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*Research Article*

# Fixed Point Theorems for $n$ Times Reasonable Expansive Mapping

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Based on previous notions of expansive mapping, $n$ times reasonable expansive mapping is defined. The existence of fixed point for $n$ times reasonable expansive mapping is discussed and some new results are obtained.



## 1. Introduction and preliminaries

The research about fixed points of expansive mapping was initiated by Machuca (see [1]). Later, Jungck discussed fixed points for other forms of expansive mapping (see [2]). In 1982, Wang et al. (see [3]) published a paper in *Advances in Mathematics* about expansive mapping which draws great attention of other scholars. Also, Zhang has done considerable work in this field. In order to generalize the results about fixed point theory, Zhang (see [4]) published his work *Fixed Point Theory and Its Applications*, in which the fixed point problem for expansive mapping is systematically presented in a chapter. As applications, he also investigated the existence of solutions of equations for locally condensing mapping and locally accretive mapping. In 1991, based on the results obtained by others, the author defined several new kinds of expansive-type mappings in [5], which expanded the expansive-type mapping from 19 to 23, and gave some new applications. Recently, the study about fixed point theorem for expansive mapping and nonexpansive mapping is deeply explored and has extended too many other directions. Motivated and inspired by the works (see [1–13]), in this paper, we define $n$ times reasonable expansive mapping and discuss the existence of fixed point for $n$ times reasonable expansive mapping. For the sake of convenience, we briefly recall some definitions.

Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a mapping.

Throughout this paper, we use $N$ to denote the set of natural numbers and $[x]$ to denote the maximum integral value that is not larger than $x$.

$T : X \to X$ is called an expansive mapping if there exists a constant $h > 1$ such that $d(Tx, Ty) \geq hd(x, y)$, for all $x, y \in X$.

$T : X \to X$ is called a two times reasonable expansive mapping if there exists a constant $h > 1$ such that $d(x, T^2x) \geq hd(x, Tx)$, for all $x \in X$.

$T : X \to X$ is called a twenty-one type expansive mapping if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq h \min\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad \forall x, y \in X. \tag{1.1}$$

$T : X \to X$ is called a twenty-three type expansive mapping if there exists a constant $h > 1$ such that

$$\begin{aligned} d^2(Tx, Ty) \geq h \min\{&d^2(x, y), d(x, y)\cdot d(x, Tx), d(x, Tx)\cdot d(y, Ty), \\ &d^2(x, Tx), d(y, Ty)\cdot d(x, Ty), d(y, Ty)\cdot d(y, Tx)\}, \quad \forall x, y \in X. \end{aligned} \tag{1.2}$$

## 2. Main results

*Definition 2.1.* Let $(X, d)$ be a complete metric space. $T : X \to X$ is called an $n$ $(n \geq 2,\ n \in N)$ times reasonable expansive mapping if there exists a constant $h > 1$ such that

$$d(x, T^n x) \geq hd(x, Tx), \quad \forall x \in X \ (n \geq 2,\ n \in N). \tag{2.1}$$

*Definition 2.2.* Let $(X, d)$ be a complete metric space. $T : X \to X$ is called an $H_1$-type $n$ $(n \geq 2,\ n \in N)$ times reasonable expansive mapping if there exists a constant $h > 1$ such that

$$\begin{aligned} d(T^{n-1}x, T^{n-1}y) \geq h \min\{&d(x, y), d(x, Tx), d(T^{n-2}y, T^{n-1}y), \\ &d(x, T^{n-1}y), d(T^{n-2}y, T^{n-1}x)\}, \quad \forall x, y \in X \ (n \geq 2,\ n \in N). \end{aligned} \tag{2.2}$$

*Definition 2.3.* Let $(X, d)$ be a complete metric space. $T : X \to X$ is called an $H_2$-type $n$ $(n \geq 2,\ n \in N)$ times reasonable expansive mapping if there exists a constant $h > 1$ such that

$$\begin{aligned} d^2(T^{n-1}x, T^{n-1}y) \geq h \min\{&d^2(x, y), d(x, y)\cdot d(x, Tx), d(x, Tx)\cdot d(T^{n-2}y, T^{n-1}y), d^2(x, Tx), \\ &d(T^{n-2}y, T^{n-1}y)\cdot d(x, T^{n-1}y), d(T^{n-2}y, T^{n-1}y)\cdot d(T^{n-2}y, T^{n-1}x)\}, \\ &\forall x, y \in X \ (n \geq 2,\ n \in N). \end{aligned} \tag{2.3}$$

