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Research Article

Well-Posedness and Fractals via Fixed Point Theory

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The purpose of this paper is to present existence, uniqueness, and data dependence results for the strict fixed points of a multivalued operator of Reich type, as well as, some sufficient conditions for the well-posedness of a fixed point problem for the multivalued operator.

1. Introduction

Let (X,d) be a metric space. We will use the following symbols (see also [1]):

$P(X)=\{Y\subset X \mid Y\neq\varnothing\}$;

$P_b(X)=\{Y\in P(X) \mid Y \text{ is bounded}\}$;

$P_{\text{cl}}(X)=\{Y\in P(X) \mid Y \text{ is closed}\}$;

$P_{\text{cp}}(X)=\{Y\in P(X) \mid Y \text{ is compact}\}$.

If $T: X\to P(X)$ is a multivalued operator, then for $Y\in P(X)$, $T(Y)=\bigcup_{x\in Y}T(x)$ we will denote the image of the set Y through T.

Throughout the paper $F_T:=\{x\in X \mid x\in T(x)\}$ (resp., $(\text{SF})_T:=\{x\in X \mid \{x\}=T(x)\}$) denotes the fixed point set (resp., the strict fixed point set) of the multivalued operator T.

We introduce the following generalized functionals.

The δ generalized functional

$$\delta_d: P(X)\times P(X)\longrightarrow \mathbb{R}_+\cup\{+\infty\},$$
$$\delta_d(A,B)=\sup\{d(a,b) \mid a\in A,\ b\in B\}. \tag{1.1}$$

The gap functional

$$D_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\},$$
$$D_d(A,B) = \inf\{d(a,b) \mid a \in A,\, b \in B\}. \tag{1.2}$$

The excess generalized functional

$$\rho_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\},$$
$$\rho_d(A,B) = \sup\{D_d(a,B) \mid a \in A\}. \tag{1.3}$$

The Pompeiu-Hausdorff generalized functional

$$H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\},$$
$$H_d(A,B) = \max\{\rho_d(A,B), \rho_d(B,A)\}. \tag{1.4}$$

The first purpose of this paper is to present existence, uniqueness, and data dependence results for the strict fixed point of a multivalued operator of Reich type. Since, in our approach, the strict fixed point is constructed by iterations, this generates the possibility to give some sufficient conditions for the well-posedness of a fixed point problem for the multivalued operator mentioned below.

Definition 1.1. Let (X,d) be a metric space and $T : X \to P_{\mathrm{cl}}(X)$. Then T is called a multivalued δ-contraction of Reich type, if there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that

$$\delta(T(x),T(y)) \leq ad(x,y) + b\delta(x,T(x)) + c\delta(y,T(y)), \tag{1.5}$$

for all $x,y \in X$.

The notion of well-posed fixed point problem for single valued and multivalued operator was defined and studied by F.S. De Blasi and J. Myjak, S. Reich and A.J. Zaslavski, Rus and Petruşel [2], Petruşel et al. [3].

Definition 1.2 (see Petruşel and Rus [2] and [3]). (A) Let (X,d) be a metric space, $Y \in P(X)$ and $T : Y \to P_{\mathrm{cl}}(X)$ be a multivalued operator.

Then the fixed point problem is well posed for T with respect to D_d if

(a_1) $F_T = \{x^*\}$ (i.e., $x^* \in T(x^*)$);

(b_1) If $x_n \in Y$, $n \in \mathbb{N}$ and $D_d(x_n, T(x_n)) \to 0$ as $n \to \infty$ then $x_n \to x^*$ as $n \to \infty$.

(B) Let (X,d) be a metric space, $Y \in P(X)$ and $T : Y \to P_{\mathrm{cl}}(X)$ be a multivalued operator.

Then the fixed problem is well posed for T with respect to H_d if

(a_2) $(\mathrm{SF})_T = \{x^*\}$ (i.e., $\{x^*\} = T(x^*)$);

(b_2) If $x_n \in Y$, $n \in \mathbb{N}$ and $H_d(T(x_n)) \to 0$ as $n \to \infty$ then $x_n \to x^*$ as $n \to \infty$.

