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Research Article

Weak Convergence Theorems of Three Iterative Methods for Strictly Pseudocontractive Mappings of Browder-Petryshyn Type

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Let E be a real q-uniformly smooth Banach space which is also uniformly convex (e.g., L_p or l_p spaces $(1 < p < \infty)$), and K a nonempty closed convex subset of E. By constructing nonexpansive mappings, we elicit the weak convergence of Mann's algorithm for a κ-strictly pseudocontractive mapping of Browder-Petryshyn type on K in condition thet the control sequence $\{\alpha_n\}$ is chosen so that (i) $\mu \leq \alpha_n < 1, n \geq 0$; (ii) $\sum_{n=0}^{\infty}(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] = \infty$, where $\mu \in [\max\{0, 1-(q\kappa/C_q)^{1/(q-1)}\}, 1)$. Moreover, we consider to find a common fixed point of a finite family of strictly pseudocontractive mappings and consider the parallel and cyclic algorithms for solving this problem. We will prove the weak convergence of these algorithms.

1. Introduction

Let E be a real Banach space and let J_q $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by $J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\}$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. If E^* is strictly convex then J_q is single-valued. In the sequel, we will denote the single-valued generalized duality mapping by j_q and $F(T) = \{x \in E : Tx = x\}$.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* of Browder-Petryshyn type [1], if for all $x, y \in D(T)$, there exists $\kappa \in [0, 1)$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^q - \kappa\|x-y-(Tx-Ty)\|^q. \tag{1.1}$$

(If (1.1) holds, we also say that T is κ-strictly pseudocontractive.)

Remark 1.2. If I denotes the identity operator, then (1.1) can be written in the form

$$\langle (I-T)x-(I-T)y, j_q(x-y)\rangle \geq \kappa \|(I-T)x-(I-T)y\|^q. \tag{1.2}$$

In Hilbert spaces, (1.1) (and hence (1.2)) is equivalent to the inequality

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|x-y-(Tx-Ty)\|^2, \quad k=(1-2\kappa)<1, \tag{1.3}$$

and we can assume also that $k \geq 0$, so that $k \in [0,1)$. Note that the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings which are mappings T on $D(T)$ such that $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in D(T)$. That is, T is nonexpansive if and only if T is 0-strictly pseudocontractive.

The class of strictly pseudocontractive mappings has been studied by several authors (see, e.g., [1–7]). However their iterative methods are far less developed though Browder and Petryshyn [1] initiated their work in 1967. As a matter of fact, strictly pseudocontractive mappings have more powerful applications in solving inverse problems (see Scherzer [8]). Therefore it is interesting to develop the theory of iterative methods for strictly pseudocontractive mappings.

Browder and Petryshyn proved the following theorem.

Theorem BP (see [1]). *Let H be a real Hilbert space and K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be a κ-strictly pseudocontractive map. Then for any fixed $\gamma \in (1-\kappa, 1)$, the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = \gamma x_n + (1-\gamma)Tx_n, \quad n \geq 1 \tag{1.4}$$

converges weakly to a fixed point of T.

Recently Marino and Xu [9] have extended Browder and Petryshyn's above-mentioned result by proving that the sequence $\{x_n\}$ generated by the following Mann's algorithm [10]:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0. \tag{1.5}$$

Theorem MX (see [9]). *Let K be a closed convex subset of a Hilbert space H. Let $T : K \to K$ be a κ-strictly pseudocontractive mapping for some $0 \leq \kappa < 1$ and $F(T) \neq \varnothing$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm* (1.5). *Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\kappa < \alpha_n < 1$ for all n and*

$$\sum_{n=0}^{\infty} (\alpha_n - \kappa)(1-\alpha_n) = \infty. \tag{1.6}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

Meanwhile, Marino and Xu raised the open question: whether Theorem MX can be extended to Banach spaces which are uniformly convex and have a Frechet differentiable norm. As a partial affirmative answer, Osilike and Udomene [2] proved the following theorem.

Theorem OU. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and let $T : K \to K$ be a κ-strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the conditions:*

(i*) $0 \le \alpha_n \le 1,\ n \ge 0$;

(ii*) $0 < a \le \alpha_n \le b < (q\kappa/C_q)^{1/(q-1)},\ n \ge 0$ *and for some constants* $a, b \in (0,1)$.

