Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2007, Article ID 76040, 15 pages
doi:10.1155/2007/76040

Research Article

An Algorithm Based on Resolvant Operators for Solving Positively Semidefinite Variational Inequalities

Juhe Sun, Shaowu Zhang, and Liwei Zhang

Received 16 June 2007; Accepted 19 September 2007

Recommended by Nan-Jing Huang

A new monotonicity, M-monotonicity, is introduced, and the resolvant operator of an M-monotone operator is proved to be single-valued and Lipschitz continuous. With the help of the resolvant operator, the positively semidefinite general variational inequality (VI) problem VI $(S^n_+, F+G)$ is transformed into a fixed point problem of a nonexpansive mapping. And a proximal point algorithm is constructed to solve the fixed point problem, which is proved to have a global convergence under the condition that F in the VI problem is strongly monotone and Lipschitz continuous. Furthermore, a convergent path Newton method is given for calculating ϵ-solutions to the sequence of fixed point problems, enabling the proximal point algorithm to be implementable.

1. Introduction

In recent years, the variational inequality has been addressed in a large variety of problems arising in elasticity, structural analysis, economics, transportation equilibrium, optimization, oceanography, and engineering sciences [1, 2]. Inspired by its wide applications, many researchers have studied the classical variational inequality and generalized it in various directions. Also, many computational methods for solving variational inequalities have been proposed (see [3–8] and the references therein). Among these methods, resolvant operator technique is an important one, which was studied in the 1990s by many researchers (such as [4, 6, 9]), and further studies developed recently [3, 10, 11].

As monotonicity plays an important role in the theory of variational inequality and its generalizations, in this paper, we introduce a new class of monotone operator: M-monotone operator. The resolvant operator associated with an M-monotone operator is proved to be Lipschitz-continuous. Applying the resolvant operator technique, we transform the positively semidefinite variational inequality (VI) problem $VI(S_+^n, F+G)$ into a fixed point problem of a nonexpansive mapping and suggest a proximal point algorithm to solve the fixed point problem. Under the condition that F in the VI problem is strongly monotone and Lipschitz-continuous, we prove that the algorithm has a global convergence. To ensure the proposed proximal point algorithm is implementable, we introduce a path Newton algorithm whose step size is calculated by Armijo rule.

In the next section, we recall some results and concepts that will be used in this paper. In Section 3, we introduce the definition of an M-monotone operator, and discuss properties of this kind of operators, especially the Lipschitz continuity of the resolvant operator of an M-monotone operator. In Section 4, we construct a proximal point algorithm, based on the results in Section 3, for $VI(S_+^n, F+G)$, and prove its global convergence. To ensure that the proposed proximal point algorithm in Section 4 is implementable, we introduce a path Newton algorithm, in Section 5, in which the step size is calculated by Armijo rule.

2. Preliminaries

Throughout this paper, we assume that S^n denotes the space of $n \times n$ symmetric matrices and S_+^n denote the cone of $n \times n$ symmetric positive semidefinite matrices. For $A, B \in S^n$, we define an inner product $\langle A, B\rangle = \operatorname{tr}(AB)$ which induces the norm $\|A\| = \sqrt{\langle A, A\rangle}$. Let 2^{S^n} denote the family of all the nonempty subsets of S^n. We recall the following concepts, which will be used in the sequel.

Definition 2.1. Let $A, B, C : S^n \to S^n$ be single-valued operators and let $M : S^n \times S^n \to S^n$ be mapping.

(i) $M(A, \cdot)$ is said to be α-strongly monotone with respect to A if there exists a constant $\alpha > 0$ satisfying

$$\langle M(Ax, u) - M(Ay, u), x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y, u \in S^n; \tag{2.1}$$

(ii) $M(\cdot, B)$ is said to be β-relaxed monotone with respect to B if there exists a constant $\beta > 0$ satisfying

$$\langle M(u, Bx) - M(u, By), x - y\rangle \geq -\beta\|x - y\|^2, \quad \forall x, y, u \in S^n; \tag{2.2}$$

