Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 791514, 7 pages
doi:10.1155/2009/791514

Research Article

Approximate Fixed Point Theorems for the Class of Almost S-KKM_C Mappings in Abstract Convex Uniform Spaces

Tong-Huei Chang, Chi-Ming Chen, and Yueh-Hung Huang

Department of Applied Mathematics, National Hsinchu University of Education, Hsin-Chu, Taiwan

Correspondence should be addressed to Chi-Ming Chen, ming@mail.nhcue.edu.tw

Received 25 February 2009; Accepted 11 June 2009

Recommended by Hichem Ben-El-Mechaiekh

We use a concept of abstract convexity to define the almost S-KKM_C property, *al*-S-$KKM_C(X, Y)$ family, and almost Φ-spaces. We get some new approximate fixed point theorems and fixed point theorems in almost Φ-spaces. Our results extend some results of other authors.

1. Introduction and Preliminaries

In 1929, Knaster et al. [1] proved the well-known KKM theorem for an n-simplex. Ky Fan's generalization of the KKM theorem to infinite dimensional topological vector spaces in 1961 [2] proved to be a very versatile tool in modern nonlinear analysis with many far-reaching applications.

Chang and Yen [3] undertook a systematic study of the KKM property, and Chang et al. [4] generalized this property as well as the notion of a $KKM(X, Y)$ family of [4] to the wider concepts of the S-KKM property and its related S-$KKM(X, Y, Z)$ family.

Among the many contributions in the study of the KKM property and related topics, we mention the work by Amini et al. [5] where the classes of KKM and S-KKM mappings have been introduced in the framework of abstract convex spaces. The authors of [5] also define a concept of convexity that contains a number of other concepts of abstract convexities and obtain fixed point theorems for multifunctions verifying the S-KKM property on Φ-spaces that extend results of Ben-El-Mechaiekh et al. [6] and Horvath [7], motivated by the works of Ky Fan [2] and Browder [8]. We refer for the study of these notions to Ben-El-Mechaiekh et al. [9], and more recently, to Park [10], and Kim and Park [11].

In this paper, we use a concept of abstract convexity to define the almost S-$KKM_{\mathcal{C}}$ property, the corresponding notion of almost S-$KKM_{\mathcal{C}}(X,Y)$ family as well as the concept of almost Φ-spaces.

Let X and Y be two sets, and let $T : X \to 2^Y$ be a set-valued mapping. We will use the following notations in the sequel;

(i) $T(x) = \{y \in Y : y \in T(x)\}$,

(ii) $T(A) = \cup_{x\in A} T(x)$,

(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,

(iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$, and

(v) if D is a nonempty subset of X, then $\langle D\rangle$ denotes the class of all nonempty finite subsets of D.

For the case where X and Y are two topological spaces, a set-valued map $T : X \to 2^Y$ is said to be closed if its graph $\mathcal{G}_T = \{(x,y) \in X \times Y : y \in T(x)\}$ is closed. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y.

Definition 1.1. An abstract convex space $(E,\mathcal{C})$ consists of a nonempty topological space E, and a family $\mathcal{C}$ of subsets of E such that E and $\emptyset$ belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. This kind of abstract convexity was widely studied; see [5, 9, 12, 13].

Suppose that A is a nonempty subset of an abstract convex space $(E,\mathcal{C})$. Then

(i) a natural definition of $\mathcal{C}$-convex hull of A is

$$co_{\mathcal{C}}(A) = \cap\{B \in \mathcal{C} : A \subset B\}, \text{and} \tag{1.1}$$

(ii) we say that A is $\mathcal{C}$-convex if for each $B \in \langle A\rangle$, $co_{\mathcal{C}}(B) \subset A$.

Remark 1.2. It is clear that if $A \in \mathcal{C}$, then A is $\mathcal{C}$-convex. That is, each member of $\mathcal{C}$ is $\mathcal{C}$-convex.

Definition 1.3. We list some properties of a uniform space. A uniformity [14] for a set E is a nonempty family $\mathcal{U}$ of subsets of $E \times E$ such that

(i) each member of $\mathcal{U}$ contains the diagonal Δ where the diagonal Δ denotes the set of all pairs (x,x) for x in E;

(ii) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(iii) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$;

(iv) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;

(v) if $U \in \mathcal{U}$ and $U \subset V \subset E \times E$, then $V \in \mathcal{U}$.

The pair $(E,\mathcal{C})$ is called a uniform space. Every member V in $\mathcal{U}$ is called an entourage. An entourage is said to be symmetric if $(x,y) \in V$ whenever $(y,x) \in V$.