Then, the sequence $\{x_n\}$ is generated by the Mann's algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{1.7}$$

converges weakly to a fixed point of T.

We would like to point out that Osilike's and Udomene's condition (ii^*) excludes the natural choice $1 - 1/n$ for α_n. This is overcome by our paper. We prove that if α_n satisfies the conditions

$$\begin{gathered} \mu \le \alpha_n < 1; \\ \sum_{n=0}^{\infty} (1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right] = \infty; \end{gathered} \tag{1.8}$$

where $\mu \in [\max\{0, 1 - (q\kappa/C_q)^{1/(q-1)}\}, 1)$, then the iterative sequence (1.5) converges weakly to a fixed point of T.

Moreover, we are concerned with the problem of finding a point x such that

$$x \in \bigcap_{i=1}^{N} F(T_i), \tag{1.9}$$

where $N \ge 1$ is a positive integer and $\{T_i\}_{i=1}^N$ are N strictly pseudocontractive mappings defined on a closed convex subset K of a real Banach space E which is q-uniformly smooth and uniformly convex. Assume that $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i = 1$. We will show that the sequence $\{x_n\}$ generated by the following parallel algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i T_i x_n, \quad n \ge 0 \tag{1.10}$$

will converge weakly to a solution to the problem (1.9).

We will consider a more general situation by allowing the weights $\{\lambda_i\}_{i=1}^N$ in (1.10) to depend on n, the number of steps of the iteration. That is we consider the algorithm which generates a sequence $\{x_n\}$ in the following way:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n, \quad n \ge 0. \tag{1.11}$$

Under appropriate assumptions on the sequences of the weights $\{\lambda_i^{(n)}\}_{i=1}^N$ we will also prove the weak convergence, to a solution of the problem (1.9), of the algorithm (1.11).

Another approach to the problem (1.9) is the cyclic algorithm [11]. (For convenience, we relabel the mappings $\{T_i\}_{i=1}^N$ as $\{T_i\}_{i=0}^{N-1}$.) This means that beginning with an $x_0 \in K$, we define the sequence $\{x_n\}$ cyclically by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n, \quad n \geq 0, \tag{1.12}$$

where $T_{[n]} = T_i$, with $i = n(\mathrm{mod}N)$, $0 \leq i \leq N-1$. We will show that this cyclic algorithm (1.12) is also weakly convergent if the sequence $\{\alpha_n\}$ of parameters is appropriately chosen.

We will use the notations:

(1) $\rightharpoonup$ for weak convergence;

(2) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

Let E be a real Banach space. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1,\ \|y\| \leq \tau\right\}. \tag{2.1}$$

E is *uniformly smooth* if and only if $\lim_{\tau\to 0}(\rho_E(\tau)/\tau) = 0$.

Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. Hilbert spaces, L_p (or l_p) spaces ($1 < p < \infty$), and the Sobolev spaces, W_m^p ($1 < p < \infty$) are q-uniformly smooth. Hilbert spaces are 2 uniformly smooth, while

$$L_p \left(\text{or } l_p\right) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth if } 1 < p \leq 2, \\ 2\text{-uniformly smooth if } p \geq 2. \end{cases} \tag{2.2}$$

Theorem HKX (see [12, page 1130]). *Let $q > 1$ and let E be a real q-uniformly smooth Banach space. Then there exists a constant $C_q > 0$ such that for all $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q\ . \tag{2.3}$$

E is said to have a Frechet differentiable norm if for all $x \in U = \{x \in E : \|x\| = 1\}$

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.4}$$

exists and is attained uniformly in $y \in U$. In this case there exists an increasing function $b : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0}b(t) = 0$ such that for all $x, h \in E$,

$$\frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle \leq \frac{1}{2}\|x+h\|^2 \leq \frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle + b(\|h\|). \tag{2.5}$$

It is well known (see, e.g., [13, page 107]) that q-uniformly smooth Banach space has a Frechet differentiable norm.

Lemma 2.1 (see [2]). *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T : K \to K$ a strictly pseudocontractive mapping of Browder-Petryshyn type. Then $(I - T)$ is demiclosed at zero, that is, $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{(I - T)x_n\}$ converges strongly to 0, then $Tx = x$.*

Lemma 2.2 (see [14, 15]). *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{\delta_n\}_{n=1}^{\infty}$ be nonnegative sequences satisfying the following inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad \forall n \geq 1. \tag{2.6}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let T be a self-mapping on K with $F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence satisfying the following conditions:*

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists for every* $p \in F(T)$;

(b) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$;

(c) $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ *exists for all* $t \in [0,1]$ *and for all* $p_1, p_2 \in F(T)$.

