Research Article

Fixed Point Theorems for Random Lower Semi-Continuous Mappings

Raúl Fierro, 1, 2 Carlos Martínez, 1 and Claudio H. Morales 3

1 Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Cerro Barón, Valparaíso, Chile
2 Laboratorio de Análisis Estocástico CIMFAV, Universidad de Valparaíso, Casilla 5030, Valparaíso, Chile
3 Department of Mathematics, University of Alabama in Huntsville, Huntsville, AL 35899, USA

Correspondence should be addressed to Claudio H. Morales, morales@math.uah.edu

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We prove a general principle in Random Fixed Point Theory by introducing a condition named \( \mathcal{P} \) which was inspired by some of Petryshyn’s work, and then we apply our result to prove some random fixed points theorems, including generalizations of some Bharucha-Reid theorems.

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1. Introduction

Let \( (X, d) \) be a metric space and \( S \) a closed and nonempty subset of \( X \). Denote by \( 2^X \) (resp., \( C(X) \)) the family of all nonempty (resp., nonempty and closed) subsets of \( X \). A mapping \( T : S \to 2^X \) is said to satisfy \( \text{condition(} \mathcal{D} \text{)} \) if, for every closed ball \( B \) of \( S \) with radius \( r \geq 0 \) and any sequence \( \{x_n\} \) in \( S \) for which \( d(x_n, B) \to 0 \) and \( d(x_n, T(x_n)) \to 0 \) as \( n \to \infty \), there exists \( x_0 \in B \) such that \( x_0 \in T(x_0) \) where \( d(x, B) = \inf\{d(x, y) : y \in B\} \). If \( \Omega \) is any nonempty set, we say that the operator \( T : \Omega \times S \to 2^X \) satisfies \( \text{condition(} \mathcal{D} \text{)} \) if for each \( \omega \in \Omega \), the mapping \( T(\omega, \cdot) : S \to 2^X \) satisfies \( \text{condition(} \mathcal{D} \text{)} \). We should observe that this latter condition is related to a condition that was originally introduced by Petryshyn [1] for single-valued operators, in order to prove existence of fixed points. However, in our case, the condition is used to prove the measurability of a certain operator. On the other hand, in the year 2001, Shahzad (cf. [2]) using an idea of Itoh (cf. [3]), see also ([4]), proved that under a somewhat more restrictive condition, named condition (A), the following result.

**Theorem S.** Let \( S \) be a nonempty separable complete subset of a metric space \( X \) and \( T : \Omega \times C \to C(X) \) a continuous random operator satisfying condition (A). Then \( T \) has a deterministic fixed point if and only if \( T \) has a random fixed point.
We shall show that the above result is still valid if the operator $T$ is only lower semi-continuous. In addition, the assumption that each value $T(x)$ is closed has been relaxed without an extra assumption. Furthermore we state a new condition which generalizes condition (A) and allow us to generalize several known results, such as, Bharucha-Reid [5, Theorem 7], Domínguez Benavides et al. [6, Theorem 3.1] and Shahzad [2, Theorem 2.1].

2. Preliminaries

Let $(\Omega, \mathcal{A})$ be a measurable space and let $(X, d)$ be a metric space. A mapping $F : \Omega \rightarrow 2^X$, is said to be measurable if $F^{-1}(G) = \{ \omega \in \Omega : F(\omega) \cap G \neq \emptyset \}$ is measurable for each open subset $G$ of $X$. This type of measurability is usually called weakly (cf. [7]), but since this is the only type of measurability we use in this paper, we omit the term “weakly”. Notice that if $X$ is separable and if, for each closed subset $C$ of $X$, the set $F^{-1}(C)$ is measurable, then $F$ is measurable.

Let $C$ be a nonempty subset of $X$ and $F : C \rightarrow 2^X$, then we say that $F$ is lower (upper) semi-continuous if $F^{-1}(A)$ is open (closed) for all open (closed) subsets $A$ of $X$. We say that $F$ is continuous if $F$ is lower and upper semi-continuous.

A mapping $F : \Omega \times X \rightarrow Y$ is called a random operator if, for each $x \in X$, the mapping $F(\cdot, x) : \Omega \rightarrow Y$ is measurable. Similarly a multivalued mapping $F : \Omega \times X \rightarrow 2^Y$ is also called a random operator if, for each $x \in X$, $F(\cdot, x) : \Omega \rightarrow 2^Y$ is measurable. A measurable mapping $\xi : \Omega \rightarrow Y$ is called a measurable selection of the operator $F : \Omega \rightarrow 2^Y$ if $\xi(\omega) \in F(\omega)$ for each $\omega \in \Omega$. A measurable mapping $\xi : \Omega \rightarrow X$ is called a random fixed point of the random operator $F : \Omega \times X \rightarrow X$ (or $F : \Omega \times X \rightarrow 2^X$) if for every $\omega \in \Omega$, $\xi(\omega) = F(\omega, \xi(\omega))$ (or $\xi(\omega) \in F(\omega, \xi(\omega))$). For the sake of clarity, we mention that $F(\omega, \xi(\omega)) = F(\omega, \cdot)(\xi(\omega))$.

