strong convergence theorems(weak convergence theorems). Some methods have been proposed to solve the problem (1.2); see, for instance, [3–5, 11–18] and the references therein. Recently, S. Takahashi and W. Takahashi [12] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem. Su et al. [13] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping in a Hilbert space. Tada and Takahashi [14] introduced two iterative schemes for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. Ceng et al. [15] introduced an iterative algorithm for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a strict pseudocontraction mapping. Chang et al. [16] introduced some iterative processes based on the extragradient method for finding the common element of the set of fixed points of a family of infinitely nonexpansive mappings, the set of problem (1.2), and the set of solutions of a variational inequality problem for an α-inverse strongly monotone mapping. Colao et al. [17] introduced an iterative method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a finite family of nonexpansive mappings in a Hilbert space and proved the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem. To the best of our knowledge, there is not any algorithms for solving problem (1.1).

On the other hand, Marino and Xu [19] and Zhou [20] introduced and researched some iterative scheme for finding a fixed point of a strict pseudocontraction mapping. Acedo and Xu [21] introduced some parallel and cyclic algorithms for finding a common fixed point of a family of finite strict pseudocontraction mappings and obtained both weak and strong convergence theorems for the sequences generated by the iterative schemes.

In the present paper, we introduce a new approximation scheme combining the viscosity method with parallel method for finding a common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of a family of finitely strict pseudocontractions. We obtain a strong convergence theorem for the sequences generated by these processes. Based on this result, we also get some new and interesting results. The results in this paper extend and improve some well-known results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty-closed convex subset of H. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C x \in C$ and

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$.

For each $B \subseteq H$, we denote by $\operatorname{conv}(B)$ the convex hull of B. A multivalued mapping $G : B \to 2^H$ is said to be a KKM map if, for every finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq B$, $\operatorname{conv}(\{x_1, x_2, \ldots, x_n\}) \subseteq \bigcup_{n=1}^{\infty} G(x_i)$.

We will use the following results in the sequel.

Lemma 2.1 (see [22]). *Let B be a nonempty subset of a Hausdorff topological vector space X and let $G : B \to 2^X$ be a KKM map. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x\in B} G(x) \neq \emptyset$.*

For solving the generalized equilibrium problem, let us give the following assumptions for the bifunction F, φ, and the set C:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) for each $y \in C$, $x \mapsto F(x,y)$ is weakly upper semicontinuous;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex;

(A5) for each $x \in C$, $y \mapsto F(x,y)$ is lower semicontinuous;

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \operatorname{dom}\varphi$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z); \tag{2.3}$$

(B2) C is a bounded set.

Lemma 2.2. *Let C be a nonempty-closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let $\varphi : H \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \operatorname{dom}\varphi \neq \emptyset$. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \left\{ z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z),\ \forall y \in C \right\} \tag{2.4}$$

for all $x \in H$. Assume that either (B1) or (B2) holds. Then, the following conclusions hold:

(1) *for each $x \in H$, $S_r(x) \neq \emptyset$;*

(2) *S_r is single-valued;*

(3) *S_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| S_r(x) - S_r(y) \right\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle; \tag{2.5}$$

(4) $\operatorname{Fix}(S_r) = \operatorname{GEP}(F, \varphi)$;

(5) $\operatorname{GEP}(F,\varphi)$ *is closed and convex.*

Proof. Let x_0 be any given point in E. For each $y \in C$, we define

$$G(y) = \left\{ z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x_0\rangle \geq \varphi(z) \right\}. \tag{2.6}$$

Note that for each $y \in C \cap \operatorname{dom}\varphi$, $G(y)$ is nonempty since $y \in G(y)$ and for each $y \in C \setminus \operatorname{dom}\varphi$, $G(y) = C$. We will prove that G is a KKM map on $C \cap \operatorname{dom}\varphi$. Suppose that there exists a finite subset $\{y_1, y_2, \dots, y_n\}$ of $C \cap \operatorname{dom}\varphi$ and $\mu_i \geq 0$ for all $i = 1, 2, \dots, n$ with $\sum_{i=1}^{n} \mu_i = 1$ such that $\widehat{z} = \sum_{i=1}^{n} \mu_i y_i \notin G(y_i)$ for each $i = 1, 2, \dots, n$. Then we have

$$F(\widehat{z}, y_i) + \varphi(y_i) - \varphi(\widehat{z}) + \frac{1}{r}\langle y_i - \widehat{z}, \widehat{z} - x_0 \rangle < 0 \tag{2.7}$$

for each $i = 1, 2, \dots, n$. By (A4) and the convexity of φ, we have

$$\begin{aligned} 0 &= F(\widehat{z}, \widehat{z}) + \varphi(\widehat{z}) - \varphi(\widehat{z}) + \frac{1}{r}\langle \widehat{z} - \widehat{z}, \widehat{z} - x_0 \rangle \\ &\leq \sum_{i=1}^{n} \mu_i \left[F(\widehat{z}, y_i) + \varphi(y_i) - \varphi(\widehat{z}) \right] + \frac{1}{r}\left[\sum_{i=1}^{n} \mu_i \langle y_i - \widehat{z}, \widehat{z} - x_0 \rangle \right] < 0, \end{aligned} \tag{2.8}$$

which is a contradiction. Hence, G is a KKM map on $C \cap \operatorname{dom}\varphi$. Note that $\overline{G(y)}^{w}$ (the weak closure of $G(y)$) is a weakly closed subset of C for each $y \in C$. Moreover, if (B2) holds, then $\overline{G(y)}^{w}$ is also weakly compact for each $y \in C$. If (B1) holds, then for $x_0 \in E$, there exists a bounded subset $D_{x_0} \subseteq C$ and $y_{x_0} \in C \cap \operatorname{dom}\varphi$ such that for any $z \in C \setminus D_{x_0}$,

$$F(z, y_{x_0}) + \varphi(y_{x_0}) + \frac{1}{r}\langle y_{x_0} - z, z - x_0 \rangle < \varphi(z). \tag{2.9}$$

