Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2008, Article ID 324575, 12 pages
doi:10.1155/2008/324575

Research Article

Implicit Iteration Process for Common Fixed Points of Strictly Asymptotically Pseudocontractive Mappings in Banach Spaces

You Xian Tian,[1] Shih-sen Chang,[2] Jialin Huang,[2] Xiongrui Wang,[2] and J. K. Kim[3]

[1] *College of Mathematics and Physics, Chongqing University of Post Telecommunications, Chongqing 400065, China*
[2] *Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China*
[3] *Department of Mathematics, Kyungnam University, Masan 631-701, South Korea*

Correspondence should be addressed to You Xian Tian, tianyx@cqupt.edu.cn

Received 25 May 2008; Accepted 3 September 2008

Recommended by Nanjing Huang

In this paper, a new implicit iteration process with errors for finite families of strictly asymptotically pseudocontractive mappings and nonexpansive mappings is introduced. By using the iterative process, some strong convergence theorems to approximating a common fixed point of strictly asymptotically pseudocontractive mappings and nonexpansive mappings are proved. The results presented in the paper are new which extend and improve some recent results of Osilike et al. (2007), Liu (1996), Osilike (2004), Su and Li (2006), Gu (2007), Xu and Ori (2001).

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, C is a nonempty closed convex subset of E, E^* is the dual space of E, and $J : E \to E^*$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}, \quad x \in E. \tag{1.1}$$

Recall that a set $C \subset E$ is said to be *closed, convex, and pointed cone* if it is a closed set and satisfies the following conditions: (1) $C + C \subset C$; (2) $\lambda C \subset C$ for each $\lambda \geq 0$; (3) if $x \in C$ with $x \neq 0$, then $-x \notin C$.

Definition 1.1. Let $T : C \to C$ be a mapping:

(1) T is said to be $(\lambda, \{k_n\})$-*strictly asymptotically pseudocontractive* if there exist a constant $\lambda \in (0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that for all $x, y \in C$, and for all $j(x-y) \in J(x-y)$,

$$\langle T^n x - T^n y,\, j(x-y)\rangle \le k_n \|x-y\|^2 - \lambda \left\|x - T^n x - (y - T^n y)\right\|^2 \quad \forall n \ge 1, \tag{1.2}$$

(2) T is said to be λ-*strictly pseudocontractive in the terminology of Browder-Petryshyn* [1] if there exist a constant $\lambda \in (0,1)$ such that for all $x, y \in C$,

$$\langle Tx - Ty,\, j(x-y)\rangle \le \|x-y\|^2 - \lambda \left\|x - Tx - (y - Ty)\right\|^2 \quad \forall j(x-y) \in J(x-y), \tag{1.3}$$

(3) T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \le L\|x-y\| \quad \forall n \ge 1. \tag{1.4}$$

The class of $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mappings was first introduced in Hilbert spaces by Liu [2]. In the case of Hilbert spaces, it is shown by [2] that (1.2) is equivalent to the inequality

$$\left\|T^n x - T^n y\right\|^2 \le k_n \|x-y\|^2 + \lambda \left\|(I - T^n)x - (I - T^n)y\right\|^2. \tag{1.5}$$

Concerning the convergence problem of iterative sequences for strictly pseudocontractive mappings has been studied by several authors (see, e.g., [1, 3–7]). Concerning the class of strictly asymptotically pseudocontractive mappings, Liu [2] and Osilike et al. [8] proved the following results.

Theorem 1.2 (Liu [2]). *Let H be a real Hilbert space, let C be a nonempty closed convex and bounded subset of H, and let $T : C \to C$ be a completely continuous uniformly L-Lipschitzian $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mapping such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\} \subset (0,1)$ be a sequence satisfying the following condition:*

$$0 < \epsilon \le \alpha_n \le 1 - \lambda - \epsilon \quad \forall n \ge 1 \text{ and some } \epsilon > 0. \tag{1.6}$$

Then, the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \quad \forall n \ge 1 \tag{1.7}$$

converges strongly to a fixed point of T.