The second aim is to study the existence of an attractor (i.e., the fixed point of the multifractal operator, see [4–7]) for an iterated multifunction system consisting of nonself multivalued operators.

2. Main results

We will give first another proof (a constructive one) of a result given by Reich [8] in 1972. For some similar results, see [9, 10]. In our proof, the strict fixed point will be obtained by iterations.

Theorem 2.1 (Reich's theorem). *Let (X,d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator, for which there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that*

$$\delta(T(x),T(y)) \leq ad(x,y) + b\delta(x,T(x)) + c\delta(y,T(y)), \quad \forall x,y \in X. \tag{2.1}$$

Then T has a unique strict fixed point in X, that is, $(SF)_T = \{x^*\}$.

Proof. Let $q>1$ and $x_0 \in X$ be arbitrarily chosen. Then there exists $x_1 \in T(x_0)$ such that

$$\delta(x_0,T(x_0)) \leq qd(x_0,x_1). \tag{2.2}$$

We have

$$\begin{aligned} \delta(x_1,T(x_1)) &\leq \delta(T(x_0),T(x_1)) \\ &\leq ad(x_0,x_1) + b\delta(x_0,T(x_0)) + c\delta(x_1,T(x_1)) \\ &\leq (a+bq)d(x_0,x_1) + c\delta(x_1,T(x_1)). \end{aligned} \tag{2.3}$$

It follows that

$$\delta(x_1,T(x_1)) \leq \frac{a+bq}{1-c} d(x_0,x_1). \tag{2.4}$$

For $x_1 \in T(x_0)$, there exists $x_2 \in T(x_1)$ such that

$$\delta(x_1,T(x_1)) \leq qd(x_1,x_2). \tag{2.5}$$

Then

$$\begin{aligned} \delta(x_2,T(x_2)) &\leq \delta(T(x_1),T(x_2)) \\ &\leq ad(x_1,x_2) + b\delta(x_1,T(x_1)) + c\delta(x_2,T(x_2)) \\ &\leq (a+bq)d(x_1,x_2) + c\delta(x_2,T(x_2)). \end{aligned} \tag{2.6}$$

It follows that

$$\begin{aligned}\delta(x_2,T(x_2)) &\le \frac{a+bq}{1-c}d(x_1,x_2)\\ &\le \frac{a+bq}{1-c}\delta(x_1,T(x_1))\\ &\le \left(\frac{a+bq}{1-c}\right)^2 d(x_0,x_1).\end{aligned} \tag{2.7}$$

Inductively, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ having the properties

(1) $(\alpha) x_n \in T(x_{n-1}),\ n\in\mathbb{N}^*$;

(2) $(\beta) d(x_n,x_{n+1}) \le \delta(x_n,T(x_n)) \le ((a+bq)/(1-c))^n d(x_0,x_1)$.

We will prove now that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy.
We successively have

$$\begin{aligned}d(x_n,x_{n+p}) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+p-1},x_{n+p})\\ &\le \left[\left(\frac{a+bq}{1-c}\right)^n + \left(\frac{a+bq}{1-c}\right)^{n+1} + \cdots + \left(\frac{a+bq}{1-c}\right)^{n+p-1}\right] d(x_0,x_1).\end{aligned} \tag{2.8}$$

Let us denote $\alpha := (a+bq)/(1-c)$. Then

$$d(x_n,x_{n+p}) \le \alpha^n\left(1+\alpha+\cdots+\alpha^{p-1}\right)d(x_0,x_1) = \alpha^n\frac{\alpha^p-1}{\alpha-1}d(x_0,x_1). \tag{2.9}$$

If we chose $q<(1-a-c)/b$, then $\alpha<1$.
Letting $n\to\infty$, since $\alpha^n\to 0$, it follows that

$$d(x_n,x_{n+p}) \longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{2.10}$$

Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

By the completeness of the space (X,d), we get that there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$.

Next, we will prove that $x^*\in(\mathrm{SF})_T$.