This shows that

$$G(y_{x_0}) = \left\{ z \in C : F(z, y_{x_0}) + \varphi(y_{x_0}) + \frac{1}{r}\langle y_{x_0} - z, z - x_0 \rangle \geq \varphi(z) \right\} \subseteq D_{x_0}. \tag{2.10}$$

Hence, $\overline{G(y_{x_0})}^{w}$ is weakly compact. Thus, in both cases, we can use Lemma 2.1 and have $\bigcap_{y \in C \cap \operatorname{dom}\varphi} \overline{G(y)}^{w} \neq \emptyset$.

Next, we will prove that $\overline{G(y)}^{w} = G(y)$ for each $y \in C$; that is, $G(y)$ is weakly closed. Let $z \in \overline{G(y)}^{w}$ and let z_m be a sequence in $G(y)$ such that $z_m \rightharpoonup z$. Then,

$$F(z_m, y) + \varphi(y) + \frac{1}{r}\langle y - z_m, z_m - x_0 \rangle \geq \varphi(z_m). \tag{2.11}$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, we can show that

$$\limsup_{m \to \infty} \langle y - z_m, z_m - x_0 \rangle \leq \langle z - y, x_0 - z \rangle. \tag{2.12}$$

It follows from (A3) and the weak lower semicontinuity of φ that

$$\begin{aligned}\varphi(z) &\le \liminf_{m\to\infty}\varphi(z_m) \le \limsup_{m\to\infty}\left[F(z_m,y)+\varphi(y)+\frac{1}{r}\langle y-z_m, z_m-x_0\rangle\right]\\ &\le \limsup_{m\to\infty}[F(z_m,y)+\varphi(y)]+\frac{1}{r}\limsup_{m\to\infty}\langle y-z_m, z_m-x_0\rangle \\ &\le F(z,y)+\varphi(y)+\frac{1}{r}\langle z-y, x_0-z\rangle.\end{aligned} \tag{2.13}$$

This implies that $z \in G(y)$. Hence, $G(y)$ is weakly closed. Hence, $S_r(x_0) = \bigcap_{y\in C} G(y) = \bigcap_{y\in C\cap \operatorname{dom}\varphi} G(y) = \bigcap_{y\in C\cap\operatorname{dom}\varphi}\overline{G(y)}^{w} \neq \emptyset$. Hence, from the arbitrariness of x_0, we conclude that $S_r(x)\neq\emptyset$, $\forall x\in H$.

We observe that $S_r(x) \subseteq \operatorname{dom}\varphi$. So by similar argument with that in the proof of Lemma 2.3 in [9], we can easily show that S_r is single-valued and S_r is a firmly nonexpansive-type map. Next, we claim that $\operatorname{Fix}(S_r) = \operatorname{GEP}(F,\varphi)$. Indeed, we have the following:

$$\begin{aligned} u\in \operatorname{Fix}(S_r) &\Longleftrightarrow u = S_r(u)\\ &\Longleftrightarrow F(u,y)+\varphi(y)+\frac{1}{r}\langle y-u, u-u\rangle \ge \varphi(u), \quad \forall y\in C\\ &\Longleftrightarrow F(u,y)+\varphi(y)\ge\varphi(u), \quad \forall y\in C\\ &\Longleftrightarrow u\in\operatorname{GEP}(F,\varphi).\end{aligned} \tag{2.14}$$

At last, we claim that $\operatorname{GEP}(F,\varphi)$ is a closed convex. Indeed, Since S_r is firmly nonexpansive, S_r is also nonexpansive. By [23, Proposition 5.3], we know that $\operatorname{GEP}(F,\varphi) = \operatorname{Fix}(S_r)$ is closed and convex. □

Remark 2.3. It is easy to see that Lemma 2.2 is a generalization of [9, Lemma 2.3].

Lemma 2.4 (see [24, 25]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n, \tag{2.15}$$

where γ_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

$$\begin{aligned} &\text{(i)}\quad \sum_{n=1}^{\infty}\gamma_n = \infty;\\ &\text{(ii)}\quad \limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\le 0 \quad \textit{or} \quad \sum_{n=1}^{\infty}|\delta_n|<\infty.\end{aligned} \tag{2.16}$$

Then, $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.5. *In a real Hilbert space H, there holds the following inequality:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \tag{2.17}$$

for all $x, y \in H$.

3. Strong Convergence Theorems

In this section, we show a strong convergence of an iterative algorithm based on both viscosity approximation method and parallel method which solves the problem of finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a family of finitely strict pseudocontractions in a Hilbert space.

