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Research Article

Convergence Theorems of Fixed Points for a Finite Family of Nonexpansive Mappings in Banach Spaces

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We modify the normal Mann iterative process to have strong convergence for a finite family nonexpansive mappings in the framework of Banach spaces without any commutative assumption. Our results improve the results announced by many others.

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space with the normalized duality mapping J from E into 2^{E^*} give by

$$J(x) = \left\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \|f\| = \|x\|\right\}, \quad \forall x \in E, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We assume that C is a nonempty closed convex subset of E and $T : C \to C$ a mapping. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that T is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (see [1, 2]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in C, \tag{1.2}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [1] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T that is nearest to u. Reich [2] extended Broweder's result to the setting of Banach spaces and proved that if X is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Recall that the normal Mann iterative process was introduced by Mann [3] in 1953. The normal Mann iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \end{aligned} \tag{1.3}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval (0,1). If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.3) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [4]). In an infinite-dimensional Hilbert space, the normal Mann iteration algorithm has only weak convergence, in general, even for nonexpansive mappings. Therefore, many authors try to modify normal Mann's iteration process to have strong convergence for nonexpansive mappings (see, e.g., [5–8] and the references therein).

Recently, Kim and Xu [5] introduced the following iteration process:

$$\begin{aligned} &x_0 = x \in C, \\ &y_n = \beta_n x_n + (1-\beta_n) T x_n, \\ &x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad \forall n \geq 0, \end{aligned} \tag{1.4}$$

where T is a nonexpansive mapping of C into itself and $u \in C$ is a given point. They proved that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a fixed point of T provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Concerning a family of nonexpansive mappings it has been considered by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see, for example, [9]. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance (see, e.g., [10]).

In this paper, we consider the mapping W_n defined by

$$\begin{aligned} U_{n0} &= I, \\ U_{n1} &= \gamma_{n1} T_1 U_{n0} + (1-\gamma_{n1}) I, \\ U_{n2} &= \gamma_{n2} T_2 U_{n1} + (1-\gamma_{n2}) I, \\ &\cdots \\ U_{n,N-1} &= \gamma_{n,N-1} T_{N-1} U_{n,N-2} + (1-\gamma_{n,N-1}) I, \\ W_n = U_{nN} &= \gamma_{nN} T_N U_{n,N-1} + (1-\gamma_{nN}) I, \end{aligned} \tag{1.5}$$

where $\{\gamma_{n1}\}, \{\gamma_{n2}\}, \ldots, \{\gamma_{nN}\}$ are sequences in $(0,1]$. Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\{\gamma_{n1}\}, \{\gamma_{n2}\}, \ldots, \{\gamma_{nN}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Moreover, in [11], it is shown that $F(W_n) = \bigcap_{i=1}^{N} F(T_i)$.

Motivated by Atsushiba and Takahashi [11], Kim and Xu [5], and Shang et al. [7], we study the following iterative algorithm:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n, \quad \forall n \geq 0, \end{aligned} \tag{1.6}$$

where W_n is defined by (1.5) and $u \in C$ is given point. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.6) converges to a common fixed point of the finite family nonexpansive mappings without any commutative assumptions.

In order to prove our main results, we need the following definitions and lemmas.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny (see [12, 13]) provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [12, 13]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality $\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in C$ and $y \in D$.

Reich [2] showed that if E is uniformly smooth and D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Lemma 1.1. *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t \to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$, that is, Q satisfies the property $\langle u - Qu, J(z - Qu)\rangle \leq 0$, for all $u \in C$ and $z \in F(T)$.*

Lemma 1.2 (see [14]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n) y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 1.3. *In a Banach space E, there holds the inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$ for all $x, y \in E$, where $j(x+y) \in J(x+y)$.*

Lemma 1.4 (see [15]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n$, where γ_n is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n\to\infty} \alpha_n = 0$.*

2. Main results

Theorem 2.1. *Let C be a closed convex subset of a uniformly smooth and strictly convex Banach space E. Let T_i be a nonexpansive mapping from C into itself for $i = 1, 2, \ldots, N$. Assume that*

$F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in (0,1), the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\lim_{n\to\infty} |\gamma_{n,i} - \gamma_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.6)*. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F$, where $x^* = Q(u)$ and $Q : C \to F$ is the unique sunny nonexpansive retraction from C onto F.*

Proof. We divide the proof into four parts.