**Lemma 2.4** (see [6]). *Let $(X, d)$ be a complete metric space, let $A$ be a subset of $X$, and let the mappings $f, g : A \to X$ satisfy the following conditions:*

(i) *$f$ is a surjective mapping ($f(A) = X$);*

(ii) *there exists a functional $\varphi : X \to R$ which is lower semicontinuous bounded from below such that $d(f(x), g(x)) \leq \varphi(f(x)) - \varphi(g(x))$, for all $x \in A$.*

*Then, $f$ and $g$ have a coincidence point, that is, there exists at least an $x \in A$ such that $f(x) = g(x)$.*

*Especially, if one lets $A = X, g = I_X$ (the identity mapping on $X$), then $f$ has a fixed point in $X$.*

**Theorem 2.5.** *Let $(X,d)$ be a complete metric space and let $T : X \to X$ be a continuous and surjective mapping if there exists a constant $h > 1$ such that*

$$d(T^{n-1}x, T^n x) \geq hd(x, Tx), \quad \forall x \in X \ \ (n \geq 2,\ n \in N). \tag{2.4}$$

*Then, $T$ has a fixed point in $X$.*

*Proof.* By (2.4), we have

$$d(T^{n-1}x, T^n x) - d(x, Tx) \geq hd(x, Tx) - d(x, Tx), \quad \forall x \in X. \tag{2.5}$$

Thus,

$$d(x, Tx) \leq \frac{1}{h-1}\left[d(T^{n-1}x, T^n x) - d(x, Tx)\right], \quad \forall x \in X. \tag{2.6}$$

Let $\varphi(x) = (1/(h-1))[d(T^{n-1}x, T^{n-2}x) + d(T^{n-2}x, T^{n-3}x) + \cdots + d(T^2x, Tx) + d(Tx, x)]$. Then we have $d(x, Tx) \leq \varphi(Tx) - \varphi(x)$, for all $x \in X$. From the continuity of $d$, we know that $\varphi(x)$ is continuous. Thus $\varphi(x)$ is lower semicontinuous bounded from below. Therefore the conclusion follows immediately from Lemma 2.4. □

**Theorem 2.6.** *Let $(X,d)$ be a complete metric space and let $T : X \to X$ be a continuous and surjective $n$ $(n \geq 2,\ n \in N)$ times reasonable expansive mapping. Assume that either (i) or (ii) holds:*

(i) *$T$ is an $H_1$-type $n$ times reasonable expansive mapping;*

(ii) *$T$ is an $H_2$-type $n$ times reasonable expansive mapping.*

*Then, $T$ has a fixed point in $X$.*

*Proof.* In the case of (i), taking $y = Tx$ in (2.2), we have

$$\begin{aligned} d(T^{n-1}x, T^n x) &\geq h \min\left\{d(x, Tx), d(x, Tx), d(T^{n-1}x, T^n x), d(x, T^n x), d(T^{n-1}x, T^{n-1}x)\right\} \\ &= h \min\left\{d(x, Tx), d(T^{n-1}x, T^n x), d(x, T^n x)\right\}. \end{aligned} \tag{2.7}$$

Because $T$ is an $n$ times reasonable expansive mapping, we have

$$d(x, T^n x) \geq hd(x, Tx) > d(x, Tx). \tag{2.8}$$

Thus, we obtain

$$d(T^{n-1}x, T^n x) \geq h \min\left\{d(x, Tx), d(T^{n-1}x, T^n x)\right\}. \tag{2.9}$$

If $d(T^{n-1}x, T^n x) = \min\{d(x, Tx), d(T^{n-1}x, T^n x)\}$, then $d(T^{n-1}x, T^n x) \geq hd(T^{n-1}x, T^n x)$. Hence, $d(T^{n-1}x, T^n x) = 0$ (otherwise, $d(T^{n-1}x, T^n x) > d(T^{n-1}x, T^n x)$, which is a contradiction). Therefore, $T^{n-1}x = T^n x$, that is $T^{n-1}x = T(T^{n-1}x)$, which implies that $T^{n-1}x$ is a fixed point of $T$ in $X$.