In 2007, Oslike et al. [8] proved the following theorem.

Theorem 1.3 (Oslike et al. [8]). *Let E be a real q-uniformly smooth Banach space which is also uniformly convex, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mapping such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and let $F(T) \neq \varnothing$. Let $\{\alpha_n\} \subset [0,1]$ be a real sequence satisfying the following condition:*

$$0 < a \le \alpha_n^{q-1} \le b < \frac{q(1-k)}{2c_q}(1+L)^{-(q-2)} \quad \forall n \ge 1. \tag{1.8}$$

Let $\{x_n\}$ be the sequence defined by (1.7). *Then,*

(1) $\lim_{n\to\infty}\|x_n - p\|$ *exists* $\forall p \in F(T)$,

(2) $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$,

(3) $\{x_n\}$ *converges weakly to a fixed point of* T.

It is our purpose in this paper to introduce the following new implicit iterative process with errors for a finite family of strictly asymptotically pseudocontractive mappings $\{T_i\}$ and a finite family of nonexpansive mappings $\{S_i\}$:

$$\begin{gathered} x_1 \in C, \\ x_n = \alpha_n S_n x_{n-1} + (1-\alpha_n) T_n^n x_n + u_n \quad \forall n \ge 1, \end{gathered} \tag{1.9}$$

where C is a closed convex cone of E, $S_n = S_{n(\operatorname{mod} N)}$, $T_n^n = T_{n(\operatorname{mod} N)}^n$, and $\{u_n\}$ is a bounded sequence in C. Also, we aim to prove some strong convergence theorems to approximating a common fixed point of $\{S_i\}$ and $\{T_i\}$. The results presented in the paper are new which extend and improve some recent results of [2–8].

In order to prove our main results, we need the following lemmas.

Lemma 1.4 (see [9]). *Let E be a real Banach space, let C be a nonempty subset of E, and let $T : C \to C$ be a $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mapping, then T is uniformly L-Lipschitzian.*

Lemma 1.5. *Let E be a real Banach space, let C be a nonempty closed convex subset of E, and let $T_i : C \to C$ be a $(\lambda_i, \{k_n^{(i)}\})$-strictly asymptotically pseudocontractive mapping, $i = 1, 2, \ldots, N$, then there exist a constant $\lambda \in (0,1)$, a constant $L > 0$, and a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for any $x, y \in C$ and for each $i = 1, 2, \ldots, N$ and each $n \ge 1$, the following hold:*

$$\langle T_i^n x - T_i^n y, j(x-y)\rangle \le k_n\|x-y\|^2 - \lambda\left\|x - T_i^n x - (y - T_i^n y)\right\|^2 \tag{1.10}$$

for each $j(x-y) \in J(x-y)$ and

$$\left\|T_i^n x - T_i^n y\right\| \le L\|x-y\|. \tag{1.11}$$

Proof. Since for each $i = 1, 2, \ldots, N$, T_i is $(\lambda_i, \{k_n^{(i)}\})$-strictly asymptotically pseudocontractive, where $\lambda_i \in (0,1)$ and $\{k_n^{(i)}\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n^{(i)} = 1$. By Lemma 1.4, T_i is L_i-Lipschitzian.

Taking $k_n = \max\{k_n^{(i)}, i = 1,2,\ldots,N\}$ and $\lambda = \min\{\lambda_i, i = 1,2,\ldots,N\}$, hence, for each $i = 1,2,\ldots,N$, we have

$$\begin{aligned}\langle T_i^n x - T_i^n y, j(x-y)\rangle &\le k_n^{(i)}\|x-y\|^2 - \lambda_i \left\|x - T_i^n x - (y - T_i^n y)\right\|^2 \\ &\le k_n\|x-y\|^2 - \lambda\left\|x - T_i^n x - (y - T_i^n y)\right\|^2.\end{aligned} \tag{1.12}$$