Then, the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, then $\{x_n\}$ is bounded. By (b) and Lemma 2.1, we have $\omega_{\mathcal{W}}(x_n) \subset F(T)$. Assume that $p_1, p_2 \in \omega_{\mathcal{W}}(x_n)$ and that $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p_1$ and $x_{m_j} \rightharpoonup p_2$, respectively. Since E is a real q-uniformly smooth Banach space which is also uniformly convex, then E has a Frechet differentiable norm. Set $x = p_1 - p_2$, $h = t(x_n - p_1)$ in (2.5), we obtain

$$\begin{aligned} &\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle \\ &\quad \leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(t\|x_n - p_1\|), \end{aligned} \tag{2.7}$$

where b is increasing. Since $\|x_n - p_1\| \leq M$, for all $n \geq 0$, for some $M > 0$, then

$$\begin{aligned} &\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle \\ &\quad \leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM). \end{aligned} \tag{2.8}$$

Therefore,

$$\begin{aligned} &\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle \\ &\quad \leq \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2 \leq \frac{1}{2}\|p_1 - p_2\|^2 + t\liminf_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM). \end{aligned} \tag{2.9}$$

Hence $\limsup_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM)/t$. Since $\lim_{t\to 0^+} b(tM)/t = 0$, then $\lim_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle$ exists. Since $\lim_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle = \langle p - p_1, j(p_1 - p_2)\rangle$, for all $p \in \omega_{\mathcal{W}}(x_n)$. Set $p = p_2$. We have $\langle p_2 - p_1, j(p_1 - p_2)\rangle = \|p_2 - p_1\|^2 = 0$, that is, $p_2 = p_1$. Hence $\omega_{\mathcal{W}}(x_n)$ is singleton, so that $\{x_n\}$ converges weakly to a fixed point of T. □

3. Mann's algorithm

Theorem 3.1. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let $T : K \to K$ be a κ-strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the condition* (1.8)*. Given $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm* (1.5)*. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $\beta_n = (\alpha_n - \mu)/(1-\mu)$. Since $\alpha_n \in (\mu, 1)$, then $\beta_n \in (0,1)$. We compute

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Tx_n = [\mu + (1-\mu)\beta_n]x_n + (1-\mu)(1-\beta_n)Tx_n \\ &= \beta_n x_n + (1-\beta_n)[\mu x_n + (1-\mu)Tx_n] = \beta_n x_n + (1-\beta_n)S_\mu x_n, \end{aligned} \tag{3.1}$$

where $S_\mu = \mu I + (1-\mu)T$. We will show that S_μ is a nonexpansive mapping and that $F(S_\mu) = F(T)$. Indeed, it follows from (1.2) and (2.3) that

$$\begin{aligned} \|S_\mu x - S_\mu y\|^q &= \|\mu x + (1-\mu)Tx - [\mu y + (1-\mu)Ty]\|^q = \|x - y - (1-\mu)[x - y - (Tx - Ty)]\|^q \\ &\le \|x-y\|^q - q(1-\mu)\langle (I-T)x - (I-T)y, j_q(x-y)\rangle + C_q(1-\mu)^q\|x-y-(Tx-Ty)\|^q \\ &\le \|x-y\|^q - q\kappa(1-\mu)\|x-y-(Tx-Ty)\|^q + C_q(1-\mu)^q\|x-y-(Tx-Ty)\|^q \\ &= \|x-y\|^q - (1-\mu)[q\kappa - C_q(1-\mu)^{q-1}]\|x-y-(Tx-Ty)\|^q. \end{aligned} \tag{3.2}$$

When $1 - (q\kappa/C_q)^{1/(q-1)} \le \mu < 1$, we have $\|S_\mu x - S_\mu y\|^q \le \|x-y\|^q$, that is, S_μ is nonexpansive. On the other hand, for all $x \in F(S_\mu)$, $x = S_\mu x = \mu x + (1-\mu)Tx$. Then $x = Tx$, that is, $x \in F(T)$.