Definition 1.4. If $(E,\mathcal{C})$ is an abstract convex space with a uniformity $\mathcal{U}$, then we say that $(E,\mathcal{C})$ is an abstract convex uniform space.

Definition 1.5. Let A be a nonempty subset of an abstract convex uniform space $(E, \mathcal{C})$ which has a uniformity $\mathcal{U}$, and $\mathcal{U}$ has a symmetric basis $\mathcal{N}$. Then A is called almost $\mathcal{C}$-convex if, for any $K \in \langle A \rangle$ and for any $V \in \mathcal{N}$, there exists a mapping $h_{K,V} : K \to A$ such that $h_{K,V}(x) \in V[x]$ for all $x \in K$ and $co_{\mathcal{C}}(h_{K,V}(K)) \subset A$. Moreover, we call the mapping $h_{K,V} : K \to A$ a $\mathcal{C}$-convex-inducing mapping.

Remark 1.6. It is clear that every $\mathcal{C}$-convex set must be almost $\mathcal{C}$-convex, but the converse is not true. And in general, the $\mathcal{C}$-convex-inducing mapping is not unique. If $U, V \in \mathcal{N}$ and $U \subset V$, then $h_{A,U} : A \to X$ can be regarded as $h_{A,V} : A \to X$. If $A \subset B$, then $h_{A,U} : A \to X$ can be regarded as $h_{B,U} : B \to X$.

Recently, Amini et al. [5] introduced the class of multifunctions with the $S - KKM_{\mathcal{C}}$ property in abstract convex spaces.

Definition 1.7 (see [5]). Let Z be a nonempty set, $(X, \mathcal{C})$ an abstract convex space, and Y a topological space. If $S : Z \to 2^X$, $T : X \to 2^Y$ and $F : Z \to 2^Y$ are three multifunctions satisfying

$$T(co_{\mathcal{C}}(S(A))) \subset \cup_{x \in A} F(x), \quad \text{for each } A \in \langle Z \rangle, \tag{1.2}$$

then F is called a S-$KKM_{\mathcal{C}}$ mapping with respect to T. If the multifunction $T : X \to 2^Y$ satisfies the requirement that for any S-$KKM_{\mathcal{C}}$ mapping F with respect to T, the family $\{\overline{F(x)} : x \in Z\}$ has the finite intersection property where $\overline{F(x)}$ denotes the closure of $F(x)$, then T is said to have the S-$KKM_{\mathcal{C}}$ property with respect to $\mathcal{C}$. We define

$$S - KKM_{\mathcal{C}}(Z, X, Y) := \left\{T : W \to 2^Y \mid T \text{ has the } S - KKM_{\mathcal{C}} \text{ property with respect to } \mathcal{C}\right\}. \tag{1.3}$$

We extended the $S - KKM_{\mathcal{C}}$ property to the almost $S - KKM_{\mathcal{C}}$ property, as follows.

Definition 1.8. Let Z be a nonempty set, let X be an almost $\mathcal{C}$-convex subset of an abstract convex uniform space $(E, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has a symmetric basis $\mathcal{N}$, and let Y be a topological space. If $S : Z \to 2^X$, $T : X \to 2^Y$ and $F : Z \to 2^Y$ are three multifunctions satisfying for each $A \in \langle Z \rangle$, each $B \in \langle S(A) \rangle$, and each $U \in \mathcal{N}$, there exists a $\mathcal{C}$-convex-inducing mapping $h_{B,U} : B \to W$ such that

$$T(co_{\mathcal{C}}(h_{B,U}(B))) \subset F(A), \tag{1.4}$$

then F is called an almost S-$KKM_{\mathcal{C}}$ mapping with respect to T. If the multifunction $T : X \to 2^Y$ satisfies the requirement that for any almost S-$KKM_{\mathcal{C}}$ mapping F with respect to T, the family $\{\overline{F(x)} : x \in Z\}$ has the finite intersection property, then T is said to have the almost S-$KKM_{\mathcal{C}}$ property with respect to $\mathcal{C}$. We define

$$\begin{aligned} &al - S - KKM_{\mathcal{C}}(Z, X, Y) \\ &\quad := \left\{T : W \to 2^Y \mid T \text{ has the almost } S - KKM_{\mathcal{C}} \text{ property with respect to } \mathcal{C}\right\}. \end{aligned} \tag{1.5}$$

From the above definitions, we have the following proposition of the $al - S - KKM_{\mathcal{C}}(Z, X, Y)$ family.