The conclusion (1.10) is proved. Again, taking $L = \max\{L_i : i = 1,2,\ldots N\}$ for any $x, y \in C$, we have

$$\left\|T_i^n x - T_i^n y\right\| \le L_i\|x-y\| \le L\|x-y\| \quad \forall n \ge 1. \tag{1.13}$$

This completes the proof of Lemma 1.5. □

Lemma 1.6 (see [9]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \le (1+b_n)a_n + c_n \quad \forall n \ge n_0, \tag{1.14}$$

where n_0 is some nonnegative integer such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

In addition, if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$ $(n \to \infty)$.

2. Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space, let C be a nonempty closed pointed convex cone of E, let $T_i : C \to C$, $i = 1,2,\ldots,N$, be a finite family of $(\lambda_i, \{k_n^{(i)}\})$-strictly asymptotically pseudocontractive mappings, and let $S_i : C \to C$, $i = 1,2,\ldots,N$, be a finite family of nonexpansive mappings with*

$$F = \bigcap_{i=1}^{N} F(S_i) \bigcap \bigcap_{i=1}^{N} F(T_i) \ne \emptyset \tag{2.1}$$

(the set of common fixed points of $\{S_i\}$ and $\{T_i\}$). Let $\{\alpha_n\}$ be a sequence in $(0,1)$, let $\{u_n\}$ be a bounded sequence in C, let $\lambda = \min\{\lambda_i : i = 1,2,\ldots,N\}$, $k_n = \max\{k_n^{(i)}, i = 1,2,\ldots,N\}$, and let $L = \max\{L_i : i = 1,2,\ldots,N\} > 0$ be positive numbers defined by (1.10) and (1.11), respectively. If the following conditions are satisfied:

(i) $0 < \max\{\lambda, (1 - 1/L)\} < \liminf_{n\to\infty} \alpha_n \le \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$,

(iii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $1 \le k_n < (1-\lambda)/(1 - \liminf_{n\to\infty}\alpha_n)$,

(iv) $\sum_{n=1}^{\infty}\|u_n\| < \infty$, *then the iterative sequence* $\{x_n\}$ *with errors defined by* (1.9) *has the following properties:*

(1) $\lim_{n\to\infty}\|x_n - p\|$ *exists for each* $p \in F$,

(2) $\lim_{n\to\infty} d(x_n, F)$ *exists,*

(3) $\liminf_{n\to\infty}\|x_n - T_n^n x_n\| = 0$,

(4) *the sequence* $\{x_n\}$ *converges strongly to a common fixed point* $p \in F$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.2}$$

Proof. We divide the proof of Theorem 2.1 into four steps.

(I) First, we prove that the mapping $G_n : C \to C$, $n = 1, 2, \ldots$, defined by

$$G_n(x) = \alpha_n S_n x_{n-1} + (1-\alpha_n) T_n^n x + u_n, \quad x \in C \tag{2.3}$$

is a Banach contractive mapping.

Indeed, it follows from condition (i) that $1 - 1/L < \alpha_n$, that is, $(1-\alpha_n)L < 1$. Hence, from Lemma 1.5, for any $x, y \in C$, we have

$$\begin{aligned}\|G_n x - G_n y\| &= \|\alpha_n S_n x_{n-1} + (1-\alpha_n)T_n^n x + u_n - (\alpha_n S_n x_{n-1} + (1-\alpha_n)T_n^n y + u_n)\| \\ &= (1-\alpha_n)\|T_n^n x - T_n^n y\| \\ &\le (1-\alpha_n)L\|x - y\|, \quad n = 1, 2, \ldots,\end{aligned} \tag{2.4}$$

that is, for each $n = 1, 2, \ldots$, $G_n : C \to C$ is a Banach contraction mapping. Therefore, there exists a unique fixed point $x_n \in C$ such that $x_n = G(x_n)$. This shows that the sequence $\{x_n\}$ defined by (1.9) is well defined.

