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Review Article

Some Generalizations of Fixed Point Theorems in Cone Metric Spaces

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We generalize, extend, and improve some recent fixed point results in cone metric spaces including the results of H. Guang and Z. Xian (2007); P. Vetro (2007); M. Abbas and G. Jungck (2008); Sh. Rezapour and R. Hamlbarani (2008). In all our results, the normality assumption, which is a characteristic of most of the previous results, is dispensed. Consequently, the results generalize several fixed results in metric spaces including the results of G. E. Hardy and T. D. Rogers (1973), R. Kannan (1969), G. Jungck, S. Radenovic, S. Radojevic, and V. Rakocevic (2009), and all the references therein.

1. Introduction

The recently discovered applications of ordered topological vector spaces, normal cones and topical functions in optimization theory have generated a lot of interest and research in ordered topological vector spaces (e.g., see [1, 2]). Recently, Huang and Zhang [3] introduced cone metric spaces, which is a generalization of metric spaces, by replacing the real numbers with ordered Banach spaces. They later proved some fixed point theorems for different contractive mappings. Their results have been generalized by different authors (e.g. see [4–7]). This paper generalizes, extends and improves the results of all those authors.

The following definitions are given in [3].

Let E be a real Banach space and P a subset of E. P is called a cone if and only if

(i) P is closed, nonempty, and $P \neq \{0\}$;

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \bigcap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called *normal* if there is $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$.

The least positive number M satisfying the above is called the normal constant of P.

The cone P is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. In [5] it was shown that every regular cone is normal.

In the sequel we will suppose that E is a metrizable linear topological space whose topology is defined by a real-valued function $F : X \to \mathfrak{R}$ called *F-norm* (see [8]). We will assume that P is a cone in E with $\operatorname{int} P \neq 0$ and $\leq$ is partial ordering with respect to P.

Metrizable linear topological spaces contain metrizable locally convex spaces and normed linear spaces [9]. Therefore our E generalizes the E as a normed linear space used in all the previous results on cone metric spaces.

A cone $P \subseteq E$ is therefore called normal if there is $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $F(x) \leq MF(y)$.

Definition 1.1. Let X be a nonempty set. Suppose that $d : X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 1.2 (see [3]). Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$, and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Clearly, this example shows that cone metric spaces generalize metric spaces.

We now give another example where E is a metrizable linear topological vector space that is not a normed linear space.

Example 1.3. Let $E = \ell^p$, $(0 < p < 1)$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0$, for all $n\}$, (X, ρ) a metric space and $d : X \times X \to E$ defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq 1}$. Then (X, d) is a cone metric space.

Definition 1.4. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent to $x \in X$, that is, $\lim_{n\to\infty} x_n = x$.

Definition 1.5. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

It is shown in [3] that a convergent sequence in a cone metric space (X, d) is a Cauchy sequence.

Definition 1.6. Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X, then X is called a sequentially compact metric space. Furthermore, X is compact if and only if X is sequentially compact. (see also [10]).

Proposition 1.7 (see [3]). *Let (X,d) be a cone metric space, P a normal cone. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x$, $y_n \to y$ as $n \to \infty$. Then*

(i) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$*

(ii) *The limit of $\{x_n\}$ is unique*

(iii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$*

(iv) *$d(x_n, y_n) \to d(x, y)$ as $n \to \infty$*

Huang and Zhang [3] proved the following theorems for E a Banach space.

Theorem 1.8. *Let (X,d) be a complete metric space, P a normal cone with normal constant M. Suppose that the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where $k \in [0,1)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 1.9. *Let (X,d) be a complete metric space, P a normal cone with normal constant M. Suppose that the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq k\big(d(Tx, x) + d(Ty, y)\big), \quad \forall x, y \in X, \tag{1.2}$$

where $k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 1.10. *Let (X,d) be a complete metric space, P a normal cone with normal constant M. Suppose that the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq k\big(d(Tx, y) + d(Ty, x)\big), \quad \forall x, y \in X, \tag{1.3}$$

where $k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Rezapour and Hamlbarani [5] improved on Theorems (1.8–1.10) by proving the same results without the assumption that P is a normal cone. They gave examples of non-normal cones and showed that there are no normal cones with normal constant $M < 1$. Observe that the normal constant M for Example 1.3 is 1.