We need the following assumptions for the parameters $\{\gamma_n\}, \{r_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, and $\{\beta_n\}$:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$;

(C3) $\{\gamma_n\} \subset [c,d]$ for some $c,d \in (\varepsilon,1)$ and $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$;

(C4) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(C5) $\lim_{n\to\infty}|\zeta_j^{(n+1)} - \zeta_j^{(n)}| = 0$ for all $j = 1,2,\ldots,N$.

Theorem 3.1. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)–(A5), and let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \operatorname{dom}\varphi \neq \emptyset$. Let $N \geq 1$ be an integer. For each $1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudocontraction for some $0 \leq \varepsilon_j < 1$ such that $\Omega = \bigcap_{j=1}^{N}\operatorname{Fix}(T_j) \cap \operatorname{GEP}(F,\varphi) \neq \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N}\zeta_j^{(n)} = 1$ for all n and $\inf_{n\geq 1}\zeta_j^{(n)} > 0$ for all $0 \leq j \leq N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. Assume that either (B1) or (B2) holds. Let f be a contraction of C into itself and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered}
x_1 = x \in C, \\
F(u_n, y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \quad \forall y \in C, \\
y_n = \gamma_n u_n + (1-\gamma_n)\sum_{j=1}^{N}\zeta_j^{(n)} T_j u_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) y_n
\end{gathered} \tag{3.1}$$

for every $n = 1,2,\ldots$, where $\{\gamma_n\}, \{r_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C5). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_\Omega f(w)$.

Proof. We show that $P_\Omega f$ is a contraction of C into itself. In fact, there exists $a \in [0,1)$ such that $\|f(x) - f(y)\| \leq a\|x - y\|$ for all $x, y \in C$. So, we have

$$\left\|P_\Omega f(x) - P_\Omega f(y)\right\| \leq \left\|f(x) - f(y)\right\| \leq a\|x - y\| \tag{3.2}$$

for all $x, y \in C$. Since H is complete, there exists a unique element $u_0 \in C$ such that $u_0 = P_\Omega f(u_0)$.

Let $u \in \Omega$ and let $\{S_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2. From $u_n = S_{r_n}(x_n) \in C$, we have

$$\|u_n - u\| = \|S_{r_n}(x_n) - S_{r_n}(u)\| \le \|x_n - u\|. \tag{3.3}$$

We define a mapping W_n by

$$W_n x = \sum_{j=1}^{N} \zeta_j^{(n)} T_j x, \quad \forall x \in C. \tag{3.4}$$

By [21, Proposition 2.6], we know that W_n is an ε-strict pseudocontraction and $F(W_n) = \bigcap_{j=1}^{N} \mathrm{Fix}(T_j)$. It follows from (3.3), $y_n = \gamma_n u_n + (1-\gamma_n) W_n u_n$ and $u = W_n u$ such that

$$\begin{aligned}
\|y_n - u\|^2 &= \gamma_n \|u_n - u\|^2 + (1-\gamma_n)\|W_n u_n - u\|^2 - \gamma_n(1-\gamma_n)\|u_n - W_n u_n\|^2 \\
&\le \gamma_n \|u_n - u\|^2 + (1-\gamma_n)\left[\|u_n - u\|^2 + \varepsilon\|u_n - W_n u_n\|^2\right] - \gamma_n(1-\gamma_n)\|u_n - W_n u_n\|^2 \\
&= \|u_n - u\|^2 + (1-\gamma_n)(\varepsilon - \gamma_n)\|u_n - W_n u_n\|^2 \\
&\le \|u_n - u\|^2.
\end{aligned} \tag{3.5}$$

Put $M_0 = \max\{\|x_1 - u\|, (1/(1-a))\|f(u) - u\|\}$. It is obvious that $\|x_1 - u\| \le M_0$. Suppose $\|x_n - u\| \le M_0$. From (3.3), (3.5), and $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) y_n$, we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n) y_n - u\| \\
&\le \alpha_n \|f(x_n) - f(u)\| + \alpha_n \|f(u) - u\| + \beta_n \|x_n - u\| + (1-\alpha_n-\beta_n)\|y_n - u\| \\
&\le \alpha_n a \|x_n - u\| + \alpha_n \|f(u) - u\| + \beta_n \|x_n - u\| + (1-\alpha_n-\beta_n)\|u_n - u\| \\
&\le \alpha_n a \|x_n - u\| + \alpha_n \|f(u) - u\| + (1-\alpha_n)\|x_n - u\| \\
&= (1-a)\alpha_n \frac{\|f(u) - u\|}{1-a} + \left[1 - (1-a)\alpha_n\right]\|x_n - u\|, \\
&\le (1-a)\alpha_n M_0 + \left[1 - (1-a)\alpha_n\right] M_0 = M_0
\end{aligned} \tag{3.6}$$

for every $n = 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.3) and (3.5), we also obtain that $\{y_n\}$ and $\{u_n\}$ are bounded.