(II) The proof of conclusions (1) and (2).

For any given $p \in F$ and for any $j(x_n - p) \in J(x_n - y)$ from Lemma 1.5, we have

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n(S_n x_{n-1} - p) + (1-\alpha_n)(T_n^n x_n - p) + u_n\|^2 \\ &= \alpha_n\langle S_n x_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\langle T_n^n x_n - p, j(x_n - p)\rangle + \langle u_n, j(x_n - p)\rangle \\ &\le \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1-\alpha_n)\{k_n\|x_n - p\|^2 - \lambda\|x_n - T_n^n x_n\|^2\} + \|u_n\|\|x_n - p\|.\end{aligned} \tag{2.5}$$

Simplifying it, we have

$$\|x_n - p\| \le \frac{\alpha_n}{1-(1-\alpha_n)k_n}\|x_{n-1} - p\| + \frac{\|u_n\|}{1-(1-\alpha_n)k_n} - \frac{(1-\alpha_n)\lambda}{1-(1-\alpha_n)k_n} \cdot \frac{\|x_n - T_n^n x_n\|^2}{\|x_n - p\|}. \tag{2.6}$$

By virtue of conditions (i) and (iii), we have

$$k_n \le \frac{1-\lambda}{1 - \lim\ \inf_{n\to\infty}\alpha_n} \le \frac{1-\lambda}{1-\alpha_n}, \tag{2.7}$$

and so

$$0 < \lambda \le 1 - (1-\alpha_n)k_n < 1. \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$\begin{aligned}\|x_n - p\| &\le \frac{\alpha_n}{1-(1-\alpha_n)k_n}\|x_{n-1}-p\| + \frac{\|u_n\|}{\lambda} - (1-\alpha_n)\lambda\cdot\frac{\|x_n - T_n^n x_n\|^2}{\|x_n - p\|} \\ &= \left(1 + \frac{(1-\alpha_n)(k_n-1)}{1-(1-\alpha_n)k_n}\right)\|x_{n-1}-p\| + \frac{\|u_n\|}{\lambda} - (1-\alpha_n)\lambda\cdot\frac{\|x_n - T_n^n x_n\|^2}{\|x_n - p\|}.\end{aligned} \tag{2.9}$$

Letting $b_n = (1-\alpha_n)(k_n-1)/(1-(1-\alpha_n)k_n)$ and $c_n = \|u_n\|/\lambda$, then we have

$$\|x_n - p\| \le (1+b_n)\|x_{n-1}-p\| + c_n \quad \forall n \ge 1. \tag{2.10}$$

By using (2.8),

$$b_n \le \frac{(1-\alpha_n)(k_n-1)}{\lambda} < \frac{k_n-1}{\lambda}. \tag{2.11}$$

By conditions (iii) and (iv), $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. By virtue of Lemma 1.6, $\lim_{n\to\infty}\|x_n - p\|$ exists; and so $\{x_n\}$ is a bounded sequence in C. Denote

$$M = \sup_{n\ge 1}\|x_n - p\|. \tag{2.12}$$

From (2.10), we have

$$d(x_n, F) \le (1+b_n)d(x_{n-1}, F) + c_n \quad \forall n \ge 1. \tag{2.13}$$

By using Lemma 1.6 again, we know that $\lim_{n\to\infty} d(x_n, F)$ exists.

The conclusions (1) and (2) are proved.

(III) The proof of conclusion (3).