Proposition 1.9. *Let X be a nonempty set, let Y be an almost $\mathcal{C}$-convex subset of an abstract convex uniform space $(E, \mathcal{C})$, let Z and W be two topological spaces, and let $S : X \to 2^Y$ be a multifunction. If $T \in al-S-KKM_{\mathcal{C}}(X, Y, Z)$ and if $f : Z \to W$ is continuous, then $fT \in al-S-KKM_{\mathcal{C}}(X, Y, W)$.*

The Φ-mappings and the Φ-spaces, in an abstract convex space setting, were also introduced by Amini et al. [5].

Definition 1.10 (see [5]). Let $(X, \mathcal{C})$ be an abstract convex space, and Y a topological space. A map $T : Y \to 2^X$ is called a Φ-mapping if there exists a multifunction $F : Y \to 2^X$ such that

(i) for each $y \in Y$, $A \in \langle F(y) \rangle$ implies $co_{\mathcal{C}}(A) \subset T(y)$, and

(ii) $Y = \cup_{x \in X} \operatorname{int} F^{-1}(x)$.

The mapping F is called a companion mapping of T.

Furthermore, if the abstract convex space $(X, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has a symmetric basis $\mathcal{N}$, then X is called a Φ-space if for each entourage $V \in \mathcal{N}$, there exists a Φ-mapping $T : X \to 2^X$ such that $\mathcal{G}_T \subset V$.

Remark 1.11. (i) If $T : Y \to 2^X$ is a Φ-mapping, then for each nonempty subset Y_1 of Y, $T|_{Y_1} : Y_1 \to X$ is also a Φ-mapping.

(ii) It is easy to see that if $X_1 \subset X$ and $\mathcal{C}_1 = \{C \cap X_1 : C \in \mathcal{C}\}$, then $(X_1, \mathcal{C}_1)$ is also a Φ-space.

In order to establish the main result of this paper for the multifunctions with the almost $S - KKM_{\mathcal{C}}$ property, we need the following definitions concerning the almost Φ-mappings and the almost Φ-spaces.

Definition 1.12. Let X be an almost $\mathcal{C}$-convex subset of an abstract convex uniform space $(E, \mathcal{C})$ which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has a symmetric base family $\mathcal{N}$, and Y a topological space. A map $T : Y \to 2^X$ is called an almost Φ-mapping if there exists a multifunction $F : Y \to 2^X$ such that

(i) for each $y \in Y$, $A \in \langle F(y) \rangle$ and $U \in \mathcal{N}$, there exists a $\mathcal{C}$-convex-inducing $h_{A,U} : A \to X$ such that $co_{\mathcal{C}}(h_{A,U}(A)) \subset U[T(y)]$, and

(ii) $Y = \cup_{x \in X} \operatorname{int} F^{-1}(x)$.

The mapping F is called an almost companion mapping of T.

Furthermore, X is called an almost Φ-space, if, for each entourage $V \in \mathcal{N}$, there exists an almost Φ-mapping $T : X \to 2^X$ such that $\mathcal{G}_T \subset V$.

Definition 1.13. Let X be an almost Φ-space, and let $T : X \to 2^X$. We say that T has the approximate fixed point property if, for each $U \in \mathcal{N}$, there exists $x \in X$ such that $U[x] \cap T(x) \neq \phi$.

2. Main Results

Using the above introduced concepts and definitions, we now state our main theorem.

Theorem 2.1. *Let X be an almost Φ-space, and let $s : X \to X$ be a surjective single-valued function. If $T \in al - s - KKM_{\mathcal{C}}(X, X, X)$ is compact, then T has the approximate fixed point property.*

Proof. Let $\mathcal{N}$ be a symmetric basis of the uniform structure, and let $U \in \mathcal{N}$. Take $V \in \mathcal{N}$ such that $V \circ V \subset U$. Then, by the definition of the almost Φ-space, there exists an almsot Φ-mapping $F : X \to 2^X$ such that $\mathcal{G}_F \subset V$. Since F is an almsot Φ-mapping, there exists an almost companion mapping $G : X \to 2^X$ such that $X = \cup_{x\in X} \operatorname{int} G^{-1}(x)$.

Let $K = \overline{T(X)}$. Then K is compact, since T is compact. Hence there exists $A \in \langle X\rangle$ such that $K \subset \cup_{x\in A} \operatorname{int} G^{-1}(x)$. Since s is surjective, there exists a finite subset B of X such that $K \subset \cup_{z\in B} \operatorname{int} G^{-1}(s(z))$.