If $d(x, Tx) = \min\{d(x, Tx), d(T^{n-1}x, T^n x)\}$, then $d(T^{n-1}x, T^n x) \geq hd(x, Tx)$.

By Theorem 2.5, we obtain that $T$ has a fixed point in $X$.

In the case of (ii), taking $y = Tx$ in (2.3), we have

$$\begin{aligned} d^2(T^{n-1}x, T^n x) &\geq h \min\{d^2(x,Tx), d(x,Tx)\cdot d(x,Tx), d(x,Tx)\cdot d(T^{n-1}x, T^n x), \\ &\qquad d^2(x,Tx), d(T^{n-1}x,T^n x)\cdot d(x,T^n x), d(T^{n-1}x,T^n x)\cdot d(T^{n-1}x, T^{n-1}x)\} \\ &= h \min\{d^2(x,Tx), d(x,Tx)\cdot d(T^{n-1}x,T^n x), d(T^{n-1}x,T^n x)\cdot d(x,T^n x)\}. \end{aligned} \tag{2.10}$$

Because $T$ is an $n$ ($n \geq 2$, $n \in N$) times reasonable expansive mapping, we have

$$d(x, T^n x) \geq hd(x,Tx) > d(x,Tx). \tag{2.11}$$

Hence, $d(x,T^n x)\cdot d(T^{n-1}x,T^n x) > d(x,Tx)\cdot d(T^{n-1}x,T^n x)$.
Therefore, we have

$$d^2(T^{n-1}x,T^n x) \geq h \min\{d^2(x,Tx), d(x,Tx)\cdot d(T^{n-1}x,T^n x)\}. \tag{2.12}$$

If $d^2(x,Tx) = \min\{d^2(x,Tx), d(x,Tx)\cdot d(T^{n-1}x,T^n x)\}$, then

$$d^2(T^{n-1}x,T^n x) \geq hd^2(x,Tx) \quad \forall x \in X, \tag{2.13}$$

that is, $d(T^{n-1}x,T^n x) \geq \sqrt{h}d(x,Tx)$.

Because $\sqrt{h} > 1$, by Theorem 2.5, we obtain that $T$ has a fixed point in $X$.

If $d(x,Tx)\cdot d(T^{n-1}x,T^n x) = \min\{d^2(x,Tx), d(x,Tx)\cdot d(T^{n-1}x,T^n x)\}$, then $d^2(T^{n-1}x, T^n x) \geq hd(x,Tx)\cdot d(T^{n-1}x,T^n x)$, that is

$$d(T^{n-1}x,T^n x)\cdot(d(T^{n-1}x,T^n x) - hd(x,Tx)) \geq 0. \tag{2.14}$$

If $d(T^{n-1}x,T^n x) = 0$, then $T^{n-1}x = T^n x$, that is $T^{n-1}x = T(T^{n-1}x)$, which implies that $T^{n-1}x$ is a fixed point of $T$ in $X$.

If $d(T^{n-1}x,T^n x) \neq 0$, then $d(T^{n-1}x,T^n x) \geq hd(x,Tx)$. By Theorem 2.5, we obtain that $T$ has a fixed point in $X$.

Therefore, by induction we derive that $T$ has a fixed point in $X$. □

**Corollary 2.7.** *Let $(X,d)$ be a complete metric space. If $T : X \to X$ is a continuous and surjective twenty-one type expansive mapping and $T : X \to X$ is a two times reasonable expansive mapping, then $T$ has a fixed point in $X$.*

*Proof.* We denote $y = T^o y$; taking $n = 2$ under the condition (i) of Theorem 2.6, Corollary 2.7 is proved immediately. □

Similarly, we denote $y = T^o y$; taking $n = 2$ under the condition (ii) of Theorem 2.6, we can obtain the following Corollary 2.8.