It follows from (2.9) that

$$\begin{aligned}\|x_n - p\| &\le (1+b_n)\|x_{n-1}-p\| + c_n - (1-\alpha_n)\lambda\cdot\frac{\|x_n - T_n^n x_n\|^2}{M} \\ &\le \|x_{n-1}-p\| + b_n M + c_n - (1-\alpha_n)\lambda\cdot\frac{\|x_n - T_n^n x_n\|^2}{M},\end{aligned} \tag{2.14}$$

that is,

$$(1-\alpha_n)\lambda\cdot\frac{\|x_n - T_n^n x_n\|^2}{M} \le \|x_{n-1}-p\| - \|x_n - p\| + b_n M + c_n. \tag{2.15}$$

For any positive number n_1, we have

$$\begin{aligned}\frac{\lambda}{M}\sum_{n=1}^{n_1}(1-\alpha_n)\|x_n-T_n^n x_n\|^2 &\le \|x_0-p\|-\|x_{n_1}-p\|+\sum_{n=1}^{n_1}(b_nM+c_n)\\ &\le \|x_0-p\|+\sum_{n=1}^{n_1}(b_nM+c_n).\end{aligned} \tag{2.16}$$

Letting $n_1\to\infty$, we have

$$\frac{\lambda}{M}\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-T_n^n x_n\|^2 \le \|x_0-p\|+\sum_{n=1}^{\infty}(b_nM+c_n)<\infty. \tag{2.17}$$

By condition (ii), we have

$$\liminf_{n\to\infty}\|x_n-T_n^n x_n\|=0. \tag{2.18}$$

(IV) Next, we prove the conclusion (4).

Necessity

If $\{x_n\}$ converges strongly to some point $p\in F$, then from $0\le d(x_n,F)\le\|x_n-p\|\to 0$, we have

$$\liminf_{n\to\infty} d(x_n,F)=0. \tag{2.19}$$

Sufficiency

If $\liminf_{n\to\infty} d(x_n,F)=0$, it follows from the conclusion (2) that $\lim_{n\to\infty} d(x_n,F)=0$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, since for any $t>0$, $1+t\le\exp(t)$, therefore, for any $m,n\ge 1$ and for given $p\in F$, from (2.10), we have

$$\begin{aligned}\|x_{n+m}-p\| &\le (1+b_{n+m})\|x_{n+m-1}-p\|+c_{n+m}\\ &\le \exp\{b_{n+m}\}\|x_{n+m-1}-p\|+c_{n+m}\\ &\le \exp\{b_{n+m}\}\left[\exp\{b_{n+m-1}\}\|x_{n+m-2}-p\|+c_{n+m-1}\right]+c_{n+m}\\ &= \exp\{b_{n+m}+b_{n+m-1}\}\|x_{n+m-2}-p\|+\exp\{b_{n+m}\}c_{n+m-1}+c_{n+m}\\ &\le\cdots\\ &\le \exp\left\{\sum_{i=n+1}^{n+m}b_i\right\}\|x_n-p\|+\sum_{i=n+1}^{n+m}\left(\exp\left\{\sum_{j=i+1}^{n+m}b_j\right\}\right)c_i\\ &\le K\left(\|x_n-p\|+\sum_{i=n+1}^{n+m}c_i\right)<\infty,\end{aligned} \tag{2.20}$$

where $K = \exp\{\sum_{j=1}^{\infty} b_j\} < \infty$. Since

$$\lim_{n\to\infty} d(x_n, F) = 0, \qquad \sum_{n=1}^{\infty} c_n < \infty \tag{2.21}$$

for any given $\epsilon > 0$, there exists a positive integer n_1 such that

$$d(x_n, F) < \frac{\epsilon}{4(K+1)}, \qquad \sum_{i=n+1}^{\infty} c_i < \frac{\epsilon}{2K} \quad \forall n \geq n_1. \tag{2.22}$$