(iii) $M(\cdot, \cdot)$ is said to be $\alpha\beta$-symmetric monotone with respect to A and B if $M(A, \cdot)$ is α-strongly monotone with respect to A; and $M(\cdot, B)$ is β-relaxed monotone with respect to B with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y$, for all $x, y, u \in S^n$;

(iv) $M(\cdot, \cdot)$ is said to be ξ-Lipschitz-continuous with respect to the first argument if there exists a constant $\xi > 0$ satisfying

$$\|M(x, u) - M(y, u)\| \leq \xi\|x - y\|, \quad \forall x, y, u \in S^n; \tag{2.3}$$

(iv) A is said to be t-Lipschitz-continuous if there exists a constant $t > 0$ satisfying

$$\|Ax - Ay\| \leq t\|x - y\|, \quad \forall x, y \in S^n; \tag{2.4}$$

(vi) B is said to be l-cocoercive if there exists a constant $l > 0$ satisfying

$$\langle Bx - By, x - y\rangle \geq l\|Bx - By\|^2, \quad \forall x, y \in S^n; \tag{2.5}$$

(vii) C is said to be r-strongly monotone with respect to $M(A,B)$ if there exists a constant $r > 0$ satisfying

$$\big\langle Cx - Cy, M(Ax,Bx) - M(Ay,By)\big\rangle \geq r\|x - y\|^2, \quad \forall x, y \in S^n. \tag{2.6}$$

In a similar way to (v), we can define the Lipschitz continuity of the mapping M with respect to the second argument.

Definition 2.2. Let $A, B : S^n \to S^n$, $M : S^n \times S^n \to S^n$ be mappings. M is said to be coercive with respect to A and B if

$$\lim_{\|x\|\to\infty} \frac{\langle M(Ax,Bx), x\rangle}{\|x\|} = +\infty. \tag{2.7}$$

Definition 2.3. Let $A, B : S^n \to S^n$, $M : S^n \times S^n \to S^n$ be mappings. M is said to be bounded with respect to A and B if $M(A(P), B(P))$ is bounded for every bounded subset P of S^n. M is said to be semicontinuous with respect to A and B if for any fixed $x, y, z \in S^n$, the function $t \mapsto \langle M(A(x+ty), B(x+ty)), z\rangle$ is continuous at 0^+.

Definition 2.4. $T : S^n \to 2^{S^n}$ is said to be monotone if

$$\langle x - y, u - v\rangle \geq 0, \quad \forall u, v \in S^n,\ x \in Tu,\ y \in Tv; \tag{2.8}$$

and it is said to be maximal monotone if T is monotone and $(I + cT)(S^n) = S^n$ for all $c > 0$, where I denotes the identity mapping on S^n.

3. M-Monotone operators

In this section, we introduce M-monotonicity of operators and discuss its properties.

Definition 3.1. Let $A, B : S^n \to S^n$ be single-valued operators, $M : S^n \times S^n \to S^n$ a mapping, and $T : S^n \to 2^{S^n}$ a multivalue operator. T is said to be M-monotone with respect to A and B if T is monotone and $(M(A,B) + cT)(S^n) = S^n$ holds for every $c > 0$.

Remark 3.2. If $M(A,B) = H$, then the above definition reduces to H-monotonicity, which was studied in [5]. If $M(A,B) = I$, then the definition of I-monotonicity is just the maximal monotonicity.

Remark 3.3. Let T be a monotone operator and let c be a positive constant. If $T : S^n \to 2^{S^n}$ is an M-monotone operator with respect to A and B, every matrix $z \in S^n$ can be written in exactly one way as $M(Ax,Bx) + cu$, where $u \in T(x)$.

Proposition 3.4. *Let M be $\alpha\beta$-symmetric monotone with respect to A and B and let $T : S^n \to 2^{S^n}$ be an M-monotone operator with respect to A and B, then T is maximal monotone.*

Proof. Since T is monotone, it is sufficient to prove the following property; inequality $\langle x - y, u - v\rangle \geq 0$ for $(v, y) \in \text{Graph}(T)$ implies that

$$x \in Tu. \tag{3.1}$$

Suppose, by contradiction, that there exists some $(u_0,x_0)\bar{\in}\text{Graph}(T)$ such that

$$\langle x_0 - y, u_0 - v\rangle \geq 0, \quad \forall (v, y) \in \text{Graph}(T). \tag{3.2}$$