Let $C$ be a closed subset of the Banach space $X$, and suppose that $F$ is a mapping from $C$ into the topological vector space $Y$. We say the $F$ is demiclosed at $y_0 \in Y$ if, for any sequences $\{x_n\}$ in $C$ and $\{y_n\}$ in $Y$ with $y_n \in F(x_n)$, $\{x_n\}$ converges weakly to $x_0$ and $\{y_n\}$ converges strongly to $y_0$, then it is the case that $x_0 \in C$ and $y_0 \in F(x_0)$. On the other hand, we say that $F$ is hemicom pact if each sequence $\{x_n\}$ in $C$ has a convergent subsequence, whenever $d(x_n, F(x_n)) \rightarrow 0$ as $n \rightarrow \infty$.

3. Main Results

Theorem 3.1. Let $C$ be a closed separable subset of a complete metric space $X$, and let $T : \Omega \times C \rightarrow 2^X$ be measurable in $\omega$ and enjoy condition($\mathcal{D}$). Suppose, for each $\omega \in \Omega$, that $h(\omega, x) = d(x, T(\omega, x))$ is upper semi-continuous and the set

$$F(\omega) := \{ x \in C : x \in T(\omega, x) \} \neq \emptyset.$$  \hfill (3.1)

Then $T$ has a random fixed point.

Proof. Let $Z = \{ z_n \}$ be a countable dense subset of $C$. Define $F : \Omega \rightarrow 2^C$ by $F(\omega) = \{ x \in C : x \in T(\omega, x) \}$. Firstly, we show that $F$ is measurable. To this end, let $B_0$ be an arbitrary closed ball of $C$, and set

$$L(B_0) = \bigcup_{k=1}^{\infty} \left\{ \omega \in \Omega : d(z, T(\omega, z)) < \frac{1}{k} \right\},$$  \hfill (3.2)
where $Z_k = B_k \cap Z$ and $B_k = \{x \in C : d(x, B_0) < 1/k\}$. We claim that $F^{-1}(B_0) = L(B_0)$. To see this, let $\omega \in F^{-1}(B_0)$. Then there exists $x \in B_0$ such that $x \in T(\omega, x)$. Since $h(\omega, \cdot)$ is upper semi-continuous, for each $k \in \mathbb{N}$, there exists $z_{nk} \in Z_k$ such that $d(z_{nk}, T(\omega, z_{nk})) < 1/k$. Therefore $\omega \in L(B_0)$. On the other hand, if $\omega \in L(B_0)$, then there exists a subsequence $\{z_{nk}\}$ of $\{z_n\}$ such that

$$d(z_{nk}, B_0) \leq \frac{1}{k}, \quad d(z_{nk}, T(\omega, z_{nk})) \leq \frac{1}{k} \quad (3.3)$$

for all $k \in \mathbb{N}$. This means that $d(z_{nk}, B_0) \to 0$ and $d(z_{nk}, T(\omega, z_{nk})) \to 0$ as $n \to \infty$. Consequently, by condition $(D)$, there exists $x_0 \in B_0$ such that $x_0 \in T(\omega, x_0)$. Hence $\omega \in F^{-1}(B_0)$. Then we conclude that $F^{-1}(B_0) = L(B_0)$, and thus $F^{-1}(B_0)$ is measurable. To complete the proof, let $G$ be an arbitrary open subset of $C$. Then by the separability of $C$,

$$G = \bigcup_{n=1}^{\infty} B_n \quad \text{where each } B_n \text{ is a closed ball of } C. \quad (3.4)$$

Since $F^{-1}(G) = \bigcup_{n=1}^{\infty} F^{-1}(B_n)$, we conclude that $F$ is measurable. Additionally, we show that $F(\omega)$ is closed for each $\omega \in \Omega$. To see this, let $x_n \in F(\omega)$ such that $x_n \to x \in C$. Then, let $B_0 = \{x\}$ be a degenerated ball centered at $x$ and radius $r = 0$, and since $d(x_n, T(\omega, x_n)) = 0$, condition $(D)$ implies that $x \in T(\omega, x)$. Hence $x \in F(\omega)$ and thus by the Kuratowski and Ryll-Nardzewski Theorem [8], $F$ has a measurable selection $\xi : \Omega \to C$ such that $\xi(\omega) \in T(\omega, \xi(\omega))$ for each $\omega \in \Omega$. \qed

As a consequence of Theorem 3.1, we derive a new result for a lower semi-continuous random operator.