Following [26], define $B_n : C \to C$ by

$$B_n = \gamma_n I + (1-\gamma_n) W_n. \tag{3.7}$$

As shown in [26], each B_n is a nonexpansive mapping on C. Set $M_1 = \sup_{n\geq 1}\{\|u_n - W_n u_n\|\}$, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|B_{n+1}(u_{n+1}) - B_n(u_n)\| \\ &\leq \|B_{n+1}(u_{n+1}) - B_{n+1}(u_n)\| + \|B_{n+1}(u_n) - B_n(u_n)\| \\ &\leq \|u_{n+1} - u_n\| + M_1|\gamma_{n+1} - \gamma_n| + (1-\gamma_{n+1})\|W_{n+1}(u_n) - W_n(u_n)\| \\ &\leq \|u_{n+1} - u_n\| + M_1|\gamma_{n+1} - \gamma_n| + (1-\gamma_{n+1})\sum_{j=1}^{N}\left|\zeta_j^{(n+1)} - \zeta_j^{(n)}\right|\|T_j u_n\|.\end{aligned} \tag{3.8}$$

On the other hand, from $u_n = T_{r_n}(x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1})$, we have

$$F(u_n, y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \quad \forall y \in C, \tag{3.9}$$

$$F(u_{n+1}, y) + \varphi(y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq \varphi(u_{n+1}), \quad \forall y \in C. \tag{3.10}$$

Putting $y = u_{n+1}$ in (3.9) and $y = u_n$ in (3.10), we have

$$\begin{gathered}F(u_n, u_{n+1}) + \varphi(u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq \varphi(u_n), \\ F(u_{n+1}, u_n) + \varphi(u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq \varphi(u_{n+1}).\end{gathered} \tag{3.11}$$

So, from the monotonicity of F, we get

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0, \tag{3.12}$$

hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{3.13}$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in N$. Then,

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\},\end{aligned} \tag{3.14}$$

hence

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|M_2,\end{aligned} \tag{3.15}$$

where $M_2 = \sup\{\|u_n - x_n\| : n \ge 1\}$.

It follows from (3.8) and (3.15) that

$$\begin{aligned}\|y_{n+1}-y_n\| \le \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|M_2 + M_1|\gamma_{n+1}-\gamma_n| \\ + (1-\gamma_{n+1})\sum_{j=1}^{N}\left|\zeta_j^{(n+1)}-\zeta_j^{(n)}\right|\|T_j u_n\|.\end{aligned} \tag{3.16}$$

Define a sequence $\{v_n\}$ such that

$$x_{n+1} = \beta_n x_n + (1-\beta_n)v_n, \quad \forall n \ge 1. \tag{3.17}$$

Then, we have

$$\begin{aligned}v_{n+1}-v_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1})y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n)y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n}f(x_n) + y_{n+1} - y_n + \frac{\alpha_n}{1-\beta_n}y_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}}y_{n+1}.\end{aligned} \tag{3.18}$$

From (3.18) and (3.16), we have

$$\begin{aligned}&\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\| \\ &\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|y_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|y_n\|) \\ &\quad\quad + \|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \\ &\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\| + \|y_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|f(x_n)\| + \|y_n\|) \\ &\quad\quad + \frac{1}{b}|r_{n+1}-r_n|M_2 + M_1|\gamma_{n+1}-\gamma_n| + (1-\gamma_{n+1})\sum_{j=1}^{N}\left|\zeta_j^{(n+1)}-\zeta_j^{(n)}\right|\|T_j u_n\|.\end{aligned} \tag{3.19}$$

It follows from (C1)–(C5) that

$$\limsup_{n\to\infty}\left(\left\|v_{n+1}-v_n\right\|-\left\|x_{n+1}-x_n\right\|\right)\le 0. \tag{3.20}$$

Hence, by [27, Lemma 2.2], we have $\lim_{n\to\infty}\|v_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\| = \lim_{n\to\infty}(1-\beta_n)\left\|v_n-x_n\right\| = 0. \tag{3.21}$$

Since $x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)y_n$, we have

$$\begin{aligned}\left\|x_n-y_n\right\| &\le \left\|x_{n+1}-x_n\right\|+\left\|x_{n+1}-y_n\right\| \\ &\le \left\|x_{n+1}-x_n\right\|+\alpha_n\left\|f(x_n)-y_n\right\|+\beta_n\left\|x_n-y_n\right\|,\end{aligned} \tag{3.22}$$

thus

$$\left\|x_n-y_n\right\| \le \frac{1}{1-\beta_n}\left(\left\|x_{n+1}-x_n\right\|+\alpha_n\left\|f(x_n)-y_n\right\|\right). \tag{3.23}$$

It follows from (C1) and (C2) that $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Since $x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)y_n$, for $u\in\Omega$, it follows from (3.5) and (3.3) that

$$\begin{aligned}\left\|x_{n+1}-u\right\|^2 &= \left\|\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)y_n-u\right\|^2 \\ &\le \alpha_n\left\|f(x_n)-u\right\|^2+\beta_n\left\|x_n-u\right\|^2+(1-\alpha_n-\beta_n)\left\|y_n-u\right\|^2 \\ &\le \alpha_n\left\|f(x_n)-u\right\|^2+\beta_n\left\|x_n-u\right\|^2 \\ &\quad+(1-\alpha_n-\beta_n)\left[\left\|u_n-u\right\|^2+(1-\gamma_n)(\varepsilon-\gamma_n)\left\|u_n-W_nu_n\right\|^2\right] \\ &\le \alpha_n\left\|f(x_n)-u\right\|^2+(1-\alpha_n)\left\|x_n-u\right\|^2 \\ &\quad+(1-\alpha_n-\beta_n)(1-\gamma_n)(\varepsilon-\gamma_n)\left\|u_n-W_nu_n\right\|^2,\end{aligned} \tag{3.24}$$