Since T is M-monotone with respect to A and B, $(M(A,B) + cT)(S^n) = S^n$ holds for every $c > 0$, there exists $(u_1,x_1) \in \text{Graph}(T)$ such that

$$M(Au_1,Bu_1) + cx_1 = M(Au_0,Bu_0) + cx_0 \in S^n. \tag{3.3}$$

It follows form (3.2) and (3.3) that

$$\begin{aligned} 0 &\leq c\langle x_0 - x_1, u_0 - u_1\rangle \\ &= -\langle M(Au_0,Bu_0) - M(Au_1,Bu_1), u_0 - u_1\rangle \\ &= -\langle M(Au_0,Bu_0) - M(Au_1,Bu_0), u_0 - u_1\rangle \\ &\quad - \langle M(Au_1,Bu_0) - M(Au_1,Bu_1), u_0 - u_1\rangle \\ &\leq -(\alpha - \beta)\|u_0 - u_1\| \\ &\leq 0, \end{aligned} \tag{3.4}$$

which yields $u_1 = u_0$. By (3.3), we have that $x_1 = x_0$. Hence $(u_0,x_0) \in \text{Graph}(T)$, which is a contradiction. Therefore (3.1) holds and T is maximal monotone. This completes the proof. □

The following example shows that a maximal monotone operator may not be M-monotone for some A and B.

Example 3.5. Let $S^n = S^2$, $T = I$, and $M(Ax,Bx) = x^2 + 2E - x$ for all $x \in S^2$, where E is an identity matrix. Then it is easy to see that I is maximal monotone. For all $x \in S^2$, we have that

$$\|(M(A,B) + I)(x)\|^2 = \|x^2 + 2E - x + x\|^2 = \|x^2 + 2E\|^2 = \operatorname{tr}\left[(x^2 + 2E)^2\right] \geq 8, \tag{3.5}$$

which means that $0\bar{\in}(M(A,B) + I)(S^2)$ and I is not M-monotone with respect to A and B.

Proposition 3.6. *Let $T : S^n \to 2^{S^n}$ be a maximal monotone operator and let $M : S^n \times S^n \to S^n$ be a bounded, coercive, semicontinuous, and $\alpha\beta$-symmetric monotone operator with respect to A and B. Then T is M-monotone with respect to A and B.*

Proof. For every $c > 0$, cT is maximal monotone since T is maximal monotone. Since M is bounded, coercive, semicontinuous, and $\alpha\beta$-symmetric monotone operator with respect to A and B, it follows from [9, Corollary 32.26] that $M(A,B)+cT$ is surjective, that is, $(M(A,B)+cT)(S^n)=S^n$ holds for every $c>0$. Thus, T is an M-monotone operator with respect to A and B. The proof is complete. □

Theorem 3.7. *Let M be an $\alpha\beta$-symmetric monotone with respect to A and B and let T be an M-monotone operator with respect to A and B. Then the operator $(M(A,B)+cT)^{-1}$ is single-valued.*

Proof. For any given $u\in S^n$, let $x,y\in(M(A,B)+cT)^{-1}(u)$. It follows that $-M(Ax,Bx)+u\in Tx$ and $-M(Ay,By)+u\in Ty$. The monotonicity of T and M implies that

$$\begin{aligned}
0 &\le \langle -M(Ax,Bx)+u-(-M(Ay,By)+u),x-y\rangle \\
&= -\langle M(Ay,By)-M(Ax,Bx),x-y\rangle \\
&\le -(\alpha-\beta)||u_0-u_1|| \\
&\le 0.
\end{aligned} \tag{3.6}$$

From the symmetric monotonicity of M, we get that $x=y$. Thus $(M(A,B)+cT)^{-1}$ is single-valued. This completes the proof. □