We have

$$\begin{aligned}\delta(x^*,T(x^*)) &\le d(x^*,x_n) + \delta(x_n,T(x_n)) + \delta(T(x_n),T(x^*))\\ &\le d(x^*,x_n) + \delta(x_n,T(x_n)) + ad(x_n,x^*) + b\delta(x_n,T(x_n)) + c\delta(x^*,T(x^*)).\end{aligned} \tag{2.11}$$

Then

$$\delta(x^*, T(x^*)) \leq \frac{1+a}{1-c} d(x^*, x_n) + \frac{1+b}{1-c} \delta(x_n, T(x_n)) \tag{2.12}$$

because $\delta(x_n, T(x_n)) \leq \alpha^n d(x_0, x_1) \Rightarrow \delta(x^*, T(x^*)) = 0 \Rightarrow T(x^*) = \{x^*\}$ (i.e., $x^* \in (\mathrm{SF})_T$).

For the last part of our proof, we will show the uniqueness of the strict fixed point.

Suppose that there exist $x^*, y^* \in (\mathrm{SF})_T$. Then

$$d(x^*, y^*) = \delta(T(x^*), T(y^*)) \leq a d(x^*, y^*) + b\delta(x^*, T(x^*)) + c\delta(y^*, T(y^*)). \tag{2.13}$$

If x^* and y^* are distinct points, then we get that $a \geq 1$, which contradicts our hypothesis. Thus $x^* = y^*$. The proof is complete. □

Regarding the well-posedness of a fixed point problem, we have the following result.

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator. Suppose there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$\delta(T(x), T(y)) \leq a d(x, y) + b\delta(x, T(x)) + c\delta(y, T(y)), \quad \forall x, y \in X. \tag{2.14}$$

Then the fixed point problem is well posed for T with respect to H_d.

Proof. By Reich's theorem, we get that $(\mathrm{SF})_T = \{x^*\}$.

Let $x_n \in X$, $n \in \mathbb{N}$ such that $H_d(x_n, T(x_n)) \to 0$ as $n \to \infty$. Then

$$H_d(x_n, T(x_n)) = \delta_d(x_n, T(x_n)). \tag{2.15}$$

We have to show that $x_n \to x^*$ as $n \to \infty$. We successively have

$$\begin{aligned} d(x_n, x^*) &\leq \delta_d(x_n, T(x_n)) + \delta_d(T(x_n), T(x^*)) \\ &\leq \delta_d(x_n, T(x_n)) + a d(x_n, x^*) + b\delta_d(x_n, T(x_n)) + c\delta_d(x^*, T(x^*)) \\ &= (1+b)\delta_d(x_n, T(x_n)) + a d(x_n, x^*). \end{aligned} \tag{2.16}$$

It follows that

$$d(x_n, x^*) \leq \frac{1+b}{1-a} \delta_d(x_n, T(x_n)) = \frac{1+b}{1-a} H_d(x_n, T(x_n)) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.17}$$

Hence

$$x_n \longrightarrow x^*, \quad n \longrightarrow \infty. \tag{2.18}$$

□

With respect to the same multivalued operators, a data dependence result can also be established as follows.

Theorem 2.3. *Let (X,d) be a complete metric space and let $T_1, T_2 : X \to P_b(X)$ be two multivalued operators. Suppose that*

(i) *there exist $a, b, c \in \mathbb{R}_+$ with $a+b+c<1$ such that*

$$\delta(T_1(x), T_1(y)) \leq ad(x,y) + b\delta(x, T_1(x)) + c\delta(y, T_1(y)), \quad \forall x, y \in X \tag{2.19}$$

(denote the unique strict fixed point of T_1 by x_1^*);

(ii) $(SF)_{T_2} \neq \emptyset$;

(iii) *there exists $\eta > 0$ such that $\delta(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.*

Then

$$\delta\left(x_1^*, (SF)_{T_2}\right) \leq \frac{(1+c)\eta}{1-a}. \tag{2.20}$$

Proof. Let $x_2^* \in (\mathrm{SF})_{T_2}$. Then $\delta(x_2^*, T_2(x_2^*)) = 0$.

