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Research Article

Strong Convergence Theorems for Nonexpansive Semigroups without Bochner Integrals

Satit Saejung

Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

Correspondence should be addressed to Satit Saejung, saejung@kku.ac.th

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We prove a convergence theorem by the new iterative method introduced by Takahashi et al. (2007). Our result does not use Bochner integrals so it is different from that by Takahashi et al. We also correct the strong convergence theorem recently proved by He and Chen (2007).

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let $\{T(t) : t \geq 0\}$ be a family of mappings from a subset C of H into itself. We call it a nonexpansive semigroup on C if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in C$;

(2) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;

(3) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;

(4) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$.

Motivated by Suzuki's result [1] and Nakajo-Takahashi's results [2], He and Chen [3] recently proved a strong convergence theorem for nonexpansive semigroups in Hilbert spaces by hybrid method in the mathematical programming. However, their proof of the main result ([3, Theorem 2.3]) is very questionable. Indeed, the existence of the subsequence $\{s_j\}$ such that (2.16) of [3] are satisfied, that is,

$$s_j \longrightarrow 0, \quad \frac{\|x_j - T(s_j)x_j\|}{s_j} \longrightarrow 0, \tag{1.1}$$

needs to be proved precisely. So, the aim of this short paper is to correct He-Chen's result and also to give a new result by using the method recently introduced by Takahashi et al.

We need the following lemma proved by Suzuki [4, Lemma 1].

Lemma 1.1. *Let $\{t_n\}$ be a real sequence and let τ be a real number such that $\liminf_n t_n \leq \tau \leq \limsup_n t_n$. Suppose that either of the following holds:*

(i) $\limsup_n (t_{n+1} - t_n) \leq 0$, *or*

(ii) $\liminf_n (t_{n+1} - t_n) \geq 0$.

Then τ is a cluster point of $\{t_n\}$. Moreover, for $\varepsilon > 0$, $k, m \in \mathbb{N}$, there exists $m_0 \geq m$ such that $|t_j - \tau| < \varepsilon$ for every integer j with $m_0 \leq j \leq m_0 + k$.

2. Results

2.1. The shrinking projection method

The following method is introduced by Takahashi et al. in [5]. We use this method to approximate a common fixed point of a nonexpansive semigroup without Bochner integrals as was the case in [5, Theorem 4.4].

Theorem 2.1. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C with a nonempty common fixed point F, that is, $F = \cap_{t\geq 0} F(T(t)) \neq \varnothing$. Suppose that $\{x_n\}$ is a sequence iteratively generated by the following scheme:*

$$\begin{gathered} x_0 \in H \text{ taken arbitrary}, \\ C_1 = C, \\ x_1 = P_{C_1}(x_0), \\ y_n = \alpha_n x_n + (1-\alpha_n) T(t_n) x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{gathered} \tag{2.1}$$

where $\{\alpha_n\} \subset [0, a] \subset [0, 1)$, $\liminf_n t_n = 0$, $\limsup_n t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. Then $x_n \to P_F(x_0)$.

Proof. It is well known that F is closed and convex. We first show that the iterative scheme is well defined. To see that each C_n is nonempty, it suffices to show that $F \subset C_n$. The proof is by induction. Clearly, $F \subset C_1$. Suppose that $F \subset C_k$. Then, for $z \in F \subset C_k$,

$$\begin{aligned} \|y_k - z\| &\leq \alpha_k \|x_k - z\| + (1-\alpha_k)\|T(t_k)x_k - z\| \\ &\leq \alpha_k \|x_k - z\| + (1-\alpha_k)\|x_k - z\| \\ &= \|x_k - z\|. \end{aligned} \tag{2.2}$$

That is, $z \in C_{k+1}$ as required.

Notice that

$$\widehat{C}_n := \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \tag{2.3}$$

is convex since

$$\|y_n - z\| \leq \|x_n - z\| \Longleftrightarrow 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2. \tag{2.4}$$

This implies that each subset $C_n = C \cap \widehat{C}_1 \cap \cdots \cap \widehat{C}_{n-1}$ is convex. It is also clear that C_n is closed. Hence the first claim is proved.