Definition 3.8. Let M be an $\alpha\beta$-symmetric monotone with respect to A and B and let T be an M-monotone operator with respect to A and B. The resolvant operator $J_{cT}^M:S^n\to S^n$ is defined by

$$J_{cT}^M(u)=\big(M(A,B)+cT\big)^{-1}(u),\quad \forall u\in S^n. \tag{3.7}$$

Theorem 3.9. *Let $M(A,B)$ be α-strongly monotone with respect to A and β-relaxed monotone with respect to B with $\alpha>\beta$. Suppose that $T:S^n\to 2^{S^n}$ is an M-monotone operator. Then the resolvant operator $J_{cT}^M:S^n\to S^n$ is Lipschitz-continuous with constant $1/(\alpha-\beta)$, that is,*

$$||J_{cT}^M(u)-J_{cT}^M(v)||\le\frac{1}{\alpha-\beta}\|u-v\|,\quad \forall u,v\in S^n. \tag{3.8}$$

Since the proof of Theorem 3.9 is similar as that of [5, Theorem 2.2], we here omit it.

4. An algorithm for variational inequalities

Let $F,G:S_+^n\to S^n$ be operators. Consider the general variational inequality problem $VI(S_+^n,F+G)$, defined by finding $u\in S_+^n$ such that

$$\langle F(u)+G(u),v-u\rangle\ge 0,\quad \forall v\in S_+^n. \tag{4.1}$$

We can rewrite it as the problem of finding $u\in S_+^n$ such that

$$0\in G(u)+T(u), \tag{4.2}$$

where $T\equiv F+\mathcal{N}(\cdot;S_+^n)$. Let $\mathrm{Sol}(S_+^n,F+G)$ be the set of solutions of $VI(S_+^n,F+G)$.

Proposition 4.1. *Let $F, G : S^n_+ \to S^n$ be continuous and let $M : S^n \times S^n \to S^n$ be a bounded, coercive, semicontinuous, and $\alpha\beta$-symmetric monotone operator with respect to $A : S^n \to S^n$ and $B : S^n \to S^n$. Then the following two properties hold for the map $T \equiv F + \mathcal{N}(\cdot; S^n_+)$:*

(a) *$J^M_{cT}(M(Ax,Bx) - cG(x)) = \mathrm{Sol}(S^n_+, F_{cx})$, where $F_{cx}(y) = M(Ay,By) - M(Ax,Bx) + c(F(y) + G(x))$;*

(b) *If F is monotone, then T is M-monotone with respect to A and B.*

Proof. We have that the inclusion

$$y \in J^M_{cT}(M(Ax,Bx) - cG(x)) = \left(M(A,B) + cT\right)^{-1}\left(M(Ax,Bx) - cG(x)\right) \tag{4.3}$$

is equivalent to

$$M(Ax,Bx) \in \left(M(A,B) + cF + c\mathcal{N}\left(\cdot; S^n_+\right)\right)(y) + cG(x), \tag{4.4}$$

or in other words,

$$0 \in M(Ay,By) - M(Ax,Bx) + c\left(F(y) + G(x)\right) + \mathcal{N}\left(y; S^n_+\right). \tag{4.5}$$

This establishes (a).

By [10, Proposition 12.3.6], we can deduce that T is maximal monotone, it follows from Proposition 3.6, we get that T is M-monotone with respect to A and B. This completes the proof. □

Lemma 4.2. *Let M be an $\alpha\beta$-symmetric monotone with respect to A and B and let T be an M-monotone operator with respect to A and B. Then $u \in S^n_+$ is a solution of $0 \in G(u) + T(u)$ if and only if*

$$u = J^M_{cT}(M(Au,Bu) - cG(u)), \tag{4.6}$$

where $J^M_{cT} = (M(A,B) + cT)^{-1}$ and $c > 0$ is a constant.

In order to obtain our results, we need the following assumption.