We have

$$\begin{aligned} d\left(x_1^*, x_2^*\right) &= \delta\left(T_1\left(x_1^*\right), T_2\left(x_2^*\right)\right) \\ &\leq \delta\left(T_1\left(x_1^*\right), T_1\left(x_2^*\right)\right) + \delta\left(T_1\left(x_2^*\right), T_2\left(x_2^*\right)\right) \\ &\leq ad\left(x_1^*, x_2^*\right) + b\delta\left(x_1^*, T_1\left(x_1^*\right)\right) + c\delta\left(x_2^*, T_1\left(x_2^*\right)\right) + \eta \\ &= ad\left(x_1^*, x_2^*\right) + c\delta\left(T_2\left(x_2^*\right), T_1\left(x_2^*\right) + \eta \leq ad\left(x_1^*, x_2^*\right) + (1+c)\eta. \end{aligned} \tag{2.21}$$

It follows that

$$d\left(x_1^*, x_2^*\right) \leq \frac{1+c}{1-a}\eta. \tag{2.22}$$

By taking $\sup_{x_2^* \in (\mathrm{SF})_{T_2}}$, it follows that

$$\delta\left(x_1^*, (\mathrm{SF})_{T_2}\right) \leq \frac{1+c}{1-a}\eta. \tag{2.23}$$

□

Let (X,d) be a complete metric space and let $F_1, \ldots, F_m : X \to P(X)$ be a finite family of multivalued operators.

The system $F = (F_1, \ldots, F_m)$ is said to be an iterated multifunction system.

The operator

$$\widetilde{T}_F : P(X) \longrightarrow P(X), \quad \widetilde{T}_F(Y) = \bigcup_{i=1}^{m} F_i(Y), \quad Y \in P(X) \tag{2.24}$$

is called the multifractal operator generated by the iterated multifunction system $F = (F_1, \ldots, F_m)$.

Remark 2.4. (i) If $F_i : X \to P_{cp}(X)$ are multivalued α_i-contractions for each $i \in \{1,2,\dots,m\}$, then the multifractal operator $\widetilde{T}_F$ is an α-contraction too, where $\alpha := \max\{\alpha_i \mid i \in \{1,\dots,m\}\}$ (Nadler Jr. [7]).

(ii) If $F_i : X \to P_{cp}(X)$ are multivalued φ_i-contractions (see [4]) for each $i \in \{1,2,\dots, m\}$, then the multifractal operator $\widetilde{T}_F$ is an φ-contraction too, see Andres and Fišer [4] for the definitions and the result.

(iii) If $F = (F_1,\dots,F_m)$ is an iterated multifunction system, such that $F_i : X \to P_{cp}(X)$ is upper semicontinuous for each $i \in \{1,\dots,m\}$, then the multifractal operator

$$\widetilde{T}_F : P_{cp}(X) \longrightarrow P_{cp}(X), \qquad \widetilde{T}_F(Y) = \bigcup_{i=1}^{m} F_i(Y) \tag{2.25}$$

is well defined. A fixed point $Y^* \in P_{cp}(X)$ of $\widetilde{T}_F$ is called an attractor of the iterated multifunction system F.

The following result is well known, see, for example, Granas and Dugundji [11].

Lemma 2.5. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$ and*

$$B := \widetilde{B}(x_0,r) = \{x \in X \mid d(x,x_0) \leq r\}. \tag{2.26}$$

Let $f : B \to X$ be an α-contraction.

If $d(x_0, f(x_0)) \leq (1-\alpha)r$, then f has a unique fixed point in B.

Our next result concerns with the existence of an attractor for an iterated multifunction system.

Theorem 2.6. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $F_i : \widetilde{B}(x_0,r) \to P_{cp}(X)$, $i \in \{1,\dots,m\}$ a finite family of multivalued operators.*

Suppose that

(i) *F_i is an α_i-contraction, for each $i \in \{1,\dots,m\}$;*

(ii) *$\delta(x_0, F_i(x_0)) \leq (1 - \max\{\alpha_i \mid i \in \{1,\dots,m\}\})r$, for all $i \in \{1,\dots,m\}$.*

Then there exists $Y^* \in \widetilde{B}(\{x_0\},r) \subset P_{cp}(X)$ a unique attractor of the iterated multifunction system $F = (F_1,\dots,F_m)$.