Hence, there exists $p_1 \in F$ such that

$$\|x_n - p_1\| < \frac{\epsilon}{2(K+1)} \quad \forall n \geq n_1. \tag{2.23}$$

Consequently, for any $n \geq n_1$ and $m \geq 1$, from (2.20), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\leq K\left\{\|x_n - p_1\| + \sum_{i=n+1}^{n+m} c_i\right\} + \|x_n - p_1\| \\ &\leq (K+1)\|x_n - p_1\| + K\left(\sum_{i=n+1}^{n+m} c_i\right) \\ &\leq (K+1)\frac{\epsilon}{2(K+1)} + K\frac{\epsilon}{2K} = \epsilon. \end{aligned} \tag{2.24}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. Let $x_n \to x^* \in C$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, and so $d(x^*, F) = 0$. Again, since $\{S_n\}$ is a finite family of nonexpansive mappings and $\{T_n\}$ is a finite family of strictly asymptotically pseudocontractive mappings, by Lemma 1.5, it is a finite family of uniformly Lipschitzian mappings. Hence, the set F of common fixed points of $\{S_n\}$ and $\{T_n\}$ is closed and so $x^* \in F$.

This completes the proof of Theorem 2.1. □

Remark 2.2. Theorem 2.1 is a generalization and improvement of the corresponding results in Osilike et al. [8] and Liu [2] which is also an improvement of the corresponding results in [3, 5–7].

The following theorem can be obtained from Theorem 2.1 immediately.

Theorem 2.3. *Let E be a real Banach space, let C be a nonempty closed pointed convex cone of E, let $T : C \to C$ be a $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mappings, and let $\{S_i : C \to C, i = 1, 2, \ldots, N\}$ be a finite family of nonexpansive mappings with*

$$F = \bigcap_{i=1}^{N} F(S_i) \bigcap F(T) \neq \emptyset \tag{2.25}$$

(the set of common fixed points of $\{S_i\}$ *and* T*). Let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$*, let* $\{u_n\}$ *be a bounded sequence in* C*. If the following conditions are satisfied:*

(i) $0 < \max\{\lambda,\ (1-1/L)\} < \liminf_{n\to\infty}\alpha_n \le \alpha_n < 1$*, where* $L > 0$ *is a constant appeared in Lemma 1.4,*

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$*,*

(iii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $1 \le k_n < (1-\lambda)/(1-\liminf_{n\to\infty}\alpha_n)$*,*

(iv) $\sum_{n=1}^{\infty}\|u_n\| < \infty$*,*

then the conclusions in Theorem 2.1 still hold.

Theorem 2.4. *Let* E *be a real Banach space, let* C *be a nonempty closed convex subset of* E*, and* $\{T_i : C \to C,\ i = 1,2,\ldots,N\}$ *be a finite family of* $(\lambda_i, \{k_n^{(i)}\})$*-strictly asymptotically pseudocontractive mappings, and let* $\{S_i : C \to C,\ i = 1,2,\ldots,N\}$ *be a finite family of nonexpansive mappings with*

$$F = \bigcap_{i=1}^{N} F(S_i) \bigcap \bigcap_{i=1}^{N} F(T_i) \neq \varnothing \tag{2.26}$$

(the set of common fixed points of $\{S_i\}$ *and* $\{T_i\}$*). Let* $\{x_n\}$ *be the sequence defined by the following: for any given* $x_1 \in C$*,*

$$x_n = \alpha_n S_n x_{n-1} + \beta_n T_n^n x_n + \gamma_n u_n \quad \forall n \ge 1, \tag{2.27}$$

where $S_n = S_{n(\mathrm{mod}\,N)}$*,* $T_n^n = T_{n(\mathrm{mod}\,N)}^n$*,* $\{\alpha_n\}$*,* $\{\beta_n\}$*, and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$*,* $\{u_n\}$ *is a bounded sequence in* C*,* $\lambda = \min\{\lambda_i : i = 1,2,\ldots,N\}$*,* $k_n = \max\{k_n^{(i)}, i = 1,2,\ldots,N\}$*, and* $L = \max\{L_i : i = 1,2,\ldots,N\} > 0$ *are positive numbers defined by* (1.10) *and* (1.11)*, respectively. If the following conditions are satisfied:*