Assumption 4.3. *The mappings F, G, M, A, B satisfy the following conditions.*

(1) *F is L-Lipschitz-continuous and m-strongly monotone.*

(2) *$M(A,\cdot)$ is α-strongly monotone with respect to A; and $M(\cdot,B)$ is β-relaxed monotone with respect to B with $\alpha > \beta$.*

(3) *$M(\cdot,\cdot)$ is ξ-Lipschitz-continuous with respect to the first argument and ζ-Lipschitz-continuous with respect to the second argument.*

(4) *A is τ-Lipschitz-continuous and B is t-Lipschitz-continuous.*

(5) *G is γ-Lipschitz-continuous and s-strongly monotone with respect to $M(A,B)$.*

Remark 4.4. Let Assumption 4.3 hold and

$$\left| c - \frac{s}{\gamma^2} \right| \le \frac{\sqrt{s^2 - \gamma^2\left[(\xi\tau + \zeta t)^2 - (\alpha - \beta)^2\right]}}{\gamma^2}, \quad s^2 > \gamma^2\left[(\xi\tau + \zeta t)^2 - (\alpha - \beta)^2\right]. \tag{4.7}$$

We can deduce that

$$\begin{aligned}
&\left\|J_{cT}^{M}(M(Ax,Bx)-cG(x))-J_{cT}^{M}(M(Ay,By)-cG(y))\right\| \\
&\quad\le \frac{1}{\alpha-\beta}\left\|M(Ax,Bx)-M(Ay,By)-c(G(x)-G(y))\right\| \\
&\quad\le \frac{\sqrt{(\xi\tau+\zeta t)^2-2cs+c^2\gamma^2}}{\alpha-\beta}\|x-y\| \\
&\quad\le \|x-y\|,
\end{aligned} \tag{4.8}$$

which implies that $J_{cT}^{M}(M(A,B)-cG)$ is nonexpansive. Then, it is natural to consider the recursion

$$x^{k+1}\equiv J_{cT}^{M}(M(Ax^k,Bx^k)-cG(x^k)), \tag{4.9}$$

which is desired to converge to a zero of $G+T$. Actually, this can be proved to be true. However, based on Lemma 4.2, we construct the following proximal point algorithm for $VI(S_+^n,F+G)$.

Algorithm 4.5

Data. $x^0\in S^n$, $c_0>0$, $\varepsilon_0\ge 0$, *and* $\rho_0>0$.

Step 1. *Set* $k=0$.

Step 2. *If* $x^k\in \mathrm{Sol}(S_+^n,F+G)$, *stop.*

Step 3. *Find* w^k *such that* $\|w^k-J_{c_kT}^{M}(M(Ax^k,Bx^k)-c_kG(x^k))\|\le\varepsilon_k$.

Step 4. *Set* $x^{k+1}\equiv(1-\rho_k)x^k+\rho_k w^k$ *and select* c_{k+1}, ε_{k+1} *and* ρ_{k+1}. *Set* $k\leftarrow k+1$ *and go to Step 1.*

The following theorem fully describes the convergence of Algorithm 4.5 for finding a solution to $VI(S_+^n,F+G)$.

Theorem 4.6. *Suppose that Algorithm 4.5 holds. Let M be bounded, coercive, semicontinuous, and $\alpha\beta$-symmetric monotone with respect to A and B; and let F be monotone and Lipschitz-continuous. Let $x^0\in S^n$ be given, let $\{\varepsilon_k\}\subset[0,\infty)$ satisfy $E\equiv\sum_{k=1}^{\infty}\varepsilon_k<\infty$, $\{c_k\}\subset(c_m,\infty)$, where $c_m>0$ and*

$$\left|c_k-\frac{s}{\gamma^2}\right|<\frac{\sqrt{s^2-\gamma^2[(\xi\tau+\zeta t)^2-(\alpha-\beta)^2]}}{\gamma^2},\qquad s^2>\gamma^2[(\xi\tau+\zeta t)^2-(\alpha-\beta)^2], \tag{4.10}$$

which implies that

$$\widetilde{L}=\frac{\left[\alpha-\beta-\sqrt{(\xi\tau+\zeta t)^2-2c_ks+c_k^2\gamma^2}\right]}{\left[\alpha-\beta+3\sqrt{(\xi\tau+\zeta t)^2-2c_ks+c_k^2\gamma^2}\right]^2}>0. \tag{4.11}$$

If $\{\rho_k\}\subseteq[R_m,R_M]$, where $0<R_m\le R_M\le p\widetilde{L}$, for all $p\in[2,+\infty)$, then the sequence $\{x^k\}$ generated by Algorithm 4.5 converges to a solution of $VI(S_+^n,F+G)$.