Now we show that $\|x_n - S_\mu x_n\|$ is decreasing. By (3.1), we have

$$\begin{aligned} \|x_{n+1} - S_\mu x_{n+1}\| &= \|\beta_n x_n + (1-\beta_n)S_\mu x_n - S_\mu x_{n+1}\| \\ &= \|\beta_n(x_n - S_\mu x_n) + \beta_n(S_\mu x_n - S_\mu x_{n+1}) + (1-\beta_n)(S_\mu x_n - S_\mu x_{n+1})\| \\ &\le \beta_n\|x_n - S_\mu x_n\| + \|S_\mu x_n - S_\mu x_{n+1}\| \le \beta_n\|x_n - S_\mu x_n\| + \|x_n - x_{n+1}\| \\ &= \beta_n\|x_n - S_\mu x_n\| + (1-\beta_n)\|x_n - S_\mu x_n\| = \|x_n - S_\mu x_n\|, \\ \|x_n - Tx_n\| &= \frac{1}{1-\alpha_n}\|x_{n+1} - x_n\| = \frac{1-\beta_n}{1-\alpha_n}\|x_n - S_\mu x_n\| = \frac{1}{1-\mu}\|x_n - S_\mu x_n\|. \end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\|x_n - Tx_n\| = \frac{1}{1-\mu}\|x_n - S_\mu x_n\| \le \frac{1}{1-\mu}\|x_{n-1} - S_\mu x_{n-1}\| = \|x_{n-1} - Tx_{n-1}\|. \tag{3.4}$$

Hence $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists.

Pick a $p \in F(T)$. We then show that the real sequence $\{\|x_n - p\|\}_{n=0}^{\infty}$ is decreasing, hence $\lim_{n\to\infty}\|x_n - p\|$ exists. To see this, using (1.2) and (2.3), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^q &= \|x_n - p - (1-\alpha_n)[x_n - p - (Tx_n - p)]\|^q \\ &\le \|x_n - p\|^q - q(1-\alpha_n)\langle x_n - p - (Tx_n - p), j_q(x_n - p)\rangle + C_q(1-\alpha_n)^q\|x_n - p - (Tx_n - p)\|^q \\ &\le \|x_n - p\|^q - q\kappa(1-\alpha_n)\|x_n - p - (Tx_n - p)\|^q + C_q(1-\alpha_n)^q\|x_n - p - (Tx_n - p)\|^q \\ &= \|x_n - p\|^q - (1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}]\|x_n - p - (Tx_n - p)\|^q. \end{aligned} \tag{3.5}$$

Then

$$(1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right]\left\|x_n - p - (Tx_n - p)\right\|^q \le \left\|x_n - p\right\|^q - \left\|x_{n+1} - p\right\|^q. \tag{3.6}$$

Since $\mu \le \alpha_n < 1$ for all n, where $\mu \in [\max\{0, 1-(q\kappa/C_q)^{1/(q-1)}\}, 1)$, we get $(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] \ge 0$. Therefore, (3.6) implies the sequence $\{\|x_n - p\|\}$ is decreasing (and hence $\lim_{n\to\infty}\|x_n - p\|$ exists). It follows from (3.6) that

$$\sum_{n=0}^{\infty}(1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right]\left\|x_n - p - (Tx_n - p)\right\|^q < \left\|x_0 - p\right\|^q < \infty. \tag{3.7}$$

Since $\sum_{n=0}^{\infty}(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] = \infty$, then (3.7) implies that

$$\liminf_{n\to\infty}\left\|x_n - Tx_n\right\| = 0. \tag{3.8}$$

Thus

$$\lim_{n\to\infty}\left\|x_n - Tx_n\right\| = 0. \tag{3.9}$$

Then we prove that for all $p_1, p_2 \in F(T)$, $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. Let $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$. It is obvious that $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}\|x_n - p_2\|$ exist. So we only need to consider the case of $t \in (0,1)$. Define $T_n : K \to K$ by

$$T_n x = \alpha_n x + (1-\alpha_n)Tx, \quad x \in K. \tag{3.10}$$

Then for all $x, y \in K$,

$$\begin{aligned}\left\|T_n x - T_n y\right\|^q &\le \|x-y\|^q - q(1-\alpha_n)\left\langle (I-T)x - (I-T)y, j_q(x-y)\right\rangle + C_q(1-\alpha_n)^q\left\|x-y-(Tx-Ty)\right\|^q \\ &\le \|x-y\|^q - (1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right]\left\|x-y-(Tx-Ty)\right\|^q.\end{aligned} \tag{3.11}$$