Vetro [7] recently combined the results of Theorems 1.8 and 1.9 and generalized them to two maps satisfying certain conditions, to obtain the following theorem.

Theorem 1.11. *Let (X,d) be a cone metric space, P a normal cone with normal constant M. Let $f, g : X \to X$ be mappings such that*

$$d(f(x), f(y)) \leq ad(f(x), g(x)) + bd(f(y), y) + cd(g(x), y) \tag{1.4}$$

for all $x, y \in X$ *where* $a, b, c \in [0, 1)$ *and* $a + b + c < 1$. *Suppose*

$$f(g(x)) = g(g(x)) \quad \text{if } f(x) = g(x) \tag{1.5}$$

and $f(X) \subset g(X)$ *and* $f(X)$ *or* $g(X)$ *is a complete subspace of* X, *then the mappings* f *and* g *have a unique common fixed point. Moreover, for any* $x_o \in X$, *the sequence* $\{f(x_n)\}$ *of the initial point* x_o, *where* $\{x_n\} \in X$ *is defined by* $g(x_n) = f(x_{n-1})$ *for all* n, *converges to the fixed point.*

Remark 1.12. The two maps f and g are said to be *weaklycompatible* if they satisfy condition (1.5). This concept was introduced by Huang and Zhang [3] and it is known to be the most general among all commutativity concepts in fixed point theory. For example every pair of weakly commuting self-maps and each pair of compatible self-maps are weakly compatible, but the converse is not always true. In fact, the notion of weakly compatible maps is more general than compatibility of type (A), compatibility of type (B), compatibility of type (C), and compatibility of type (P). For a review of those notions of commutativity, see ([11, 12]).

In Theorem 2.1, we unify Theorems 1.8–1.10 into a single theorem and generalize. In Theorem 2.3, we examine the situation where the sum of the coefficients, rather than less than 1, is actually 1. Theorem 3.1 generalizes Theorem 2.1 to two weakly compatible maps thus extending Theorem 1.11. Furthermore, we remove the assumption of normality of cone P in all our results and extend E to a metrizable linear topological space. Some other consequences follow.

2. Theorems on Single Maps

Theorem 2.1. *Let* (X, d) *be a complete cone metric space and* $f : X \to X$ *be mappings such that*

$$d(f(x), f(y)) \leq a_1 d(f(x), x) + a_2 d(f(y), y) + a_3 d(f(y), x) + a_4 d(f(x), y) + a_5 d(y, x) \tag{2.1}$$

for all $x, y \in X$ *where* $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ *and* $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. *Then the mappings* f *have a unique fixed point. Moreover, for any* $x \in X$, *the sequence* $\{f^n(x)\}$ *converges to the fixed point.*

Proof. We adapt the technique in [13]. Without loss of generality we may assume that $a_1 = a_2$ and $a_3 = a_4$ so that from (2.1), we have

$$\begin{aligned} d(f(x), f(y)) \leq{}& \frac{a_1 + a_2}{2}\left[d(f(x), x) + d(f(y), y)\right] + \frac{a_3 + a_4}{2}\left[d(f(y), x)\right. \\ &\left. + d(f(x), y)\right] + a_5 d(y, x). \end{aligned} \tag{2.2}$$

Set $y = f(x)$ in (2.1) and simplify to obtain

$$d\Big(f(x), f^2(x)\Big) \leq \frac{a_1 + a_5}{1 - a_2} d(x, f(x)) + \frac{a_3}{1 - a_2} d\Big(x, f^2(x)\Big). \tag{2.3}$$

By the triangle inequality, $d(f(x), f^2(x)) \geq d(f^2(x), x) - d(f(x), x)$ and so from (2.3) we get

$$d\left(f^2(x), x\right) - d(f(x), x) \leq \frac{a_1 + a_5}{1 - a_2} d(x, f(x)) + \frac{a_3}{1 - a_2} d\left(x, f^2(x)\right), \tag{2.4}$$

which on simplifying gives

$$d\left(f^2(x), x\right) \leq \frac{1 + a_1 + a_5 - a_2}{1 - a_2 - a_3} d(x, f(x)). \tag{2.5}$$

Substituting (2.5) into (2.3) we obtain

$$d\left(f(x), f^2(x)\right) \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} d(x, f(x)), \tag{2.6}$$

and by symmetry, we may exchange a_1 with a_2 and a_3 with a_4 in (2.6) to obtain

$$d\left(f(x), f^2(x)\right) \leq \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} d(x, f(x)). \tag{2.7}$$