**Theorem 3.2.** Let $C$ be a closed separable subset of a complete metric space $X$, and let $T : \Omega \times C \to 2^X$ be a lower semi-continuous random operator, which enjoys condition $(D)$. Suppose, for each $\omega \in \Omega$, that the set

$$F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset. \quad (3.5)$$

Then $T$ has a random fixed point.

**Proof.** Due to Theorem 3.1, it is enough to show that $h(\omega, \cdot)$ is upper semi-continuous. To see this, we will prove that $A = \{x \in C : d(x, T(\omega, x)) < a\}$ is open in $C$ for $a > 0$. Let $a \in A$ and select $e = a - d(a, T(\omega, a))$. Then there exists $y \in T(\omega, a)$ so that $d(a, y) < e/3 + d(a, T(\omega, a)).$ Since $T(\omega, \cdot)$ is lower semi-continuous, there exists a positive number $r < e/3$ such that $T(\omega, u) \cap B(y; e/3) \neq \emptyset$ for all $u \in B(a; r)$. Hence, we may choose $z_u \in T(\omega, u) \cap B(y; e/3)$ for which,

$$d(u, z_u) \leq d(u, a) + d(a, y) + d(y, z_u) < a, \quad (3.6)$$

and consequently, $d(u, T(\omega, u)) < a$. Therefore, $A$ is open, and proof is complete. \qed
We observe that if the mapping \( h(x) = d(x, T(x)) \) is upper semi-continuous, then not necessarily the mapping \( T \) is lower semi-continuous. Consider the following example.

Let \( T : \mathbb{R} \rightarrow 2^\mathbb{R} \) be defined by

\[
T(x) = \begin{cases} 
1, & x \neq 0 \\
[2, 3], & x = 0.
\end{cases}
\]  

(3.7)

Then \( h(x) = |x - 1| \) for \( x \neq 0 \) while \( h(0) = 2 \), which is upper semi-continuous. On the other hand, \( T \) is not lower semi-continuous.

Now, we derive several consequences of Theorem 3.2. We first obtain an extension of one of the main results of [6].

**Theorem 3.3.** Let \( C \) be a weakly compact separable subset of a Banach space \( X \), and let \( T : \Omega \times C \rightarrow 2^X \) be a lower semi-continuous random operator. Suppose, for each \( \omega \in \Omega \), that \( I - T(\omega, \cdot) \) is demiclosed at 0 and the set

\[
F(\omega) := \{ x \in C : x \in T(\omega, x) \} \neq \emptyset.
\]  

Then \( T \) has a random fixed point.

**Proof.** In order to apply Theorem 3.2, we just need to prove that \( T \) enjoys condition(\( P \)). To this end, let \( \omega \) be fixed in \( \Omega \). Suppose that \( B_0 \) is a closed ball of \( C \) with radius \( r \geq 0 \) where \( \{ x_n \} \) is a sequence in \( C \) such that \( d(x_n, B_0) \rightarrow 0 \) and \( d(x_n, T(\omega, x_n)) \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( C \) is separable, the weak topology on \( C \) is metrizable, and thus there exists a weakly convergent subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \), so that \( x_{n_k} \rightharpoonup x \) weakly, while \( d(x_{n_k}, T(\omega, x_{n_k})) \rightarrow 0 \) as \( k \rightarrow \infty \). Consequently, for each \( k \in \mathbb{N} \), there exists \( z_k \in T(\omega, x_{n_k}) \) such that

\[
\| x_{n_k} - z_k \| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]  

(3.9)

Hence, the demiclosedness of \( I - T(\omega, \cdot) \) implies that \( x \in T(\omega, x) \), and thus \( T(\omega, \cdot) \) enjoys condition(\( P \)).

Before we give an extension of the main result of [4], we observe that condition(\( P \)) is basically applied to those closed balls directly used to prove the measurability of the mapping \( F \), as will be seen in the proof of the next result. \( \square \)

**Theorem 3.4.** Let \( C \) be a closed separable subset of a complete metric space \( X \), and let \( T : \Omega \times C \rightarrow \mathcal{C}(X) \) be a continuous hemi-compact random operator. If, for each \( \omega \in \Omega \), the set

\[
F(\omega) := \{ x \in C : x \in T(\omega, x) \} \neq \emptyset,
\]  

(3.10)

then \( T \) has a random fixed point.