By the choice of α_n, we have $(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] \ge 0$, so it follows that $\|T_n x - T_n y\| \le \|x-y\|$. Set $S_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_n$, $m \ge 1$. We have

$$\begin{gathered}\left\|S_{n,m}x - S_{n,m}y\right\| \le \|x-y\| \quad \forall x, y \in K, \\ S_{n,m}x_n = x_{n+m}, \qquad S_{n,m}p = p \quad \forall p \in F(T).\end{gathered} \tag{3.12}$$

Set $b_{n,m} = \|S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\|$. Let δ denote the *modulus of convexity* of E. If $\|x_n - p_1\| = 0$ for some n_0, then $x_n = p_1$ for any $n \ge n_0$ so that $\lim_{n\to\infty}\|x_n - p_1\| = 0$, in fact $\{x_n\}$ converges strongly to $p_1 \in F(T)$. Thus we may assume $\|x_n - p_1\| > 0$ for any $n \ge 0$. It is well known (see, e.g., [16, page 108]) that

$$\left\|tx + (1-t)y\right\| \le 1 - 2\min\{t, (1-t)\}\delta(\|x-y\|) \le 1 - 2t(1-t)\delta(\|x-y\|) \tag{3.13}$$

for all $t \in [0,1]$ and for all $x, y \in E$ such that $\|x\| \le 1,\ \|y\| \le 1$. Set

$$w_{n,m} = \frac{S_{n,m}p_1 - S_{n,m}(tx_n + (1-t)p_1)}{t\|x_n - p_1\|},$$
$$z_{n,m} = \frac{S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}x_n}{(1-t)\|x_n - p_1\|}. \tag{3.14}$$

Then $\|w_{n,m}\| \le 1$ and $\|z_{n,m}\| \le 1$ so that it follows from (3.13) that

$$2t(1-t)\delta(\|w_{n,m} - z_{n,m}\|) \le 1 - \|tw_{n,m} + (1-t)z_{n,m}\|. \tag{3.15}$$

Observe that

$$\|w_{n,m} - z_{n,m}\| = \frac{b_{n,m}}{t(1-t)\|x_n - p_1\|},$$
$$\|tw_{n,m} + (1-t)z_{n,m}\| = \frac{\|S_{n,m}x_n - S_{n,m}p_1\|}{\|x_n - p_1\|}, \tag{3.16}$$

it follows from (3.15) that

$$2t(1-t)\|x_n - p_1\|\delta\left(\frac{b_{n,m}}{t(1-t)\|x_n - p_1\|}\right) \le \|x_n - p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\| = \|x_n - p_1\| - \|x_{n+m} - p_1\|. \tag{3.17}$$

Since E is uniformly convex, then $\delta(s)/s$ is nondecreasing, and since $\|x_n - p\|$ is decreasing, hence it follows from (3.17) that

$$\frac{\|x_0 - p_1\|}{2}\delta\left(\frac{4}{\|x_0 - p_1\|}b_{n,m}\right) \le \|x_n - p_1\| - \|x_{n+m} - p_1\| \quad \left(\text{since } t(1-t) \le \frac{1}{4}\ \forall t \in [0,1]\right). \tag{3.18}$$

Since $\delta(0) = 0$ and $\lim_{n\to\infty}\|x_n - p\|$ exists, then the continuity of yields $\lim_{n\to\infty} b_{n,m} = 0$ uniformly for all m. Observe that

$$\begin{aligned}
a_{n+m}(t) &\le \|tx_{n+m} + (1-t)p_1 - p_2 + (S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1)\| \\
&\quad + \|S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\| \\
&= \|S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}p_2\| + b_{n,m} \le \|tx_n + (1-t)p_1 - p_2\| + b_{n,m} = a_n(t) + b_{n,m}.
\end{aligned} \tag{3.19}$$

Hence $\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)$, this ensures that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in (0,1)$.

Now apply Lemma 2.3 to conclude that $\{x_n\}$ converges weakly to a fixed point of T. □

Remark 3.2. In particular, set $q = 2$, $C_q = 1$, our result reduces to Theorem MX. Moreover, if T is nonexpansive, then $\kappa = 0$ and our Theorem 3.1. reduces to Reich's theorem [17].