Step 1. First we observe that sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are bounded.

Indeed, take a point $p \in F$ and notice that

$$\|y_n - p\| \leq \beta_n \|x_n - p\| + (1 - \beta_n)\|W_n x_n - p\| \leq \|x_n - p\|. \tag{2.1}$$

It follows that

$$\|x_{n+1} - p\| = \|\alpha_n(u - p) + (1 - \alpha_n)(y_n - p)\| \leq \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\|. \tag{2.2}$$

By simple inductions, we have $\|x_n - p\| \leq \max\{\|x_0 - p\|, \|u - p\|\}$, which gives that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$.

Step 2. In this part, we will claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Put $l_n = (x_{n+1} - \beta_n x_n)/(1 - \beta_n)$. Now, we compute $l_{n+1} - l_n$, that is,

$$x_{n+1} = (1 - \beta_n) l_n + \beta_n x_n, \quad \forall n \geq 0. \tag{2.3}$$

Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1} u + (1 - \alpha_{n+1}) y_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + (1 - \alpha_n) y_n - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}(u - y_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n (u - y_n)}{1 - \beta_n} + W_{n+1} x_{n+1} - W_n x_n, \end{aligned} \tag{2.4}$$

we have

$$\|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|u - y_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|y_n - u\| + \|x_{n+1} - x_n\| + \|W_{n+1} x_n - W_n x_n\|. \tag{2.5}$$

From the proof of Yao [8], we have

$$\|W_{n+1} x_n - W_n x_n\| \leq M_1 \sum_{i=1}^{N} |\gamma_{n+1,i} - \gamma_{n,i}|, \tag{2.6}$$

where M_1 is an appropriate constant. Substituting (2.6) into (2.5), we have

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|u - y_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|y_n - u\| + M \sum_{i=1}^{N} |\gamma_{n+1,i} - \gamma_{n,i}|. \tag{2.7}$$

Observing the conditions (i)–(iii), we get $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. We can obtain $\lim_{n\to\infty} \|l_n - x_n\| = 0$ easily by Lemma 1.2. Observe that (2.3) yields $x_{n+1} - x_n = (1 - \beta_n)(l_n - x_n)$. Therefore, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.8}$$

Step 3. We will prove $\lim_{n\to\infty}\|W_n x_n - x_n\| = 0$.

Observing that $x_{n+1} - y_n = \alpha_n(u - y_n)$ and the condition (i), we can easily get

$$\lim_{n\to\infty}\left\|y_n - x_{n+1}\right\| = 0. \tag{2.9}$$

On the other hand, we have $\|y_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$. Combining (2.8) with (2.9), we have

$$\lim_{n\to\infty}\left\|y_n - x_n\right\| = 0. \tag{2.10}$$

Notice that $\|W_n x_n - x_n\| \leq \|x_n - y_n\| + \beta_n\|x_n - W_n x_n\|$. This implies $(1-\beta_n)\|W_n x_n - x_n\| \leq \|x_n - y_n\|$. From the condition (iii) and (2.10), we obtain

$$\lim_{n\to\infty}\left\|W_n x_n - x_n\right\| = 0. \tag{2.11}$$

Step 4. Finally, we will show $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$.