from which it follows that

$$\begin{aligned}\left\|u_n-W_nu_n\right\|^2 &\le \frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n)(\gamma_n-\varepsilon)}\left(\left\|f(x_n)-u\right\|^2-\left\|x_n-u\right\|^2\right) \\ &\quad+\frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n)(\gamma_n-\varepsilon)}\left(\left\|x_n-u\right\|^2-\left\|x_{n+1}-u\right\|^2\right). \\ &\le \frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n)(\gamma_n-\varepsilon)}\left(\left\|f(x_n)-u\right\|^2-\left\|x_n-u\right\|^2\right) \\ &\quad+\frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n)(\gamma_n-\varepsilon)}\left(\left\|x_n-u\right\|+\left\|x_{n+1}-u\right\|\right)\left\|x_{n+1}-x_n\right\|.\end{aligned} \tag{3.25}$$

It follows from (C1)–(C3) and $\|x_{n+1} - x_n\| \to 0$ that

$$\|u_n - W_n u_n\| \longrightarrow 0. \tag{3.26}$$

For $u \in \Omega$, we have from Lemma 2.2,

$$\begin{aligned}\|u_n - u\|^2 &= \|S_{r_n} x_n - S_{r_n} u\|^2 \le \langle S_{r_n} x_n - S_{r_n} u, x_n - u\rangle \\ &= \langle u_n - u, x_n - u\rangle = \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - u_n\|^2\}.\end{aligned} \tag{3.27}$$

Hence,

$$\|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2. \tag{3.28}$$

By (3.24) and (3.28), we have

$$\begin{aligned}\|x_{n+1} - u\|^2 &\le \alpha_n \|f(x_n) - u\|^2 + \beta_n \|x_n - u\|^2 + (1 - \alpha_n - \beta_n)\|u_n - u\|^2 \\ &\le \alpha_n \|f(x_n) - u\|^2 + \beta_n \|x_n - u\|^2 + (1 - \alpha_n - \beta_n)\left[\|x_n - u\|^2 - \|x_n - u_n\|^2\right].\end{aligned} \tag{3.29}$$

Hence,

$$\begin{aligned}(1 - \alpha_n - \beta_n)\|x_n - u_n\|^2 &\le \alpha_n \|f(x_n) - u\|^2 - \alpha_n \|x_n - u\|^2 + \|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\ &\le \alpha_n \|f(x_n) - u\|^2 - \alpha_n \|x_n - u\|^2 \\ &\quad + (\|x_n - u\| + \|x_{n+1} - u\|)\|x_n - x_{n+1}\|.\end{aligned} \tag{3.30}$$

It follows from (C1), (C2), and $\|x_n - x_{n+1}\| \to 0$ that $\lim_{n\to\infty}\|x_n - u_n\| = 0$.
Next, we show that

$$\limsup_{n\to\infty}\langle f(u_0) - u_0, x_n - u_0\rangle \le 0, \tag{3.31}$$

where $u_0 = P_\Omega f(u_0)$. To show this inequality, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle f(u_0) - u_0, x_{n_i} - u_0\rangle = \limsup_{n\to\infty}\langle f(u_0) - u_0, x_n - u_0\rangle. \tag{3.32}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $\{x_{n_i}\} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{n_i} \rightharpoonup w$. From $\|x_n - y_n\| \to 0$, we also obtain that $y_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

We first show that $w \in \bigcap_{k=1}^N \mathrm{Fix}(T_k)$. To see this, we observe that we may assume (by passing to a further subsequence if necessary) $\zeta_k^{(n_i)} \to \zeta_k$ (as $i \to \infty$) for $k = 1, 2, \dots, N$. It is easy to see that $\zeta_k > 0$ and $\sum_{k=1}^N \zeta_k = 1$. We also have

$$W_{n_i} x \longrightarrow Wx \quad (\text{as } i \longrightarrow \infty)\ \forall x \in C, \tag{3.33}$$

where $W = \sum_{k=1}^N \zeta_k T_k$. Note that by [21, Proposition 2.6], W is an ε-strict pseudocontraction and $\mathrm{Fix}(W) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Since

$$\begin{aligned} \left\|u_{n_i} - Wu_{n_i}\right\| &\le \left\|u_{n_i} - W_{n_i}u_{n_i}\right\| + \left\|W_{n_i}u_{n_i} - Wu_{n_i}\right\| \\ &\le \left\|u_{n_i} - W_{n_i}u_{n_i}\right\| + \sum_{k=1}^N \left|\zeta_k^{(n_i)} - \zeta_k\right| \left\|T_k u_{n_i}\right\|, \end{aligned} \tag{3.34}$$

it follows from (3.26) and $\zeta_k^{(n_i)} \to \zeta_k$ that

$$\left\|u_{n_i} - Wu_{n_i}\right\| \longrightarrow 0. \tag{3.35}$$

So by the demiclosedness principle [21, Proposition 2.6(ii)], it follows that $w \in \mathrm{Fix}(W) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

We now show $w \in \mathrm{GEP}(F, \varphi)$. By $u_n = T_{r_n} x_n$, we know that

$$F(u_n, y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge \varphi(u_n), \quad \forall y \in C. \tag{3.36}$$