Now, we define $P : X \to 2^X$ by

$$P(z) = K \setminus \operatorname{int} G^{-1}(s(z)), \text{ for each } z \in X. \tag{2.1}$$

By the definition of P, we obtain that P is not an almost $s - KKM_{\mathcal{C}}$ mapping with respect to T. Hence, there exist $N = \{z_1, z_2, \ldots, z_k\} \subset X$ and $D \in \langle s(N)\rangle$ such that for any $\mathcal{C}$-convex-inducing $h_{D,V} : D \to W_\infty$, we have

$$T(co_{\mathcal{C}}(h_{D,V}(D))) \not\subseteq \cup_{i=1}^k P(z_i). \tag{2.2}$$

So, for any $\mathcal{C}$-convex-inducing $h_{D,V} : D \to X$, there exist $x_U \in co_{\mathcal{C}}(h_{D,V}(D))$ and $y_U \in T(x_U)$ such that $y_U \notin \cup_{i=1}^k P(z_i)$. Consequently, $y_U \in \cap_{i=1}^k \operatorname{int} G^{-1}(s(z_i))$, and so $s(z_i) \in G(y_U)$ for all $i = 1, 2, \ldots, k$. Since F is an almost Φ-mapping, there exists a $\mathcal{C}$-convex-inducing $h^*_{D,V} : D \to X$ such that $co_{\mathcal{C}}(h^*_{D,V}(D)) \subset V[F(y_U)]$. So $x_U \in ad_{\mathcal{C}}(h^*_{D,V}(D))$ and $x_U \in V[F(y_U)]$. Thus, there exists $z_U \in F(y_U)$ such that $x_U \in V[z_U]$. Since X is an almost Φ-space, we have $(y_U, z_U) \in \mathcal{G}_F \subset V$, and so $(y_U, x_U) = (y_U, z_U) \circ (z_U, x_U) \in V \circ V \subset U$, that is, $y_U \in U[x_U]$. Therefore, $y_U \in U[x_U] \cap T(x_U)$. The proof is finished. □

Remark 2.2. In the case, if X is a Φ-space and $T \in s - KKM_{\mathcal{C}}(X, X, X)$, then the above theorem reduces to Amini et al. [5, Theorem 2.5]

From Theorem 2.1 above, we obtain immediately the following fixed point theorem.

Theorem 2.3. *Suppose that all of the assumptions of Theorem 2.1 hold. If T is closed, then T has a fixed point in X.*

Proof. By Theorem 2.1, for each $U \in \mathcal{N}$, there exist $x_U, y_U \in X$ such that $y_U \in U[x_U] \cap T(x_U)$. Since T is compact, without loss of generality, we may assume that y_U converges to some $\overline{y}$ in X; then x_U also converges to $\overline{y}$ since X is a Hausdorff uniform space and $(x_U, y_U) \in U$ for each $U \in \mathcal{N}$. By the closedness of T, we have that $\overline{y} \in T(\overline{y})$. □

Corollary 2.4. *Let X be an almost Φ-space, and let $s : X \to X$ be a surjective single-valued function. Suppose $T \in al - s - KKM_{\mathcal{C}}(X, X, X)$ such that $\overline{T(X)}$ is totally bounded. Then T has the approximate fixed point property.*

Corollary 2.5. *Suppose that all of the assumptions of the above Corollary 2.5 hold. If T is closed, then T has a fixed point in X.*

In case X is an almost convex subset of Hausdorff topological vector spaces and for each $A \subset X$, we have

(i) $co_{\mathcal{C}}(A) = co(A)$, and

(ii) $al - s - KKM_{\mathcal{C}}(X, X, X) = al - s - KKM(X, X, X)$.

This allows us to state the following results.

Theorem 2.6. *Let E be a Hausdorff locally convex space, let X be an almost convex subset of E, and let $s : X \to X$ be a surjective function. Assume that $T \in al - s - KKM(X, X, X)$ is compact and closed, then T has a fixed point in X.*

Proof. Let $\mathcal{C}$ be the family of all convex subsets of E, and let $\mathcal{B}_0 = \{\overline{V}_\alpha : \alpha \in \Lambda\}$ be a local basis of E such that each $\overline{V}_\alpha \in \mathcal{B}_0$ is symmetric and convex for each $\alpha \in \Lambda$. For each $x \in X$, we set $V_\alpha[x] = x + \overline{V}_\alpha$. Noting that $x \in V_\alpha[x]$. Set

$$\mathcal{N} = \left\{V_\alpha \mid V_\alpha = \cup_{x \in X}\left\{(x, y) : y \in V_\alpha[x]\right\},\ \alpha \in \Lambda\right\}. \tag{2.3}$$

Then $\mathcal{N}$ is a basis of a uniformity of E. For each $V_\beta \in \mathcal{N}$, $\beta \in \Lambda$, we define the two set-valued mappings $G, F : X \to 2^X$ by $G(x) = F(x) = V_\beta[x]$ for each $x \in X$. Then we have

(i) for each $x \in X$, $co(G(x)) = co(V_\beta[x]) \subset V_\beta[V_\beta[x]] = V_\beta[F(x)]$, and

(ii) $X = \cup_{x \in X} \operatorname{int} G^{-1}(x)$.