Proof. We introduce a new map

$$Q^k \equiv I - J^M_{c_k T}(M(A,B) - c_k G). \tag{4.12}$$

Clearly, any zero of $G + F + \mathcal{N}(\cdot; S^n_+)$, being a fixed point of $J^M_{c_k T}(M(A,B) - c_k G)$, is also a zero of Q^k. Now, let us prove that Q^k is $\widetilde{L}$-cocoercive.

For $x, y \in S^n$ we know that

$$\begin{aligned}
&\langle Q^k(x) - Q^k(y), x - y\rangle \\
&\quad = \langle x - y - (J^M_{c_k T}(M(Ax,Bx) - c_k G(x)) - J^M_{c_k T}(M(Ay,By) - c_k G(y))), x - y\rangle \\
&\quad = \|x - y\|^2 - \langle J^M_{c_k T}(M(Ax,Bx)) - J^M_{c_k T}(M(Ay,By) - c_k(G(x) - G(y))), x - y\rangle \\
&\quad \geq \|x - y\|^2 - \frac{1}{\alpha - \beta}||M(Ax,Bx) - M(Ay,By) - c_k(G(x) - G(y))||\,\|x - y\| \\
&\quad \geq \|x - y\|^2 - \frac{1}{\alpha - \beta}\sqrt{(\xi\tau + \zeta t)^2 - 2c_k s + c_k^2\gamma^2}\,\|x - y\|^2 \\
&\quad = \left(1 - \frac{\sqrt{(\xi\tau + \zeta t)^2 - 2c_k s + c_k^2\gamma^2}}{\alpha - \beta}\right)\|x - y\|^2,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&||Q^k(x) - Q^k(y)||^2 \\
&\quad = ||x - y - (J^M_{c_k T}(M(Ax,Bx) - c_k G(x)) - J^M_{c_k T}(M(Ay,By) - c_k G(y)))||^2 \\
&\quad = \|x - y\|^2 - 2\langle x - y, J^M_{c_k T}(M(Ax,Bx) - c_k G(x)) - J^M_{c_k T}(M(Ay,By) - c_k G(y))\rangle \\
&\qquad + ||J^M_{c_k T}(M(Ax,Bx) - c_k G(x)) - J^M_{c_k T}(M(Ay,By) - c_k G(y))||^2 \\
&\quad \leq \|x - y\|^2 + 2\frac{\sqrt{(\xi\tau + \zeta t)^2 - 2c_k s + c_k^2\gamma^2}}{\alpha - \beta}\|x - y\|^2 \\
&\qquad + \frac{\sqrt{(\xi\tau + \zeta t)^2 - 2c_k s + c_k^2\gamma^2}}{\alpha - \beta}\|x - y\|^2 \\
&\quad = \left(1 + 3\frac{\sqrt{(\xi\tau + \zeta t)^2 - 2c_k s + c_k^2\gamma^2}}{\alpha - \beta}\right)\|x - y\|^2.
\end{aligned} \tag{4.14}$$

Inequalities (4.13) and (4.14) imply that

$$\begin{aligned}
&\langle Q^k(x) - Q^k(y), x - y\rangle \\
&\quad \geq \left[1 - \frac{\sqrt{(\xi\tau + \zeta t)^2 - 2c_k s + c_k^2\gamma^2}}{\alpha - \beta}\right]\left[1 + 3\frac{\sqrt{(\xi\tau + \zeta t)^2 - 2c_k s + c_k^2\gamma^2}}{\alpha - \beta}\right]^{-1}||Q^k(x) - Q^k(y)||^2 \\
&\quad = \widetilde{L}||Q^k(x) - Q^k(y)||^2.
\end{aligned} \tag{4.15}$$

For all k, we denote by $\overline{x}^k$ the point computed exactly by the resolvent. That is,

$$\overline{x}^{k+1} \equiv (1-\rho_k)x^k + \rho_k J_{c_k T}^{M}(M(Ax^k, Bx^k) - c_k G(x^k)). \tag{4.16}$$