It follows from (A2) that

$$\varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n) + \varphi(u_n), \quad \forall y \in C. \tag{3.37}$$

Hence,

$$\varphi(y) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \ge F(y, u_{n_i}) + \varphi(u_n), \quad \forall y \in C. \tag{3.38}$$

It follows from (A4), (A5) and the weakly lower semicontinuity of φ, $(u_{n_i}-x_{n_i})/r_{n_i} \to 0$, and $u_{n_i} \rightharpoonup w$ that

$$F(y,w)+\varphi(w)\leq \varphi(y), \quad \forall y\in C. \tag{3.39}$$

For t with $0<t\leq 1$ and $y\in C\cap \operatorname{dom}\varphi$, let $y_t = ty+(1-t)w$. Since $y\in C\cap\operatorname{dom}\varphi$ and $w\in C\cap\operatorname{dom}\varphi$, we obtain $y_t\in C\cap\operatorname{dom}\varphi$, and hence $F(y_t,w)+\varphi(w)\leq\varphi(y_t)$. So by (A4) and the convexity of φ, we have

$$\begin{aligned}
0 &= F(y_t,y_t)+\varphi(y_t)-\varphi(y_t)\\
&\leq tF(y_t,y)+(1-t)F(y_t,w)+t\varphi(y)+(1-t)\varphi(w)-\varphi(y_t)\\
&\leq t\left[F(y_t,y)+\varphi(y)-\varphi(y_t)\right].
\end{aligned} \tag{3.40}$$

Dividing by t, we get

$$F(y_t,y)+\varphi(y)-\varphi(y_t)\geq 0. \tag{3.41}$$

Letting $t\to 0$, it follows from (A3) and the weakly lower semicontinuity of φ that

$$F(w,y)+\varphi(y)\geq\varphi(w) \tag{3.42}$$

for all $y\in C\cap\operatorname{dom}\varphi$. Observe that if $y\in C\setminus\operatorname{dom}\varphi$, then $F(w,y)+\varphi(y)\geq\varphi(w)$ holds. Moreover, hence $w\in \mathrm{GEP}(F,\varphi)$. This implies $w\in\Omega$. Therefore, we have

$$\limsup_{n\to\infty}\langle f(u_0)-u_0,x_n-u_0\rangle = \lim_{i\to\infty}\langle f(u_0)-u_0,x_{n_i}-u_0\rangle = \langle f(u_0)-u_0,w-u_0\rangle\leq 0. \tag{3.43}$$

Finally, we show that $x_n\to u_0$, where $u_0=P_\Omega f(u_0)$.

From Lemma 2.5, we have

$$\begin{aligned}
\|x_{n+1}-u_0\|^2 &= \|\alpha_n(f(x_n)-u_0)+\beta_n(x_n-u_0)+(1-\alpha_n-\beta_n)(y_n-u_0)\|^2\\
&\leq \|\beta_n(x_n-u_0)+(1-\alpha_n-\beta_n)(y_n-u_0)\|^2+2\alpha_n\langle f(x_n)-u_0,x_{n+1}-u_0\rangle\\
&\leq \left[(1-\alpha_n-\beta_n)\|y_n-u_0\|+\beta_n\|x_n-u_0\|\right]^2+2\alpha_n\langle f(x_n)-u_0,x_{n+1}-u_0\rangle\\
&\leq (1-\alpha_n)^2\|x_n-u_0\|^2+2\alpha_n\langle f(x_n)-f(u_0),x_{n+1}-u_0\rangle\\
&\quad +2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle\\
&\leq (1-\alpha_n)^2\|x_n-u_0\|^2+2\alpha_n a\|x_n-u_0\|\|x_{n+1}-u_0\|+2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle\\
&\leq (1-\alpha_n)^2\|x_n-u_0\|^2+\alpha_n a\left(\|x_n-u_0\|^2+\|x_{n+1}-u_0\|^2\right)\\
&\quad +2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle,
\end{aligned} \tag{3.44}$$

thus

$$\begin{aligned}\|x_{n+1}-u_0\|^2 \le{}& \left(1-\frac{2(1-a)\alpha_n}{1-a\alpha_n}\right)\|x_n-u_0\|^2\\&+\frac{2(1-a)\alpha_n}{1-a\alpha_n}\left\{\frac{\alpha_n}{2(1-a)}\|x_n-u_0\|^2+\frac{1}{1-a}\langle 2f(u_0)-2u_0, x_{n+1}-u_0\rangle\right\}.\end{aligned} \tag{3.45}$$

It follows from (C1), (3.43), (3.45), and Lemma 2.4 that $\lim_{n\to\infty}\|x_n-u_0\| = 0$. From $\|x_n-u_n\| \to 0$ and $\|y_n-x_n\| \to 0$, we have $u_n \to u_0$ and $y_n \to u_0$. The proof is now complete. □