**Proof.** Due to Theorem 3.2, it would be enough to show that \( T(\omega, \cdot) \) enjoys condition(\( P \)) for every \( \omega \in \Omega \). To see this, let \( B_0 \) be a closed ball of \( C \), and let \( \{ x_n \} \) be a sequence in \( C \) such that \( d(x_n, B_0) \rightarrow 0 \) and \( d(x_n, T(\omega, x_n)) \rightarrow 0 \) as \( n \rightarrow \infty \). Then by the hemi-compactness of \( T \), there exists a convergent subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \), so that \( x_{n_k} \rightharpoonup x \in B_0 \). Hence
Fixed Point Theory and Applications

\[ d(x_{n_k}, T(\omega, x_{n_k})) \to 0 \text{ as } k \to \infty. \] This means that, for each \( k \in \mathbb{N} \), there exists \( z_k \in T(\omega, x_{n_k}) \) such that

\[ d(x_{n_k}, z_k) \to 0 \text{ as } k \to \infty. \tag{3.11} \]

Consequently, \( z_k \to x \). On the other hand, since \( T \) is upper semi-continuous at \( x \), for every \( \epsilon > 0 \) there exist \( k_0 \in \mathbb{N} \) such that

\[ T(\omega, x_{n_k}) \subseteq B(T(\omega, x); \epsilon) \text{ for all } k \geq k_0. \tag{3.12} \]

Hence, \( x \in \overline{B}(T(\omega, x); \epsilon) \). Since \( \epsilon \) is arbitrary and \( T(\omega, x) \) is closed, we derive that \( x \in T(\omega, x) \), and thus \( T \) satisfies condition (\( D \)).

**Corollary 3.5.** Let \( C \) be a locally compact separable subset of a complete metric space \( X \), and let \( T : \Omega \times C \to C(X) \) be a continuous random operator. Suppose, for each \( \omega \in \Omega \), that the set

\[ F(\omega) := \{ x \in C : x \in T(\omega, x) \} \neq \emptyset. \tag{3.13} \]

Then \( T \) has a random fixed point.

**Proof.** Let \( G \) be an arbitrary open subset of \( C \), and let \( x \in G \). Since \( C \) is locally compact, there exists a compact ball \( B \) centered at \( x \) such that \( B \subseteq G \). Now, we prove that condition (\( D \)) holds with respect to \( B \). To see this, let \( \omega \in \Omega \), and let \( \{ x_n \} \) be a sequence in \( X \) such that \( d(x_n, B) \to 0 \) and \( d(x_n, T(\omega, x_n)) \to 0 \) as \( n \to \infty \). Then there exists a sequence \( \{ y_n \} \) in \( B \) so that \( d(x_n, y_n) \to 0 \) as \( n \to \infty \). Since \( B \) is compact, there exists a convergent subsequence \( \{ y_{n_k} \} \) of \( \{ y_n \} \) such that \( y_{n_k} \to x \), and consequently \( x_{n_k} \to x \) with \( x \in B \) as well as \( d(x_{n_k}, T(\omega, x_{n_k})) \to 0 \) as \( k \to \infty \). Since \( T \) is upper semi-continuous, we derive, as in the proof of Theorem 3.4, that \( x \in T(x) \). In addition, since \( T \) is lower semi-continuous, we may follow the proof of Theorem 3.1, to conclude that \( F^{-1}(B) \) is measurable. Hence, the separability of \( C \) implies that we can select countably many compact balls \( B_i \) centered at corresponding points \( x_i \in G \) such that

\[ F^{-1}(G) = \bigcup_{i \in \mathbb{N}} F^{-1}(B_i). \tag{3.14} \]

Therefore, \( F \) is measurable. \( \square \)

Next, we get a stochastic version of Schauder’s Theorem, which is also an extension of a Theorem of Bharucha-Reid (see [5, Theorem 10]). We also observe that our proof is much easier and quite short.

**Corollary 3.6.** Let \( C \) be a compact and convex subset of a Fréchet space \( X \), and let \( T : \Omega \times C \to C \) be a continuous random operator. Then \( T \) has a random fixed point.