4. Parallel algorithm

The following proposition lists some useful properties for strictly pseudocontractive mappings.

Proposition 4.1. *Let K be a closed convex subset of a Banach space E. Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : K \to K$ is a κ_i-strictly pseudocontractive mapping for some $0 \leq \kappa_i < 1$. Assume $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then.*

(i) *$\sum_{i=1}^N \lambda_i T_i$ is a κ-strictly pseudocontractive mapping, with $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$.*

(ii) *Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point. Then*

$$F\left(\sum_{i=1}^N \lambda_i T_i\right) = \bigcap_{i=1}^N F(T_i). \tag{4.1}$$

Proof. To prove (i), we only need to consider the case of $N = 2$ (the general case can be proved by induction). Set $G = (1-\lambda)T_1 + \lambda T_2$, where $\lambda \in (0,1)$ and for $i = 1,2$, T_i is a κ_i-strictly pseudocontractive mapping. Set $\kappa = \min\{\kappa_1, \kappa_2\}$;

$$\begin{aligned}
&\langle Gx - Gy, j_q(x-y)\rangle \\
&\quad\leq (1-\lambda)\langle T_1x - T_1y, j_q(x-y)\rangle + \lambda\langle T_2x - T_2y, j_q(x-y)\rangle \\
&\quad\leq (1-\lambda)\left[\|x-y\|^q - \kappa_1\left\|(I-T_1)x-(I-T_1)y\right\|^q\right] + \lambda\left[\|x-y\|^q - \kappa_2\left\|(I-T_2)x-(I-T_2)y\right\|^q\right] \\
&\quad\leq \|x-y\|^q - \kappa\left[(1-\lambda)\left\|(I-T_1)x-(I-T_1)y\right\|^q + \lambda\left\|(I-T_2)x-(I-T_2)y\right\|^q\right] \\
&\quad\leq \|x-y\|^q - \kappa\left\|(I-G)x-(I-G)y\right\|^q.
\end{aligned} \tag{4.2}$$

Hence G is a κ-strictly pseudocontractive mapping.

To prove (ii), again we can assume $N = 2$. It suffices to prove that $F(G) \subset F(T_1) \cap F(T_2)$, where $G = (1-\lambda)T_1 + \lambda T_2$, with $\lambda \in (0,1)$. Let $x \in F(G)$ and take $z \in F(T_1) \cap F(T_2)$ to deduce that

$$\begin{aligned}
\|x-z\|^q &= (1-\lambda)\langle T_1x - z, j_q(x-z)\rangle + \lambda\langle T_2x - z, j_q(x-z)\rangle \\
&\leq (1-\lambda)\left[\|x-z\|^q - \kappa\left\|(I-T_1)x-(I-T_1)z\right\|^q\right] + \lambda\left[\|x-z\|^q - \kappa\left\|(I-T_2)x-(I-T_2)z\right\|^q\right] \\
&= \|x-z\|^q - \kappa\left[(1-\lambda)\left\|(I-T_1)x\right\|^q + \lambda\left\|(I-T_2)x\right\|^q\right].
\end{aligned} \tag{4.3}$$

Since $\kappa > 0$, we get $(1-\lambda)\|(I-T_1)x\|^q + \lambda\|(I-T_2)x\|^q \leq 0$. This together with $0 < \lambda < 1$ implies that $T_1x = x$ and $T_2x = x$. Thus $x \in F(T_1) \cap F(T_2)$. □

Theorem 4.2. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : K \to K$ be a κ_i-strictly pseudocontractive mapping for some $0 \leq \kappa_i < 1$. Let $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^N F(T_i)$ is nonempty. Assume also $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i = 1$. Given $x_0 \in K$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by Mann's algorithm* (1.10)*:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^N \lambda_i T_i x_n, \quad n \geq 0. \tag{4.4}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ *be a real sequence satisfying the conditions* (1.8). *Then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^{N}$.

Proof. Put

$$A = \sum_{i=1}^{N} \lambda_i T_i. \tag{4.5}$$

Then by Proposition 4.1, A is a κ-strictly pseudocontractive mapping and $F(A) = \bigcap_{i=1}^{N} F(T_i)$.