Theorem 3.2. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)–(A5), and let $\varphi : H \to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function such that $C\cap \operatorname{dom}\varphi \neq \emptyset$. Let $N \ge 1$ be an integer. For each $1 \le j \le N$, let $T_j : C \to C$ be an ε_j-strict pseudocontraction for some $0 \le \varepsilon_j < 1$ such that $\Omega = \bigcap_{j=1}^{N}\mathrm{Fix}(T_j)\cap \mathrm{GEP}(F,\varphi)\neq\emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N}\zeta_j^{(n)} = 1$ for all n and $\inf_{n\ge1}\zeta_j^{(n)} > 0$ for all $0 \le j \le N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \le j \le N\}$. Assume that either (B1) or (B2) holds. Let v be an arbitrary point in C and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered}x_1 = x\in C,\\ F(u_n,y)+\varphi(y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge \varphi(u_n), \quad \forall y\in C,\\ y_n = \gamma_n u_n + (1-\gamma_n)\sum_{j=1}^{N}\zeta_j^{(n)}T_j u_n,\\ x_{n+1} = \alpha_n v + \beta_n x_n + (1-\alpha_n-\beta_n)y_n\end{gathered} \tag{3.46}$$

for every $n = 1,2,\dots$, where $\{\gamma_n\},\{r_n\},\{\alpha_n\},\{\zeta_1^{(n)}\},\{\zeta_2^{(n)}\},\dots,\{\zeta_N^{(n)}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C5). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_\Omega v$.

Proof. Let $f(x) = v$ for all $x\in C$, by Theorem 3.1, we obtain the desired result. □

4. Applications

By Theorems 3.1 and 3.2, we can obtain many new and interesting strong convergence theorems. Now, give some examples as follows: for $j = 1,2,\dots,N$, let $T_1 = T_2 = \cdots = T_N = T$, by Theorems 3.1 and 3.2, respectively, we have the following results.

Theorem 4.1. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)–(A5), and let $\varphi : H \to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function such that $C\cap\operatorname{dom}\varphi\neq\emptyset$. Let $T : C \to C$ be an ε-strict pseudocontraction for some $0 \le \varepsilon < 1$ such that $\mathrm{Fix}(T)\cap\mathrm{GEP}(F,\varphi)\neq\emptyset$. Assume that either (B1) or*

(B2) holds. Let f be a contraction of C into itself and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by

$$\begin{gathered} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \quad \forall y \in C, \\ y_n = \gamma_n u_n + (1-\gamma_n)Tu_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)y_n \end{gathered} \tag{4.1}$$

for every $n = 1, 2, \ldots$, where $\{\gamma_n\}$, $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C4). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_{\mathrm{Fix}(T)\cap \mathrm{GEP}(F,\varphi)} f(w)$.

Theorem 4.2. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)–(A5), and let $\varphi : H \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \operatorname{dom}\varphi \neq \emptyset$. Let $T : C \to C$ be an ε-strict pseudocontraction for some $0 \leq \varepsilon < 1$ such that $\mathrm{Fix}(T) \cap \mathrm{GEP}(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let v be an arbitrary point in C, and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \quad \forall y \in C, \\ y_n = \gamma_n u_n + (1-\gamma_n)Tu_n, \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1-\alpha_n-\beta_n)y_n \end{gathered} \tag{4.2}$$

for every $n = 1, 2, \ldots$, where $\{\gamma_n\}$, $\{r_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C4). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_{\mathrm{Fix}(T)\cap \mathrm{GEP}(F,\varphi)} v$.

We need the following two assumptions.

(B3) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0. \tag{4.3}$$

(B4) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \operatorname{dom}\varphi$ such that for any $z \in C \setminus D_x$,

$$g(y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z) + g(z). \tag{4.4}$$

Let $\varphi(x) = \delta_C(x)$, $\forall x \in H$, by Theorems 3.1 and 3.2, respectively, we obtain the following results.

Theorem 4.3. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)–(A5). Let $N \geq 1$ be an integer. For each $1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudocontraction for some $0 \leq \varepsilon_j < 1$ such that $\Gamma = \bigcap_{j=1}^{N} \mathrm{Fix}(T_j) \cap EP(F) \neq \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ for all n and $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ for all $0 \leq j \leq N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. Assume that either (B3) or (B2) holds. Let f be a contraction of H into itself, and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$
\begin{gathered}
x_1 = x \in C, \\
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \gamma_n u_n + (1-\gamma_n)\sum_{j=1}^{N} \zeta_j^{(n)} T_j u_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n) y_n
\end{gathered}
\tag{4.5}
$$

for every $n = 1, 2, \ldots$, where $\{\gamma_n\}, \{r_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C5). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_\Gamma f(w)$.

Theorem 4.4. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)–(A5). Let $N \geq 1$ be an integer. For each $1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudocontraction for some $0 \leq \varepsilon_j < 1$ such that $\Gamma = \bigcap_{j=1}^{N} \mathrm{Fix}(T_j) \cap EP(F) \neq \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ for all n and $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ for all $0 \leq j \leq N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. Assume that either (B3) or (B2) holds. Let v be an arbitrary point in C, and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$
\begin{gathered}
x_1 = x \in C, \\
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \gamma_n u_n + (1-\gamma_n)\sum_{j=1}^{N} \zeta_j^{(n)} T_j u_n, \\
x_{n+1} = \alpha_n v + \beta_n x_n + (1-\alpha_n-\beta_n) y_n
\end{gathered}
\tag{4.6}
$$

for every $n = 1, 2, \ldots$, where $\{\gamma_n\}, \{r_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C5). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_\Gamma v$.

Let $F(x, y) = g(y) - g(x)$ for all $x, y \in C$, by Theorems 3.1 and 3.2, respectively, we obtain the following results.