Next, we prove that $\{x_n\}$ is bounded. As $x_n = P_{C_n}(x_0)$,

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall z \in C_n. \tag{2.5}$$

In particular, for $z \in F \subset C_n$ for all $n \in \mathbb{N}$, the sequence $\{x_n - x_0\}$ is bounded and hence so is $\{x_n\}$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. As $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n}(x_0)$,

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n. \tag{2.6}$$

Moreover, since the sequence $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} \|x_n - x_0\| \text{ exists.} \tag{2.7}$$

Note that

$$\langle x_0 - x_n, x_n - v\rangle \geq 0 \quad \forall v \in C_n. \tag{2.8}$$

In particular, since $x_{n+k} \in C_{n+k} \subset C_n$ for all $k \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+k} - x_n\|^2 &= \|x_{n+k} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+k} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+k} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{2.9}$$

It then follows from the existence of $\lim_n \|x_n - x_0\|^2$ that $\{x_n\}$ is a Cauchy sequence. In fact, for $\varepsilon > 0$, there exists a natural number N such that, for all $n \geq N$,

$$\left| \|x_n - x_0\|^2 - a \right| < \frac{\varepsilon}{2}, \tag{2.10}$$

where $a = \lim_n \|x_n - x_0\|^2$. In particular, if $n \geq N$ and $k \in \mathbb{N}$, then

$$\begin{aligned} \|x_{n+k} - x_n\|^2 &\leq \|x_{n+k} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\leq a + \frac{\varepsilon}{2} - \left(a - \frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned} \tag{2.11}$$

Moreover,

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{2.12}$$

We now assume that $x_n \to p$ for some $p \in C$. Now since $\alpha_n \leq a < 1$ for all $n \in \mathbb{N}$ and $x_{n+1} \in C_n$,

$$\begin{aligned}\|x_n - T(t_n)x_n\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\leq \frac{1}{1-a}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\leq \frac{2}{1-a}\|x_{n+1} - x_n\| \longrightarrow 0.\end{aligned} \tag{2.13}$$

The last convergence follows from (2.12). We choose a sequence $\{t_{n_k}\}$ of positive real number such that

$$t_{n_k} \longrightarrow 0, \quad \frac{1}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| \longrightarrow 0. \tag{2.14}$$

We now show that how such a special subsequence can be constructed. First we fix $\delta > 0$ such that

$$\liminf_n t_n = 0 < \delta < \limsup_n t_n. \tag{2.15}$$

From (2.13), there exists $m_1 \in \mathbb{N}$ such that $\|T(t_n)x_n - x_n\| < 1/3^2$ for all $n \geq m_1$. By Lemma 1.1, $\delta/2$ is a cluster point of $\{t_n\}$. In particular, there exists $n_1 > m_1$ such that $\delta/3 < t_{n_1} < \delta$. Next, we choose $m_2 > n_1$ such that $\|T(t_n)x_n - x_n\| < 1/4^2$ for all $n \geq m_2$. Again, by Lemma 1.1, $\delta/3$ is a cluster point of $\{t_n\}$ and this implies that there exists $n_2 > m_2$ such that $\delta/4 < t_{n_2} < \delta/2$. Continuing in this way, we obtain a subsequence $\{n_k\}$ of $\{n\}$ satisfying

$$\|T(t_{n_k})x_{n_k} - x_{n_k}\| < \frac{1}{(k+2)^2}, \quad \frac{\delta}{k+2} < t_{n_k} < \frac{\delta}{k} \quad \forall k \in \mathbb{N}. \tag{2.16}$$

Consequently, (2.14) is satisfied.

We next show that $p \in F$. To see this, we fix $t > 0$,

$$\begin{aligned}&\|x_{n_k} - T(t)p\| \\ &\leq \sum_{j=0}^{[t/t_{n_k}]-1} \|T(jt_{n_k})x_{n_k} - T((j+1)t_{n_k})x_{n_k}\| \\ &\quad + \left\|T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)x_{n_k} - T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)p\right\| + \left\|T\left(\left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)p - T(t)p\right\| \\ &\leq \left[\frac{t}{t_{n_k}}\right]\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - p\| + \left\|T\left(t - \left[\frac{t}{t_{n_k}}\right]t_{n_k}\right)p - p\right\| \\ &\leq \frac{t}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - p\| + \sup\{\|T(s)p - p\| : 0 \leq s \leq t_{n_k}\}.\end{aligned} \tag{2.17}$$

As $x_{n_k} \to p$ and (2.14), we have $x_{n_k} \to T(t)p$ and so $T(t)p = p$.

Finally, we show that $p = P_F(x_0)$. Since $F \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}(x_0)$,

$$\|x_{n+1} - x_0\| \leq \|q - x_0\| \quad \forall n \in \mathbb{N}, \ q \in F. \tag{2.18}$$

But $x_n \to p$; we have

$$\|p - x_0\| \leq \|q - x_0\| \quad \forall q \in F. \tag{2.19}$$

Hence $p = P_F(x_0)$ as required. This completes the proof. □

2.2. The hybrid method

We consider the iterative scheme computing by the hybrid method (some authors call the CQ-method). The following result is proved by He and Chen [3]. However, the important part of the proof seems to be overlooked. Here we present the correction under some additional restriction on the parameter $\{t_n\}$.