For every zero x^* of T, we obtain

$$\begin{aligned}
||\overline{x}^{k+1} - x^*||^2 &= ||x^k - \rho_k Q^k(x^k) - x^*||^2 \\
&= ||x^k - x^*||^2 - 2\rho_k \langle Q^k(x^k) - Q^k(x^*), x^k - x^* \rangle + \rho_k^2 ||Q^k(x^k)||^2 \\
&\le ||x^k - x^*||^2 - 2\rho_k \widetilde{L} ||Q^k(x^k)||^2 + \rho_k^2 ||Q^k(x^k)||^2 \\
&\le ||x^k - x^*||^2 - \rho_k (2\widetilde{L} - \rho_k) ||Q^k(x^k)||^2 \\
&\le ||x^k - x^*||^2 - R_m (2\widetilde{L} - R_M) ||Q^k(x^k)||^2 \\
&\le ||x^k - x^*||^2.
\end{aligned} \tag{4.17}$$

Since $\|x^k - \overline{x}^k\| \le \rho_k \varepsilon_k$, we get that

$$\begin{aligned}
||x^{k+1} - x^*|| &\le ||\overline{x}^{k+1} - x^*|| + ||x^{k+1} - \overline{x}^{k+1}|| \\
&\le ||x^k - x^*|| + \rho_k \varepsilon_k \\
&\le ||x^0 - x^*|| + \sum_{i=0}^{k} \rho_i \varepsilon_i \\
&\le ||x^0 - x^*|| + p\widetilde{L}E.
\end{aligned} \tag{4.18}$$

Therefore, the sequence $\{x^k\}$ is bounded. On the other hand, we have that

$$\begin{aligned}
||x^{k+1} - x^*||^2 &= ||\overline{x}^{k+1} - x^* + (x^{k+1} - \overline{x}^{k+1})||^2 \\
&= ||\overline{x}^{k+1} - x^*||^2 + 2\langle \overline{x}^{k+1} - x^*, x^{k+1} - \overline{x}^{k+1} \rangle + ||x^{k+1} - \overline{x}^{k+1}||^2 \\
&\le ||\overline{x}^{k+1} - x^*||^2 + 2||\overline{x}^{k+1} - x^*|| \, ||x^{k+1} - \overline{x}^{k+1}|| + ||x^{k+1} - \overline{x}^{k+1}||^2 \\
&\le ||x^k - x^*||^2 + 2\rho_k \varepsilon_k (||x^0 - x^*|| + p\widetilde{L}E) + \rho_k^2 \varepsilon_k^2 \\
&\quad - R_m (2\widetilde{L} - R_M) ||Q^k(x^k)||^2.
\end{aligned} \tag{4.19}$$

Letting $E_2 = \sum_{i=0}^{\infty} \varepsilon_k^2 < \infty$, we have for every k,

$$\begin{aligned}
||x^{k+1} - x^*||^2 &\le ||x^0 - x^*||^2 + 2p\widetilde{L}E(||x^0 - x^*|| + p\widetilde{L}E) \\
&\quad + p^2 \widetilde{L}^2 E_2 - R_m (2\widetilde{L} - R_M) \sum_{i=0}^{k} ||Q^k(x^k)||^2.
\end{aligned} \tag{4.20}$$

Passing to the limit $k \to \infty$, one has that $\sum_{i=0}^{\infty} \|Q^k(x^k)\|^2 < \infty$, implying that

$$\lim_{k \to \infty} Q^k(x^k) = 0. \tag{4.21}$$

According to Remark 3.3, for every k, there exists a unique pair (y^k, v^k) in gph T such that $z^k = M(Ax^k, Bx^k) - c_k G(x^k) = M(Ay^k, By^k) + c_k v^k$. Then $J^M_{c_k T}(M(Ax^k, Bx^k) - c_k G(x^k)) = y^k$. So that $Q^k(x^k) \to 0$ implies that $(x^k - y^k) \to 0$, $v^k \to 0$.