**Proof.** As we know, every Fréchet space is a complete metric space, and since \( C \) is compact, \( C \) itself is a complete separable metric space. In addition, for each \( \omega \in \Omega \), there exists \( x \in C \) such that \( T(\omega, x) = x \). This means that the set \( F(\omega) \), defined in Theorem 3.1, is nonempty.
Since $C$ is compact, any sequence in $C$ contains a convergent subsequence, which means that $T$ is trivially a hemicompact operator. Consequently, by Theorem 3.4, $T$ has a random fixed point.

Before obtaining an extension of Bharucha-Reid [5, Theorem 3.7], we define a contraction mapping for metric spaces. Let $X$ be a metric space, and let $Ω$ be a measurable space. A random operator $T : Ω \times X \to X$ is said to be a random contraction if there exists a mapping $k : Ω \to [0, 1)$ such that

$$d(T(ω, x), T(ω, y)) \leq k(ω)d(x, y) \quad \text{for all } x, y \in X.$$  \hfill (3.15)

**Theorem 3.7.** Let $X$ be a complete separable metric space, and let $T : Ω \times X \to X$ be a continuous random operator such that $T^2$ is a contraction with constant $k(ω)$ for each $ω \in Ω$. Then $T$ has a unique random fixed point.

**Proof.** For each $ω \in Ω$, the mapping $T^2$ has a unique fixed point, $ξ(ω)$, which is also the unique fixed point of $T$. It remains to show that the mapping $ξ : Ω \to X$ defined by $T(ω, ξ(ω)) = ξ(ω)$ is measurable. To see this, let $f_0 : Ω \to X$ be an arbitrary measurable function. Then, we claim that $T(ω, f_0(ω))$ is measurable. To this end, let $Z = \{z_n\}$ be a countable dense set of $X$. Let $ω \in Ω$ and let $k \in \mathbb{N}$. Define

$$h_k : Ω \to X \quad \text{by} \quad h_k(ω) = z_m,$$  \hfill (3.16)

where $m$ is the smallest natural number for which $d(z_m, f_0(ω)) < 1/k$. Since $f_0$ is measurable, so are the sets $E_m = \{ω \in Ω : d(z_m, f_0(ω)) < 1/k\}$, which, as a matter of fact, conform a disjoint covering of $Ω$. Consequently, $\{h_k\}$ is a sequence of measurable functions that converges pointwise to $f_0$. On the other hand, the range of each $h_k$ is a subset of $Z$, and hence constant on each set $E_m$. Since the mapping $T$ is measurable in $ω$, then, for each $k \in \mathbb{N}$, $T(ω, h_k(ω))$ is also measurable. Therefore the continuity of $T$ on the second variable implies that

$$T(ω, h_k(ω)) \to T(ω, f_0(ω)) \quad \text{as } k \to \infty,$$  \hfill (3.17)

for each $ω \in Ω$. Hence $T(ω, f_0(ω))$ is measurable. Define the sequence

$$f_n(ω) = T(ω, f_{n-1}(ω)), \quad n \in \mathbb{N}.$$  \hfill (3.18)

Then $\{f_n\}$ is a sequence of measurable functions. Since $f_n(ω) = T^n(ω, f_0(ω))$, the fact that $T^2$ is a contraction implies that $f_n(ω) \to ξ(ω)$. Therefore, the mapping $ξ$ is measurable, which completes the proof.

As a direct consequence of Theorem 3.7, we derive the extension mentioned earlier where the space $X$ is more general, and the randomness on the mapping $k$ has been removed.

**Corollary 3.8.** Let $X$ be a complete separable metric space, and let $T : Ω \times X \to X$ be a random contraction operator with constant $k(ω)$ for each $ω \in Ω$. Then $T$ has a unique random fixed point.
Next, one can derive a corollary of the proof of Theorem 3.7, which is a theorem of Hans [9].

**Corollary 3.9.** Let $X$ be a complete separable metric space, and let $T : \Omega \times X \to X$ be a continuous random operator. Suppose, for each $\omega \in \Omega$, that there exists $n \in \mathbb{N}$ such that $T^n$ is a contraction with constant $k(\omega)$. Then $T$ has a unique random fixed point.

**Proof.** As in the proof of the theorem, the mapping $T$ has a unique fixed point for each $\omega \in \Omega$. The rest of the proof follows the proof of the theorem with the appropriate changes of the second power of $T$ by the $n$th power of $T$. \qed

Notice that Theorem 3.7 holds for single-valued operators. The following question is formulated for multivalued operators taking closed and bounded values in $X$.

**Open Question**

Suppose that $X$ is a complete separable metric space, and let $T : \Omega \times X \to CB(X)$ be a continuous random operator such that $T^2$ is a contraction with constant $k(\omega)$ for each $\omega \in \Omega$. Then does $T$ have a unique random fixed point?

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