If $\alpha = \min\{(a_1 + a_3 + a_5)/(1 - a_2 - a_3), (a_2 + a_4 + a_5)/(1 - a_1 - a_4)\}$, then

$$d\left(f(x), f^2(x)\right) \leq \alpha d(x, f(x)), \tag{2.8}$$

where $\alpha \in [0, 1)$. Let $m > n$, then in view of (2.8), we obtain

$$\begin{aligned} d(f^m(x), f^n(x)) &\leq d\left(f^m(x), f^{m-1}(x)\right) + \cdots + d\left(f^{n+1}(x), f^n(x)\right) \\ &\leq \alpha^n \left(1 + \alpha + \cdots + \alpha^{m-n}\right) d(x, f(x)) \\ &\leq \frac{\alpha^n}{1 - \alpha} d(x, f(x)). \end{aligned} \tag{2.9}$$

Let $0 \ll c$ be given and choose a natural number N_1 such that $(\alpha^n/(1-\alpha)) d(x, f(x)) \ll c$ for all $n \geq N_1$. Thus,

$$d(f^m(x), f^n(x)) \ll c \tag{2.10}$$

for $n > m$. Therefore, $\{f^n(x)\}_{n \geq 1}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $x^* \in X$ such that $f^n(x) \to x^*$. Choose a natural number N_2 such that for all $n \geq N_2$,

$$\begin{aligned} d(f^n(x), x^*) &\ll \frac{c(1 - (a_2 + a_3))}{2(a_1 + a_4 + 1)}, \\ d(f^{n-1}(x), x^*) &\ll \frac{c(1 - (a_2 + a_3))}{2(a_1 + a_3 + a_5)}. \end{aligned} \tag{2.11}$$

Then

$$\begin{aligned}
d(f(x^*),x^*) &\leq d(f^n(x),f(x^*)) + d(f^n(x),x^*) \\
&\leq a_1 d\Big(f^n(x),f^{n-1}(x)\Big) + a_2 d(f(x^*),x^*) + a_3 d\Big(f(x^*),f^{n-1}(x)\Big) \\
&\quad + a_4 d(f^n(x),x^*) + a_5 d\Big(f^{n-1}(x),x^*\Big) + d(f^n(x),x^*) \\
&\leq a_1 d(f^n(x),x^*) + a_1 d\Big(f^{n-1}(x),x^*\Big) + a_2 d(f(x^*),x^*) \\
&\quad + a_3 d(f(x^*),x^*) + a_3 d\Big(f^{n-1}(x),x^*\Big) + a_4 d(f^n(x),x^*) \\
&\quad + a_5 d\Big(f^{n-1}(x),x^*\Big) + d(f^n(x),x^*) \\
&\leq \frac{a_1+a_3+a_5}{1-(a_2+a_3)} d\Big(f^{n-1}(x),x^*\Big) + \frac{a_1+a_4+1}{1-(a_2+a_3)} d(f^n(x),x^*) \\
&\ll \frac{c}{2} + \frac{c}{2} = c.
\end{aligned} \tag{2.12}$$

Thus, $d(f(x^*),x^*) \ll c/m$, for all $m \geq 1$. So $c/m - d(f(x^*),x^*) \in P$, for all $m \geq 1$. Since $c/m \to 0$ as $m \to \infty$, and P is closed, $-d(f(x^*),x^*) \in P$. But $d(f(x^*),x^*) \in P$ and so $d(f(x^*),x^*) = 0$. Hence $f(x^*) = x^*$. The uniqueness follows from the contractive definition of f in (2.1). □

Remark 2.2. The theorem is valid if we replace the completeness of X with the condition that $f(X)$ is complete. If E is restricted to a normed linear space and $a_1 = a_2 = a_3 = a_4 = 0$ in Theorem 2.1 we have [5, Theorem 2.3]; if $a_3 = a_4 = a_5 = 0$ in Theorem 2.1, we obtain [5, Theorem 2.6]; if $a_1 = a_2 = a_5 = 0$, we obtain [5, Theorem 2.7] and if $a_1 = a_2 = a_3 = 0$, we obtain [5, Theorem 2.8]. Furthermore, if we add the normality assumption to Theorem 2.1, then [3, Theorems 1, 2, and 4] there are special cases of Theorem 2.1.