(i) $0 < \lambda < \liminf_{n\to\infty}\alpha_n \le \alpha_n < 1$*,*

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$*,*

(iii) $0 < \beta_n \le \limsup_{n\to\infty}\beta_n \le \min\{1-\lambda,\ 1/L\} < 1$*,*

(iv) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $1 \le k_n < (1-\lambda)/(1-\liminf_{n\to\infty}\alpha_n)$*,*

(v) $\sum_{n=1}^{\infty}\gamma_n < \infty$*,*

then the conclusions of Theorem 2.1 for sequence $\{x_n\}$ *defined by* (2.27) *still hold.*

Proof. By the same method as given in the proof of Theorem 2.1, we can prove that the mapping $W_n : C \to C$ defined by

$$W_n(x) = \alpha_n S_n x_{n-1} + \beta_n T_n^n x + \gamma_n u_n, \quad x \in C,\ n \ge 1, \tag{2.28}$$

is a Banach contractive mapping. Hence, there exists a unique $x_n \in C$ such that $x_n = W(x_n)$. This implies that the sequence $\{x_n\}$ defined by (2.27) is well defined.

For each $p \in F$, we have

$$\begin{aligned}\|x_n - p\|^2 &= \alpha_n \langle S_n x_{n-1} - p, j(x_n - p)\rangle + \beta_n \langle T_n^n x_n - p, j(x_n - p)\rangle + \gamma_n \langle u_n - p, j(x_n - p)\rangle \\ &\le \alpha_n \|x_{n-1} - p\| \|x_n - p\| + \beta_n \left\{ k_n \|x_n - p\|^2 - \lambda \|x_n - T_n^n x_n\|^2 \right\} + \gamma_n \|u_n - p\| \|x_n - p\|.\end{aligned} \tag{2.29}$$

Simplifying it, we have

$$\|x_n - p\|^2 \le \frac{\alpha_n \|x_{n-1} - p\| \|x_n - p\|}{1 - \beta_n k_n} - \frac{\beta_n \lambda}{1 - \beta_n k_n} \|x_n - T_n^n x_n\|^2 + \frac{\gamma_n}{1 - \beta_n k_n} \|u_n - p\| \|x_n - p\|. \tag{2.30}$$

Since

$$\limsup_{n \to \infty} \beta_n = \limsup_{n \to \infty} (1 - \alpha_n - \gamma_n) \le \limsup_{n \to \infty} (1 - \alpha_n) = 1 - \liminf_{n \to \infty} \alpha_n, \tag{2.31}$$

by conditions (i), (iii), and (iv), we have

$$k_n \le \frac{1 - \lambda}{1 - \liminf_{n \to \infty} \alpha_n} \le \frac{1 - \lambda}{\limsup_{n \to \infty} \beta_n} \le \frac{1 - \lambda}{\beta_n}, \tag{2.32}$$

that is, $1 - \beta_n k_n \ge \lambda > 0$. Hence, we have

$$\begin{aligned}\|x_n - p\| &\le \frac{\alpha_n \|x_{n-1} - p\|}{1 - \beta_n k_n} + \frac{\gamma_n}{\lambda} \|u_n - p\| \\ &= \left(1 + \frac{\beta_n k_n - \beta_n - \gamma_n}{1 - \beta_n k_n}\right) \|x_{n-1} - p\| + \frac{\gamma_n}{\lambda} \|u_n - p\| \\ &\le \left(1 + \frac{\beta_n k_n - \beta_n}{1 - \beta_n k_n}\right) \|x_{n-1} - p\| + \frac{\gamma_n}{\lambda} \|u_n - p\|.\end{aligned} \tag{2.33}$$

By condition (iv),

$$\sum_{n=1}^{\infty} \frac{\beta_n k_n - \beta_n}{1 - \beta_n k_n} \le \frac{1}{\lambda} \sum_{n=1}^{\infty} (k_n - 1) < \infty. \tag{2.34}$$

Again, since $\{\|u_n - p\|\}$ is bounded, by condition (v), we have

$$\sum_{n=1}^{\infty} \frac{\gamma_n \|u_n - p\|}{\lambda} < \infty. \tag{2.35}$$

It follows from (2.33) and Lemma 1.6 that $\lim_{n \to \infty} \|x_n - p\|$ exists, and so $\{x_n\}$ is bounded. Since $\{T_i\}$ is uniformly Lipschitzian, $\{T_n^n x_n\}$ is bounded.