Theorem 4.5. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let $g : C \to R$ be a lower semicontinuous and convex function, and let $\varphi : H \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \mathrm{dom}\,\varphi \neq \emptyset$. Let $N \geq 1$ be an integer. For each*

$1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudocontraction for some $0 \leq \varepsilon_j < 1$ such that $\Theta = \bigcap_{j=1}^{N} \mathrm{Fix}(T_j) \cap \mathrm{Argmin}(g,\varphi) \neq \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ for all n and $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ for all $0 \leq j \leq N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. Assume that either (B4) or (B2) holds. Let f be a contraction of H into itself, and let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by

$$
\begin{gathered}
x_1 = x \in C, \\
g(y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq g(u_n) + \varphi(u_n), \quad \forall y \in C, \\
y_n = \gamma_n u_n + (1-\gamma_n) \sum_{j=1}^{N} \zeta_j^{(n)} T_j u_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) y_n
\end{gathered}
\tag{4.7}
$$

for every $n = 1, 2, \ldots$, where $\{\gamma_n\}, \{r_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C5). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_\Theta f(w)$.

Theorem 4.6. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let $g : C \to R$ be a lower semicontinuous and convex function, and let $\varphi : H \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \mathrm{dom}\, \varphi \neq \emptyset$. Let $N \geq 1$ be an integer. For each $1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudocontraction for some $0 \leq \varepsilon_j < 1$ such that $\Theta = \bigcap_{j=1}^{N} \mathrm{Fix}(T_j) \cap \mathrm{Argmin}(g,\varphi) \neq \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ for all n and $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ for all $0 \leq j \leq N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. Assume that either (B4) or (B2) holds. Let v be an arbitrary point in C, and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$
\begin{gathered}
x_1 = x \in C, \\
g(y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n) + g(u_n), \quad \forall y \in C, \\
y_n = \gamma_n u_n + (1-\gamma_n) \sum_{j=1}^{N} \zeta_j^{(n)} T_j u_n, \\
x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \alpha_n - \beta_n) y_n
\end{gathered}
\tag{4.8}
$$

for every $n = 1, 2, \ldots$, where $\{\gamma_n\}, \{r_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions (C1)–(C5). Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_\Theta v$.

Let $\varphi(x) = \delta_C(x)$, $\forall x \in H$, and let $F(x, y) = 0$ for all $x, y \in C$. Then $u_n = P_C x_n = x_n$. By Theorems 3.1 and 3.2, we obtain the following results.

Theorem 4.7. *Let C be a nonempty-closed convex subset of a real Hilbert space H. Let $N \geq 1$ be an integer. For each $1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudocontraction for some $0 \leq \varepsilon_j < 1$*

such that $\bigcap_{j=1}^N \mathrm{Fix}(T_j) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1}\zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Let* f *be a contraction of* H *into itself, and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$
\begin{aligned}
x_1 &= x \in C,\\
y_n &= \gamma_n x_n + (1-\gamma_n)\sum_{j=1}^N \zeta_j^{(n)} T_j x_n,\\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n) y_n
\end{aligned}
\tag{4.9}
$$

for every $n = 1, 2, \ldots$, *where* $\{\gamma_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, *and* $\{\beta_n\}$ *are sequences of numbers satisfying the conditions (C1)–(C3) and (C5). Then,* $\{x_n\}$, *and* $\{y_n\}$ *converge strongly to* $w = P_{\cap_{j=1}^N \mathrm{Fix}(T_j)} f(w)$.

Theorem 4.8. *Let* C *be a nonempty-closed convex subset of a real Hilbert space* H. *Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudocontraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\bigcap_{j=1}^N \mathrm{Fix}(T_j) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1}\zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Let* v *be an arbitrary point in* C, *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$
\begin{aligned}
x_1 &= x \in C,\\
y_n &= \gamma_n x_n + (1-\gamma_n)\sum_{j=1}^N \zeta_j^{(n)} T_j x_n,\\
x_{n+1} &= \alpha_n v + \beta_n x_n + (1-\alpha_n-\beta_n) y_n
\end{aligned}
\tag{4.10}
$$

for every $n = 1, 2, \ldots$, *where* $\{\gamma_n\}, \{\alpha_n\}, \{\zeta_1^{(n)}\}, \{\zeta_2^{(n)}\}, \ldots, \{\zeta_N^{(n)}\}$, *and* $\{\beta_n\}$ *are sequences of numbers satisfying the conditions (C1)–(C3) and (C5). Then,* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w = P_{\cap_{j=1}^N \mathrm{Fix}(T_j)} v$.

Remark 4.9. (1) Since the nonexpansive mappings have been replaced by the strict pseudocontractions, Theorems 3.1, 3.2, 4.1 and 4.2 extend and improve [6, Theorem 3.1], [8, Theorem 3.5], [9, Theorems 4.1 and 4.2], [18, Theorem 4.1], and the main results in [9–11, 13–16].

(2) Since the weak convergence has been replaced by strong convergence, Theorems 3.1, 3.2, 4.1–4.4 extend and improve [12, Theorem 3.1], [10, Corollary 4.1].

(3) Theorems 4.7 and 4.8 are strong convergence theorems for strict pseudocontractions without CQ constraints and hence they improve the corresponding results in [19, 21]. Theorems 3.1 and 3.2 also improve [10, Corollary 3.1].

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