Thus Theorem 2.1 is both an extension generalization and an improvement of the results of [3, 5].

We now consider the situation where $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ in Theorem 2.1.

Theorem 2.3. *Let (X,d) be a sequentially compact cone metric space and $f : X \to X$ be a continuous mapping such that*

$$\begin{aligned}
d(f(x),f(y)) &< a_1 d(f(x),x) + a_2 d(f(y),y) + a_3 d(f(y),x) + a_4 d(f(x),y) \\
&\quad + a_5 d(y,x),
\end{aligned} \tag{2.13}$$

for all $x,y \in X, x \neq y$ where $a_1,a_2,a_3,a_4,a_5 \in [0,1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 = 1$. Then the mappings f have a unique fixed point.

Proof. We follow the same argument as Theorem 2.1. Without loss of generality, we may assume that $a_1 + a_4$ and $a_2 + a_3$ are less than 1. Hence (2.8) becomes

$$d\Big(f(x),f^2(x)\Big) < d(x,f(x)). \tag{2.14}$$

Since X is sequentially compact, then it is compact [10]. The fact that f is continuous and X is compact implies that $f(X)$ is compact and hence $\inf\{d(x, f(x)) : x \in X\}$ exists and $\inf\{d(x, f(x)) : x \in X\} = d(y, f(y)$ for some $y \in X$. From (2.14), it can be infered that y is fixed under f and uniqueness follows from (2.13). □

Remark 2.4. If $a_1 = a_2 = a_3 = a_4 = 0$, with the additional assumption that P is a regular cone in Theorem 2.3, we obtain [3, Theorem 2]. Thus Theorem 2.3 is both an extension and improvement of [3, Theorem 2].

3. Common Fixed Points

Theorem 3.1. *Let (X, d) be a cone metric space and let $f, g : X \to X$ be mappings such that*

$$\begin{aligned} d(f(x), f(y)) \le{}& a_1 d(f(x), g(x)) + a_2 d(f(y), g(y)) + a_3 d(f(y), g(x)) \\ &+ a_4 d(f(x), g(y)) + a_5 d(g(y), g(x)) \end{aligned} \tag{3.1}$$

for all $x, y \in X$ where $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Suppose f and g are weakly compatible and $f(X) \subset g(X)$ such that $f(X)$ or $g(X)$ is a complete subspace of X, then the mappings f and g have a unique common fixed point. Moreover, for any $x_o \in X$, the sequence $\{x_n\} \subset X$ defined by $g(x_n) = f(x_{n-1})$ for all n, converges to the fixed point.

Proof. Observe that if f satisfies (3.1), it also satisfies

$$\begin{aligned} d(f(x), f(y)) \le{}& kd(f(x), g(x)) + kd(f(y), g(y)) + ld(f(y), g(x)) \\ &+ ld(f(x), g(y)) + md(g(y), g(x)) \end{aligned} \tag{3.2}$$

for all $x, y \in X$ where $k, l, m \in [0, 1)$ and $2k + 2l + m < 1$, $(2k = a_1 + a_2, 2l = a_3 + a_4, a_5 = m)$.

If $f(x_n) = f(x_{n-1})$ for all $n \in N$, then $\{f(x_n)\}$ is a Cauchy sequence. Suppose $f(x_n) \neq f(x_{n-1})$ for all $n \in N$. Using (3.2) and the fact that $g(x_n) = f(x_{n-1})$ for all n, we have

$$\begin{aligned} d(f(x_{n+1}), f(x_n)) \le{}& kd(f(x_{n+1}), f(x_n)) + kd(f(x_n), f(x_{n-1})) \\ &+ ld(f(x_n), f(x_n)) + ld(f(x_{n+1}), f(x_n)) \\ &+ ld(f(x_n), f(x_{n-1})) + md(f(x_{n-1}), f(x_n)) \\ \le{}& \frac{k + l + m}{1 - (k + l)} d(f(x_{n-1}), f(x_n)). \end{aligned} \tag{3.3}$$

Consequently

$$d(f(x_{n+1}), f(x_n)) \le \left(\frac{k + l + m}{1 - (k + l)}\right)^n d(f(x_o), f(x_1)). \tag{3.4}$$

Now, for all $m, n \in N$, with $n > m$, we have

$$
\begin{aligned}
d\big(f(x_n), f(x_m)\big) &\le kd\big(f(x_n), f(x_{n-1})\big) + kd\big(f(x_{n-1}), f(x_{n-2})\big) + \cdots + d\big(f(x_{m+1}), f(x_m)\big) \\
&= \Big(k^{n-1} + k^{n-2} + \cdots k^m\Big) d\big(f(x_o), f(x_1)\big) \\
&\le \frac{k^m}{1-k} df(x_o), f(x_1),
\end{aligned}
\tag{3.5}
$$

where $k = (k + l + m)/(1 - (k + l)) \in [0, 1)$.