Now, we rewrite (2.27) as follows:

$$x_n = \alpha_n S_n x_{n-1} + (1-\alpha_n) T_n^n x_n + v_n \quad \forall n \geq 1, \tag{2.36}$$

where $v_n = \gamma_n (u_n - T_n^n x_n)$. By condition (v),

$$\sum_{n=1}^{\infty} \|v_n\| < \infty. \tag{2.37}$$

These imply that all conditions in Theorem 2.1 are satisfied. Therefore, the conclusion of Theorem 2.4 can be obtained from Theorem 2.1 immediately.

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let E be a real Banach space, let C be a nonempty closed convex subset of E, and let $\{T_i : C \to C, i = 1, 2, \ldots, N\}$ be a finite family of $(\lambda_i, \{k_n^{(i)}\})$-strictly asymptotically pseudocontractive mappings with*

$$F = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing \tag{2.38}$$

(the set of common fixed points of $\{T_i\}$). Let $\{x_n\}$ be the sequence defined by the following:

for any given $x_1 \in C$,

$$x_n = \alpha_n x_{n-1} + \beta_n T_n^n x_n + \gamma_n u_n \quad \forall n \geq 1, \tag{2.39}$$

where $T_n^n = T_{n(\bmod N)}^n$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in [0, 1] with $\alpha_n + \beta_n + \gamma_n = 1$, $\{u_n\}$ is a bounded sequence in C, $\lambda = \min\{\lambda_i : i = 1, 2, \ldots, N\}$, $k_n = \max\{k_n^{(i)}, i = 1, 2, \ldots, N\}$, and $L = \max\{L_i : i = 1, 2, \ldots, N\} > 0$ are positive numbers defined by (1.10) *and* (1.11)*, respectively. If the following conditions are satisfied:*

(i) $0 < \lambda < \liminf_{n\to\infty} \alpha_n \leq \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty} (1-\alpha_n) = \infty$,

(iii) $0 < \beta_n \leq \limsup_{n\to\infty} \beta_n \leq \min\{1-\lambda, 1/L\} < 1$,

(iv) $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ *and* $1 \leq k_n < (1-\lambda)/(1-\liminf_{n\to\infty} \alpha_n)$,

(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$,

then the conclusions of Theorem 2.1 for sequence $\{x_n\}$ defined by (2.39) *still hold.*

References

[1] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, no. 2, pp. 197–228, 1967.

[2] Q. Liu, "Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 11, pp. 1835–1842, 1996.

[3] F. Gu, “The new composite implicit iterative process with errors for common fixed points of a finite family of strictly pseudocontractive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 329, no. 2, pp. 766–776, 2007.

[4] M. O. Osilike, “Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps,” *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 73–81, 2004.

[5] M. O. Osilike, “Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps,” *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 73–81, 2004.

[6] Y. Su and S. Li, “Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps,” *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 882–891, 2006.

[7] H.-K. Xu and R. G. Ori, “An implicit iteration process for nonexpansive mappings,” *Numerical Functional Analysis and Optimization*, vol. 22, no. 5-6, pp. 767–773, 2001.

[8] M. O. Osilike, A. Udomene, D. I. Igbokwe, and B. G. Akuchu, “Demiclosedness principle and convergence theorems for k-strictly asymptotically pseudocontractive maps,” *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1334–1345, 2007.

[9] M. O. Osilike, S. C. Aniagbosor, and B. G. Akuchu, “Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces,” *Pan-American Mathematical Journal*, vol. 12, no. 2, pp. 77–88, 2002.