First, we claim that

$$\limsup_{n\to\infty}\left\langle u - x^*, J\left(x_n - x^*\right)\right\rangle \leq 0, \tag{2.12}$$

where $x^* = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto tu + (1-t)W_n x$. Then x_t solves the fixed point equation $x_t = tu + (1-t)W_n x_t$. Thus we have

$$\|x_t - x_n\| = \|(1-t)(W_n x_t - x_n) + t(u - x_n)\|. \tag{2.13}$$

It follows from Lemma 1.3 that

$$\begin{aligned}\left\|x_t - x_n\right\|^2 = \left\|(1-t)\left(W_n x_t - x_n\right) + t\left(u - x_n\right)\right\|^2 &\leq (1-2t+t)^2\left\|x_t - x_n\right\|^2 \\ &+ f_n(t) + 2t\left\langle u - x_t, J\left(x_t - x_n\right)\right\rangle + 2t\left\langle x_t - x_n, J\left(x_t - x_n\right)\right\rangle,\end{aligned} \tag{2.14}$$

where

$$f_n(t) = \left(2\left\|x_t - x_n\right\| + \left\|x_n - W_n x_n\right\|\right)\left\|x_n - W_n x_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow 0. \tag{2.15}$$

It follows from (2.14) that

$$\left\langle x_t - u, J\left(x_t - x_n\right)\right\rangle \leq \frac{t}{2}\left\|x_t - x_n\right\| + \frac{1}{2t}f_n(t). \tag{2.16}$$

Letting $n \to \infty$ in (2.16) and noting (2.15) yield

$$\limsup_{n\to\infty}\left\langle x_t - u, J\left(x_t - x_n\right)\right\rangle \leq \frac{t}{2}M_2, \tag{2.17}$$

where M_2 is an appropriate constant. Taking $t \longrightarrow 0$ in (2.17), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\left\langle x_t - u, J\left(x_t - x_n\right)\right\rangle \leq 0. \tag{2.18}$$

On the other hand, we have

$$\begin{aligned}\langle u-x^*, J(x_n-q)\rangle = {} & \langle u-x^*, J(x_n-q)\rangle - \langle u-x^*, J(x_n-x_t)\rangle + \langle u-x^*, J(x_n-x_t)\rangle \\ & - \langle u-x_t, J(x_n-x_t)\rangle + \langle u-x_t, J(x_n-x_t)\rangle.\end{aligned} \tag{2.19}$$

It follows that

$$\begin{aligned}&\limsup_{n\to\infty}\langle u-x^*, J(x_n-q)\rangle \\ &\quad \leq \sup_{n\in N}\langle u-x^*, J(x_n-q)-J(x_n-x_t)\rangle + \|x_t-x^*\|\limsup_{n\to\infty}\|x_n-x_t\| + \limsup_{n\to\infty}\langle u-x_t, J(x_n-x_t)\rangle.\end{aligned} \tag{2.20}$$

Noticing that J is norm-to-norm uniformly continuous on bounded subsets of C and from (2.18), we have $\lim_{t\to 0}\sup_{n\in N}\langle u-x^*, J(x_n-q)-J(x_n-x_t)\rangle = 0$. It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-x^*, J(x_n-q)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle u-x^*, J(x_n-q)\rangle \\ &\leq \limsup_{t\to 0}\limsup_{n\to\infty}\langle u-x_t, J(x_n-x_t)\rangle \leq 0.\end{aligned} \tag{2.21}$$

Hence, (2.12) holds. Now, from Lemma 1.3, we have

$$\|x_{n+1}-x^*\|^2 \leq (1-\alpha_n)\|x_n-x^*\|^2 + 2\alpha_n\langle u-x^*, J(x_{n+1}-x^*)\rangle. \tag{2.22}$$

Applying Lemma 1.4 to (2.22) we have $x_n \to q$ as $n \to \infty$. □

Remark 2.2. Theorem 2.1 improves the results of Kim and Xu [5] from a single nonexpansive mapping to a finite family of nonexpansive mappings.

Remark 2.3. If $f : C \to C$ is a contraction map and we replace u by $f(x_n)$ in the recursion formula (1.6), we obtain what some authors now call viscosity iteration method. We note that our theorem in this paper carries over trivially to the so-called viscosity process. Therefore, our results also include Yao et al. [16] as a special case.

Remark 2.4. Our results partially improve Shang et al. [7] from a Hilbert space to a Banach space.

Remark 2.5. If W_n is a single nonexpansive mapping, then the strict convexity of E may not be needed.

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