Let $0 \ll c$ be given and choose a natural number N_1 such that $(k^m/(1-k))d(x, f(x)) \ll c$ for all $m \ge N_1$. Thus,

$$
d\big(f(x_m), f(x_n)\big) \ll c \tag{3.6}
$$

for $n > m$. Therefore, $\{f(x_n)\}_{n\ge1}$ is a Cauchy sequence. Since $f(\mathrm{X})$ or $g(X)$ is complete, then there exists $x^* \in g(\mathrm{X})$ such that $f(x_n) \to x^*$ and $g(x_n) \to x^*$. Let $y \in \mathrm{X}$ such that $g(y) = x^*$. We claim that $f(y) = g(y)$. From (3.2), we have

$$
\begin{aligned}
d\big(f(x_n), f(y)\big) \le{}& kd\big(f(x_n), g(x_n)\big) + kd\big(f(y), g(y)\big) + ld\big(f(y), g(x_n)\big) \\
&+ ld\big(f(x_n), g(y)\big) + md\big(g(y), g(x_n)\big).
\end{aligned}
\tag{3.7}
$$

As $n \to \infty$ we obtain

$$
\begin{aligned}
d\big(x^*, f(y)\big) &\le kd\big(f(y), g(y)\big) + ld\big(f(y), x^*\big) + ld\big(x^*, g(y)\big) + md\big(g(y), x^*\big) \\
&= (k + l)d\big(x^*, f(y)\big), \quad \text{and hence } x^* = f(y) = g(y).
\end{aligned}
\tag{3.8}
$$

Since $f(y) = g(y)$ and f and g are weakly compatible, then

$$
f(x^*) = f\big(g(y)\big) = g\big(g(y)\big) = g(x^*). \tag{3.9}
$$

Next we show that $x^* = f(x^*) = g(x^*)$. Suppose $f(x^*) \ne x^*$, from (3.2), we have

$$
\begin{aligned}
d\big(f(x^*), f(y)\big) &\le kd\big(f(x^*), g(x^*)\big) + kd\big(f(y), g(y)\big) + ld\big(f(y), g(x^*)\big) \\
&\quad + ld\big(f(x^*), g(y)\big) + md\big(g(y), g(x^*)\big) \\
&= 2ld\big(f(y), g(x^*)\big) = 2ld\big(f(y), f(x^*)\big).
\end{aligned}
\tag{3.10}
$$

This is a contradiction and hence $f(x^*) = x^* = g(x^*)$. Thus x^* is a common fixed point of f and g. The uniqueness follows from (3.1). □

Remark 3.2. (i) If $a_3 = a_4 = 0$ and E is restricted to normed linear spaces in Theorem 3.1, with the additional normality assumption, we obtain the common fixed point Theorem of Vetro [7].

(ii) Suppose E is restricted to normed linear spaces, with the additional normality assumption, if $a_1 = a_2 = a_3 = a_4 = 0$, then Theorem 3.1 gives [4, Theorem 2.1]; if $a_3 = a_4 = a_5 = 0$, we obtain [4, Theorem 2.3], and if $a_1 = a_2 = a_5 = 0$, we obtain [4, Theorem 2.4]. Thus our theorem is both an extension, generalization and an improvement of the results of [4, 7].

(iii) If E is restricted to normed linear spaces, Theorem 3.1 reduces to [14, Theorem 2.8].

(iv) If in Theorem 3.1 we choose choose $g = I_X$ the identity mapping on X, we have Theorem 2.1.

Open Question

Theorem 2.3 was proved for the usual metric space by the author in [15] without the assumptions that f is continuous and X is compact. Is the above Theorem 2.3 still valid if we remove the assumption that f is continuous and X is compact?.

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