**Corollary 2.8.** *Let $(X,d)$ be a complete metric space. If $T : X \to X$ is a continuous and surjective twenty-three type expansive mapping and $T : X \to X$ is a two times reasonable expansive mapping, then $T$ has a fixed point in $X$.*

*Remark 2.9.* Corollaries 2.7 and 2.8 are Theorems 2.3 and 2.5 in [5], respectively. Thus, Theorems 2.3 and 2.5 in [5] are the special examples of Theorem 2.6.

**Theorem 2.10.** *Let* $(X, d)$ *be a complete metric space and let* $T : X \to X$ *be a continuous and surjective* $n$ $(n \geq 2,\ n \in N)$ *times reasonable expansive mapping. If there exists a constant* $h > 1$ *such that*

$$d(T^n x, T^n y) \geq h \min\{d(x,y), d(y, T^n y)\}, \quad \forall x, y \in X \ (n \geq 2,\ n \in N), \tag{2.15}$$

*then* $T$ *has a fixed point.*

*Proof.* Letting $x = Ty$ in (2.15), we have

$$d(T^{n+1} y, T^n y) \geq h \min\{d(Ty, y), d(y, T^n y)\}, \quad \forall y \in X. \tag{2.16}$$

Since $T$ is an $n$ $(n \geq 2,\ n \in N)$ times reasonable expansive mapping, then

$$d(y, T^n y) \geq h d(y, Ty) > d(y, Ty), \quad \forall y \in X. \tag{2.17}$$

By (2.16) and (2.17), we have $d(T^{n+1} y, T^n y) \geq h d(Ty, y)$ for all $y \in X$.
It follows from Theorem 2.5 that $T$ has a fixed point in $X$. □

*Remark 2.11.* Generally speaking, $n$ $(n \geq 2,\ n \in N)$ times reasonable expansive mapping does not necessarily have a fixed point. This can be illustrated by the following examples.

*Example 2.12.* We denote by $B_1$ the unit circle which takes the original point as its center and 1 as its radius on the complex plane, that is, $B_1 = \{Z \mid |Z| = 1,\ Z \in C\}$. $B_1$ can also be written as $\{e^{i\theta} \mid e^{i\theta} \in C, -\infty < \theta < +\infty\}$. Suppose that $T : B_1 \to B_1$ is a mapping defined as follows:

$$TZ = Te^{i\theta} = e^{i(\theta + 2\pi/3n)}. \tag{2.18}$$

For every $Z \in B_1$, that is, $Z = e^{i\theta}$, we have

$$\begin{aligned}
TZ &= Te^{i\theta} = e^{i(\theta + 2\pi/3n)}, \\
T^2 Z &= T(TZ) = T\left(Te^{i\theta}\right) = Te^{i(\theta + 2\pi/3n)} = e^{i(\theta + 2(2\pi/3n))}, \\
&\cdots \\
T^k Z &= e^{i(\theta + k(2\pi/3n))}, \\
&\cdots \\
T^n Z &= e^{i(\theta + n(2\pi/3n))} = e^{i(\theta + 2\pi/3)}.
\end{aligned} \tag{2.19}$$

From the above equations, we obtain

$$\begin{aligned}
d(Z, T^n Z) &= \left|T^n Z - Z\right| = \left|e^{i(\theta + 2\pi/3)} - e^{i\theta}\right| = \left|e^{i\theta}\right| \cdot \left|e^{i(2\pi/3)} - 1\right| \\
&= \left|\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} - 1\right| = \left|-\frac{1}{2} + \frac{\sqrt{3}}{2} i - 1\right| = \sqrt{3}, \\
d(Z, TZ) &= |TZ - Z| = \left|e^{i(\theta + 2\pi/3n)} - e^{i\theta}\right| = \left|e^{i\theta}\right| \cdot \left|e^{i(2\pi/3n)} - 1\right| = \left|\cos\frac{2\pi}{3n} + i \sin\frac{2\pi}{3n} - 1\right| \\
&= \sqrt{2 - 2\cos\frac{2\pi}{3n}} = 2\sqrt{\sin^2\frac{\pi}{3n}} = 2\sin\frac{\pi}{3n} \quad (n \geq 2,\ n \in N).
\end{aligned} \tag{2.20}$$