We can rewrite the algorithm (1.10) as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) A x_n, \quad n \geq 0. \tag{4.6}$$

Now apply Theorem 3.1 to conclude that sequence $\{x_n\}$ converges weakly to a fixed point of A. □

Theorem 4.3. *Let* E *be a real* q*-uniformly smooth Banach space which is also uniformly convex and let* K *be a nonempty closed convex subset of* E. *Let* $N \geq 1$ *be an integer. Let, for each* $1 \leq i \leq N$, $T_i : K \to K$ *be a* κ_i*-strictly pseudocontractive mapping for some* $0 \leq \kappa_i < 1$. *Let* $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$. *Assume the common fixed point set* $\bigcap_{i=1}^{N} F(T_i)$ *is nonempty. Assume also for each* n, $\{\lambda_i^{(n)}\}_{i=1}^{N}$ *is a finite sequence of positive numbers such that* $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \lambda_i^{(n)} > 0$ *for all* $1 \leq i \leq N$. *Given* $x_0 \in K$, *let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by the algorithm* (1.11):

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n, \quad n \geq 0. \tag{4.7}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ *be a real sequence satisfying the condition* (1.8). *Assume also that*

$$\sum_{n=0}^{\infty} \left(\sum_{i=1}^{N} \left| \lambda_i^{(n+1)} - \lambda_i^{(n)} \right| \right) < \infty. \tag{4.8}$$

Then $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^{N}$.

Proof. Write, for each $n \geq 1$,

$$A_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_i. \tag{4.9}$$

By Proposition 4.1, each A_n is a κ-strictly pseudocontractive mapping with $F(A_n) = \bigcap_{i=1}^{N} F(T_i)$, and the algorithm (1.11) can be rewritten as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) A_n x_n, \quad n \geq 0. \tag{4.10}$$

As Theorem 3.1, if set $\beta_n = (\alpha_n - \mu)/(1-\mu)$, then $\{x_n\}_{n=0}^{\infty}$ can also be generated by the following algorithm:

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_{\mu,n} x_n, \tag{4.11}$$

where $S_{\mu,n} = \mu I + (1-\mu) A_n$ and $S_{\mu,n}$ is a nonexpansive mapping with $F(S_{\mu,n}) = F(A_n)$. Similarly, we can prove that $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in \bigcap_{i=1}^{N} F(T_i)$, and that

$$\liminf_{n\to\infty} \left\| x_n - A_n x_n \right\| = 0. \tag{4.12}$$

Since we can write $A_{n+1}x_{n+1} = A_n x_{n+1} + y_n$, where $y_n = \sum_{i=1}^{N}(\lambda_i^{(n+1)} - \lambda_i^{(n)})T_i x_{n+1}$, then by (4.11) we obtain

$$\begin{aligned}\|x_{n+1} - S_{\mu,n+1}x_{n+1}\| &= \|\beta_n x_n + (1-\beta_n)S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1}\| \\ &= \|\beta_n(x_n - S_{\mu,n}x_n) + \beta_n(S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1}) + (1-\beta_n)(S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1})\| \\ &\le \beta_n\|x_n - S_{\mu,n}x_n\| + \|S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1}\| \\ &\le \beta_n\|x_n - S_{\mu,n}x_n\| + \|S_{\mu,n}x_n - S_{\mu,n}x_{n+1}\| + \|S_{\mu,n}x_{n+1} - S_{\mu,n+1}x_{n+1}\| \\ &\le \beta_n\|x_n - S_{\mu,n}x_n\| + \|x_n - x_{n+1}\| + (1-\mu)\|A_n x_{n+1} - A_{n+1}x_{n+1}\| \\ &\le \beta_n\|x_n - S_{\mu,n}x_n\| + (1-\beta_n)\|x_n - S_{\mu,n}x_n\| + (1-\mu)\|y_n\| \\ &= \|x_n - S_{\mu,n}x_n\| + (1-\mu)\|y_n\|.\end{aligned} \tag{4.13}$$

Assumption (4.8) implies that

$$\sum_{n=0}^{\infty}\|y_n\| < \infty. \tag{4.14}$$

Using Lemma 2.2, we conclude that $\lim_{n\to\infty}\|x_n - S_{\mu,n}x_n\|$ exists. Then $\lim_{n\to\infty}\|x_n - A_n x_n\|$ exists. Thus, by (4.12) we have $\lim_{n\to\infty}\|x_n - A_n x_n\| = 0$.