So, G is an almost companion mapping of F. This implies that F is an almost Φ-mapping such that $\mathcal{G}_F \subset V_\beta$. Therefore, X is an almost Φ-space.

All conditions of Theorems 2.1 and 2.3 are therefore fulfilled; the result follows from an argument similar to those in the proofs of Theorems 2.1 and 2.3. □

Theorem 2.7. *Let E be a topological vector space, let X be an almost convex subset of E, and let $s : X \to X$ be a surjective function. Suppose that $T \in al - s - KKM(X, X, X)$ is compact, then for any symmetric convex neighborhood $\overline{V}$ of 0 in E, there is $x_V \in X$ such that $(x_V + \overline{V}) \cap T(x_V) \neq \phi$.*

Proof. Let $\mathcal{C}$ be the family of all convex subsets of E, and let $\mathcal{B}_0 = \{a\overline{V} : a > 0\}$ be a new local basis of E. We will use $\mathcal{B}_0$ to construct a weaker topology on E such that E becomes a new topological vector space. For each $x \in X$, we set $V_a[x] = x + a\overline{V}$. Noting that $x \in V_a[x]$. Set

$$\mathcal{N} = \left\{V_a \mid V_a = \cup_{x \in X}\left\{(x, y) : y \in V_a[x]\right\},\ a > 0\right\}. \tag{2.4}$$

Then $\mathcal{N}$ is a basis of a uniformity of E. In vein of the reasonings similar to those of Theorems 2.1 and 2.6, we complete the proof. □

References

[1] B. Knaster, C. Kuratowski, and S. Mazurkiewicz, "Ein Beweis des Fixpunksatzes fur n-dimensionale Simplexe," *Fundamenta Mathematicae*, vol. 14, pp. 132–137, 1929.

[2] K. Fan, "A generalization of Tychonoff's fixed point theorem," *Mathematische Annalen*, vol. 142, pp. 305–310, 1961.

[3] T.-H. Chang and C.-L. Yen, "KKM property and fixed point theorems," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 1, pp. 224–235, 1996.

[4] T.-H. Chang, Y.-Y. Huang, and J.-C. Jeng, "Fixed-point theorems for multifunctions in S-KKM class," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 8, pp. 1007–1017, 2001.

[5] A. Amini, M. Fakhar, and J. Zafarani, "Fixed point theorems for the class *S-KKM* mappings in abstract convex spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 1, pp. 14–21, 2007.

[6] H. Ben-El-Mechaiekh, P. Deguire, and A. Granas, "Points fixes et cocidences pour les fonctions multivoques (applications de type ϕ et ϕ^*). II," *Comptes Rendus de l'Académie des Sciences*, vol. 295, no. 5, pp. 381–384, 1982.

[7] C. D. Horvath, "Contractibility and generalized convexity," *Journal of Mathematical Analysis and Applications*, vol. 156, no. 2, pp. 341–357, 1991.

[8] F. E. Browder, "The fixed point theory of multi-valued mappings in topological vector spaces," *Mathematische Annalen*, vol. 177, pp. 283–301, 1968.

[9] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J.-V. Llinares, "Abstract convexity and fixed points," *Journal of Mathematical Analysis and Applications*, vol. 222, no. 1, pp. 138–150, 1998.

[10] S. Park, "Fixed points of better admissible maps on generalized convex spaces," *Journal of the Korean Mathematical Society*, vol. 37, no. 6, pp. 885–899, 2000.

[11] J.-H. Kim and S. Park, "Comments on some fixed point theorems in hyperconvex metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 291, no. 1, pp. 154–164, 2004.

[12] D. C. Kay and E. W. Womble, "Axiomatic convexity theory and relationships between the Caratheodory, Helly, and Radon numbers," *Pacific Journal of Mathematics*, vol. 38, pp. 471–485, 1971.

[13] J.-V. Llinares, "Abstract convexity, some relations and applications," *Optimization*, vol. 51, no. 6, pp. 797–818, 2002.

[14] J. L. Kelly, *General Topology*, Van Nostrand, Princeton, NJ, USA, 1955.