Theorem 2.2. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C with a nonempty common fixed point F, that is, $F = \cap_{t\geq 0} F(T(t)) \neq \varnothing$. Suppose that $\{x_n\}$ is a sequence iteratively generated by the following scheme:*

$$\begin{gathered}
x_0 \in C \text{ taken arbitrary}, \\
y_n = \alpha_n x_n + (1 - \alpha_n) T(t_n) x_n, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - x_0, z - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{2.20}$$

where $\{\alpha_n\} \subset [0, a] \subset [0, 1)$, $\liminf_n t_n = 0$, $\limsup_n t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. Then $x_n \to P_F(x_0)$.

Proof. For the sake of clarity, we give the whole sketch proof even though some parts of the proof are the same as [3]. To see that the scheme is well defined, it suffices to show that both C_n and Q_n are closed and convex, and $C_n \cap Q_n \neq \varnothing$ for all $n \in \mathbb{N}$. It follows easily from the definition that C_n and Q_n are just the intersection of C and the half-spaces, respectively,

$$\begin{aligned}
\widehat{C}_n &:= \left\{z \in H : 2\langle x_n - y_n, z \rangle \leq \|x_n\|^2 - \|y_n\|^2 \right\}, \\
\widehat{Q}_n &:= \{z \in H : \langle x_n - x_0, z - x_n \rangle \geq 0\}.
\end{aligned} \tag{2.21}$$

As in the proof of the preceding theorem, we have $F \subset C_n$ for all $n \in \mathbb{N}$. Clearly, $F \subset C = Q_1$. Suppose that $F \subset Q_k$ for some $k \in \mathbb{N}$, we have $p \in C_k \cap Q_k$. In particular, $\langle x_{k+1} - x_0, p - x_{k+1} \rangle \geq 0$, that is, $p \in Q_{k+1}$. It follows from the induction that $F \subset Q_n$ for all $n \in \mathbb{N}$. This proves the claim.

We next show that $x_n - T(t_n)x_n \to 0$. To see this, we first prove that

$$x_{n+1} - x_n \longrightarrow 0. \tag{2.22}$$

As $x_{n+1} \in Q_n$ and $x_n = P_{Q_n}(x_0)$,

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n \in \mathbb{N}. \tag{2.23}$$

For fixed $z \in F$. It follows from $F \subset Q_n$ for all $n \in \mathbb{N}$ that

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall n \in \mathbb{N}. \tag{2.24}$$

This implies that sequence $\{x_n\}$ is bounded and

$$\lim_{n \to \infty} \|x_n - x_0\| \text{ exists.} \tag{2.25}$$

Notice that

$$\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0. \tag{2.26}$$

This implies that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0. \end{aligned} \tag{2.27}$$

It then follows from $x_{n+1} \in C_n$ that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$ and hence

$$\begin{aligned} \|T(t_n)x_n - x_n\| &= \frac{1}{\alpha_n}\|y_n - x_n\| \\ &\leq \frac{1}{\alpha_n}\left(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|\right) \longrightarrow 0. \end{aligned} \tag{2.28}$$

As in Theorem 2.1, we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that

$$x_{n_k} \xrightarrow{w} p \in C, \; t_{n_k} \longrightarrow 0, \quad \frac{1}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| \longrightarrow 0. \tag{2.29}$$

Consequently, for any $t > 0$,

$$\|x_{n_k} - T(t)p\| \leq \frac{t}{t_{n_k}}\|x_{n_k} - T(t_{n_k})x_{n_k}\| + \|x_{n_k} - p\| + \sup\left\{\|T(s)p - p\| : 0 \leq s \leq t_{n_k}\right\}. \tag{2.30}$$

This implies that

$$\limsup_{k \to \infty}\|x_{n_k} - T(t)p\| \leq \limsup_{k \to \infty}\|x_{n_k} - p\|. \tag{2.31}$$

In virtue of Opial's condition of H, we have $p = T(t)p$ for all $t > 0$, that is, $p \in F$. Next, we observe that

$$\|x_0 - P_F(x_0)\| \leq \|x_0 - p\| \leq \liminf_{k \to \infty}\|x_0 - x_{n_k}\| \leq \limsup_{k \to \infty}\|x_0 - x_{n_k}\| \leq \|x_0 - P_F(x_0)\|. \tag{2.32}$$

This implies that

$$\lim_{k \to \infty}\|x_0 - x_{n_k}\| = \|x_0 - P_F(x_0)\| = \|x_0 - p\|. \tag{2.33}$$

Consequently,

$$x_{n_k} \longrightarrow P_F(x_0) = p. \tag{2.34}$$

Hence the whole sequence must converge to $P_F(x_0) = p$, as required. □

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