If we define $T_n : K \to K$ by

$$T_n x = \alpha_n x + (1-\alpha_n)A_n x, \quad x \in K. \tag{4.15}$$

According to the corresponding deductive process of Theorem 3.1, we can prove that $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$ and for all $p_1, p_2 \in F(A_n)$.

Consequently, $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{N}$ by Lemma 2.3. □

5. Cyclic algorithm

Theorem 5.1. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $T_i : K \to K$ be a κ_i-strictly pseudocontractive mapping for some $0 \le \kappa_i < 1$. Let $\kappa = \min\{\kappa_i : 0 \le i \le N-1\}$. Assume the common fixed point set $\bigcap_{i=0}^{N-1}F(T_i)$ is nonempty. Given $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm* (1.12)*:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n, \quad n \ge 0, \tag{5.1}$$

where $T_{[n]} = T_i$, with $i = n(\mathrm{mod}\, N)$, $0 \le i \le N-1$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the condition

$$\mu \le \alpha_n < 1 - \varepsilon \tag{5.2}$$

for all n and some $\varepsilon \in (0, 1-\mu)$, where $\mu \in [\max\{0, 1-(q\kappa/C_q)^{1/(q-1)}\}, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=0}^{N-1}$.

Proof. Pick a $p \in F = \bigcap_{i=0}^{N-1} F(T_i)$. We first show that the real sequence $\{\|x_n - p\|\}_{n=0}^{\infty}$ is decreasing, hence $\lim_{n\to\infty}\|x_n - p\|$ exists. To see this, using (1.2) and (2.3), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^q &= \|x_n - p - (1-\alpha_n)[x_n - p - (T_{[n]}x_n - p)]\|^q \\
&\le \|x_n - p\|^q - q(1-\alpha_n)\langle x_n - p - (T_{[n]}x_n - p), j_q(x_n - p)\rangle + C_q(1-\alpha_n)^q \|x_n - p - (T_{[n]}x_n - p)\|^q \\
&\le \|x_n - p\|^q - q\kappa(1-\alpha_n)\|x_n - p - (T_{[n]}x_n - p)\|^q C_q(1-\alpha_n)^q \|x_n - p - (T_{[n]}x_n - p)\|^q \\
&= \|x_n - p\|^q - (1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right]\|x_n - p - (T_{[n]}x_n - p)\|^q.
\end{aligned} \tag{5.3}$$

Since $\mu \le \alpha_n < 1 - \varepsilon$, we get by (5.3)

$$\varepsilon\left[q\kappa - C_q(1-\mu)^{q-1}\right]\|x_n - p - (T_{[n]}x_n - p)\|^q \le \|x_n - p\|^q - \|x_{n+1} - p\|^q. \tag{5.4}$$

It follows that the sequence $\{\|x_n - p\|\}$ is decreasing (and hence $\lim_{n\to\infty}\|x_n - p\|$ exists) and that $\lim_{n\to\infty}\|x_n - T_{[n]}x_n\| = 0$. This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\alpha_n)\|x_n - T_{[n]}x_n\| = 0. \tag{5.5}$$

Claim: $\omega_{\mathcal{W}}(x_n) \subset F$.

Indeed, assume $x^* \in \omega_{\mathcal{W}}(x_n)$ and $x_{n_i} \rightharpoonup x^*$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We may further assume $n_i = l(\operatorname{mod} N)$ for all i. Since by (5.5), we also have $x_{n_{i+j}} \rightharpoonup x^*$ for all $j \ge 0$, we deduce that

$$\|x_{n_{i+j}} - T_{[l+j]}x_{n_{i+j}}\| = \|x_{n_{i+j}} - T_{[n_i+j]}x_{n_{i+j}}\| \longrightarrow 0. \tag{5.6}$$

Then Lemma 2.1 implies that $x^* \in F(T_{[l+j]})$ for all j. This ensures that $x^* \in F$.

If we define $T_n : K \to K$ by

$$T_n x = \alpha_n x + (1-\alpha_n)T_{[n]}x, \quad x \in K. \tag{5.7}$$

According to the corresponding deductive process of Theorem 3.1, we can prove that $\lim_{n\to\infty}\|t x_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$ and for all $p_1, p_2 \in F$.

Consequently, we conclude that $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=0}^{N-1}$ by using Lemma 2.3. This completes the proof. □

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