Proof. Since $F_i : \widetilde{B}(x_0,r) \to P_{cp}(X)$ is an α_i-contraction, for each $i \in \{1,\dots,m\}$ it follows that F_i is upper semicontinuous, for each $i \in \{1,\dots,m\}$. By Remark 2.4(iii), we get that the operator $\widetilde{T}_F : \widetilde{B}(\{x_0\},r) \subset P_{cp}(X) \to P_{cp}(X)$, $\widetilde{T}_F(Y) = \bigcup_{i=1}^{m} F_i(Y)$, $Y \in \widetilde{B}(\{x_0\},r)$ is well defined.

Any fixed point $Y^* \in \widetilde{B}(\{x_0\},r) \subset P_{cp}(X)$ of $\widetilde{T}_F$ is an attractor of the iterated multifunction system $F = (F_1,\dots,F_m)$.

Notice first that, if $Y \in \widetilde{B}(\{x_0\},r) \subset (P_{cp}(X),H)$, then $H(\{x_0\},Y) \leq r$, which implies that $d(x_0,y) \leq r$, for all $y \in Y$. Thus $y \in \widetilde{B}(x_0,r)$, for all $y \in Y$.

We will show that $\widetilde{T}_F$ satisfies the following two conditions:

(i) $\widetilde{T}_F$ is an α-contraction, with $\alpha := \max\{\alpha_i \mid i \in \{1,\ldots,m\}\}$, that is,

$$H(\widetilde{T}_F(Y_1),\widetilde{T}_F(Y_2)) \le \alpha H(Y_1,Y_2), \quad \forall Y_1,Y_2 \in \widetilde{B}(\{x_0\},r) \subset P_{cp}(X); \tag{2.27}$$

(ii) $H(\{x_0\},\widetilde{T}_F(\{x_0\})) \le (1-\alpha)r$.

Indeed, we have

(i) Let $Y_1,Y_2 \in \widetilde{B}(\{x_0\},r) \subset P_{cp}(X)$ şi $u \in \widetilde{T}_F(Y_1)$. By the definition of $\widetilde{T}_F$, it follows that there exists $j \in \{1,\ldots,m\}$ and there exists $y_1 \in Y_1$ such that $u \in F_j(y_1)$. Since $Y_1,Y_2 \in P_{cp}(X)$, there exists $y_2 \in Y_2$ such that $d(y_1,y_2) \le H(Y_1,Y_2)$.

Since, for arbitrary $\varepsilon > 0$ and each $A,B \in P_{cp}(X)$ with $H(A,B) \le \varepsilon$, we have that for all $a \in A$ there exists $b \in B$ such that $d(a,b) \le \varepsilon$, by the following relations

$$H(F_j(y_1),F_j(y_2)) \le \alpha_j d(y_1,y_2) \le \alpha_j H(Y_1,Y_2), \tag{2.28}$$

we obtain that for $u \in F_j(y_1) \subset \widetilde{T}_F(Y_1)$, there exists $v \in F_j(y_2) \subset \widetilde{T}_F(Y_2)$ such that $d(u,v) \le \alpha_j H(Y_1,Y_2) \le \alpha H(Y_1,Y_2)$.

By the above relation and by the similar one (where the roles of $\widetilde{T}_F(Y_1)$ and $\widetilde{T}_F(Y_2)$ are reversed), the first conclusion follows.

(ii) We have to show that

$$\delta(\{x_0\},\widetilde{T}_F(\{x_0\})) \le (1-\alpha)r \tag{2.29}$$

or equivalently for all $u \in \widetilde{T}_F(\{x_0\})$, we have $d(x_0,u) \le (1-\alpha)r$. Since $u \in \widetilde{T}_F(\{x_0\})$ it follows that there exists $j \in \{1,\ldots,m\}$ such that $u \in F_j(x_0)$. Then

$$d(x_0,u) \le \delta(x_0,F_j(x_0)) \le (1-\alpha)r. \tag{2.30}$$

By Lemma 2.5, applied to $\widetilde{T}_F$, we get that there exists $Y^* \in \widetilde{B}(\{x_0\},r) \subset P_{cp}(X)$ a unique fixed point for $\widetilde{T}_F$, that is, a unique attractor of the iterated multifunction system $F = (F_1,\ldots,F_m)$. The proof is complete. □

Remark 2.7. An interesting extension of the above results could be the case of a set endowed with two metrics, see [12] for other details.

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