Since $n \geq 2$, then $\sin(\pi/3n) \leq 1/2$. Thus $d(Z, T^n Z)/d(Z, TZ) \geq \sqrt{3}$, for all $Z \in B_1$, that is, $d(Z, T^n Z) \geq \sqrt{3} d(Z, TZ)$, for all $Z \in B_1$. We can take a constant $h = \sqrt{3}$, which means that there exists a constant $h > 1$ such that $d(Z, T^n Z) \geq h d(Z, TZ)$, for all $Z \in B_1$ ($n \geq 2$, $n \in N$). Therefore, $T$ is an $n$ times reasonable expansive mapping. Since $e^{i\theta} \neq e^{i(\theta + 2\pi/3)}$, then $TZ \neq Z$, for all $Z \in B_1$. It implies that $T$ does not have a fixed point.

*Example 2.13.* Suppose that $T : R \to R$ is a mapping defined as $Tx = x + 1$.

It is obvious that $T$ is continuous and surjective and $T$ does not have a fixed point.

Now, we prove $T$ is an $n$ times reasonable expansive mapping.

In fact, by the definition of $T$, we have $T^n x = x + n$ $(n \geq 2, n \in N)$ .

Because $d(x, T^n x) = |x + n - x| = n \geq 2$ and $d(x, Tx) = |x + 1 - x| = 1$, we have $d(x, T^n x) \geq 2 d(x, Tx)$. Thus, we can take a constant $h = 2$, which means that there exists a constant $h > 1$ such that $d(x, T^n x) \geq h d(x, Tx)$, for all $x \in R$ $(n \geq 2, n \in N)$ .

Therefore, $T$ is an $n$ times reasonable expansive mapping.

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## References

[1] R. Machuca, "A coincidence theorem," *The American Mathematical Monthly*, vol. 74, no. 5, p. 569, 1967.
[2] G. Jungck, "Commuting mappings and fixed points," *The American Mathematical Monthly*, vol. 83, no. 4, pp. 261–263, 1976.
[3] S. Wang, B. Li, and Z. Gao, "Expansive mappings and fixed-point theorems for them," *Advances in Mathematics*, vol. 11, no. 2, pp. 149–153, 1982 (Chinese).
[4] S. Zhang, *Fixed Point Theory and Its Applications*, Chongqing Press, Chongqing, China, 1984.
[5] C. Zhu, "Several new mappings of expansion and fixed point theorems," *Journal of Jiangxi Normal University*, vol. 15, no. 8, pp. 244–248, 1991 (Chinese).
[6] S. Park, "On extensions of the Caristi-Kirk fixed point theorem," *Journal of the Korean Mathematical Society*, vol. 19, no. 2, pp. 143–151, 1983.
[7] C. Zhu, "Several nonlinear operator problems in the Menger PN space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1281–1284, 2006.
[8] D. Guo, *Nonlinear Functional Analysis*, Shandong Science and Technology Press, Jinan, China, 1985.
[9] C. Zhu and Z. Xu, "Several problems of random nonlinear equations," *Acta Analysis Functionalis Applicata*, vol. 5, no. 3, pp. 217–219, 2003 (Chinese).
[10] C. Zhu, "Generalizations of Krasnoselskii's theorem and Petryshyn's theorem," *Applied Mathematics Letters*, vol. 19, no. 7, pp. 628–632, 2006.
[11] C. Zhu and Z. Xu, "Random ambiguous point of random $k(\omega)$-set-contractive operator," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 1, pp. 2–6, 2007.
[12] C. Zhu and X. Huang, "The topological degree of A-proper mapping in the Menger PN-space—II," *Bulletin of the Australian Mathematical Society*, vol. 73, no. 2, pp. 169–173, 2006.
[13] C. Zhu and Z. Xu, "A class of random operator equations in the Hilbert space," *Acta Mathematica Sinica*, vol. 47, no. 4, pp. 641–646, 2004 (Chinese).