Since c_k is bounded away from zero, it follows that $c_k^{-1} Q^k(x^k) \to 0$. Since x_k is bounded, it has at least a limit point. Let x^∞ be such a limit point and assume that the subsequence $\{x^{k_i} : k_i \in k\}$ converges to x^∞. It follows that $\{y^{k_i} : k_i \in k\}$ also converges to x^∞. For every (y, v) in gph T by the monotonicity of T, we have that $\langle y - y^k, v - v^k \rangle \geq 0$. Letting $k_i (\in k) \to \infty$, we get that $\langle y - y^\infty, v - v^k \rangle \geq 0$. We see that T is M-monotone due to Proposition 4.1, this implies that $(x^\infty, -G(x^\infty)) \in$ gph T, that is, $-G(x^\infty) \in T(x^\infty)$. This completes the proof. □

5. Solving an approximate fixed point to $J^M_{c_k T}$

How to calculate w^k at Step 3 is the key in Algorithm 4.5. If $\varepsilon_k = 0$, this amounts to the exact solution of $VI(S^n_+, F_k)$, where

$$F_k(x) = M(Ax, Bx) - M(Ax^k, Bx^k) + c_k(F(x) + G(x^k)). \tag{5.1}$$

Now, we consider the case of $\varepsilon_k > 0$. We can prove that $J^M_{c_k T}(M(Ax^k, Bx^k) - c_k G(x^k))$ is the unique solution of the $VI(S^n_+, F_k)$. Hence, w^k is an inexact solution of the $VI(S^n_+, F_k)$ satisfying $\operatorname{dist}(w^k, \operatorname{Sol}(S^n_+, F_k)) \leq \varepsilon_k$.

Lemma 5.1. *Let F, G, M, A, B satisfy all the conditions of Assumption 4.3. Then a constant $c(k) > 0$ exists such that*

$$\operatorname{dist}(w^k, \operatorname{Sol}(S^n_+, F_k)) \leq c(k) \| (F_k)^{\text{nat}}_{S^n_+}(w^k) \|. \tag{5.2}$$

Proof. By Assumption 4.3, we can easily get that F_k is $L'(k)$-Lipschitz-continuous and $\eta(k)$-strongly monotone, where $L'(k) = \xi\tau + \zeta t + c_k L$ and $\eta(k) = \alpha - \beta + c_k m$, that is,

$$\begin{gathered} \|F_k(x) - F_k(y)\| \leq L'(k) \|x - y\|, \\ \langle F_k(x) - F_k(y), x - y \rangle \geq \eta(k) \|x - y\|^2, \quad \forall x, y \in S^n_+. \end{gathered} \tag{5.3}$$

Let $r = (F_k)^{\text{nat}}_{S^n_+}(w^k)$, where $(F_k)^{\text{nat}}_{S^n_+}$ is the natural map associated with the $VI(S^n_+, F_k)$. We have that $w^k - r = \Pi_{S^n_+}(w^k - F_k(w^k))$, that is,

$$\langle y - w^k + r, F_k(w^k) - r \rangle \geq 0, \quad \forall y \in S^n_+. \tag{5.4}$$

For all $x^* \in \operatorname{Sol}(S^n_+, F_k)$ and $w^k - r \in S^n_+$, we also have that

$$\langle w^k - r - x^*, F_k(x^*) \rangle \geq 0. \tag{5.5}$$

From (5.4) and (5.5), we get that

$$\begin{aligned} &\langle x^* - w^k, F_k(w^k) - r \rangle + \langle r, F_k(w^k) - r \rangle \\ &\qquad = \langle x^* - w^k, F_k(w^k) \rangle - \langle x^* - w^k, r \rangle + \langle r, F_k(w^k) \rangle + \|r\|^2 \geq 0, \end{aligned} \tag{5.6}$$

$$\langle x^* - w^k, F_k(x^*) \rangle - \langle r, F_k(x^*) \rangle \geq 0. \tag{5.7}$$

Adding (5.6) and (5.7), we deduce that

$$\begin{aligned}\eta(k)\|w^k - x^*\|^2 &\le \langle x^* - w^k, F_k(w^k) - F_k(x^*)\rangle \\ &\le -\langle w^k - x^*, r\rangle - \|r\|^2 + \langle r, F_k(x^*) - F_k(w^k)\rangle \\ &\le \|r\|\,\|w^k - x^*\| + \|r\| L'(k)\|w^k - x^*\| \\ &= (1 + L'(k))\|w^k - x^*\|\,\|r\|.\end{aligned} \tag{5.8}$$

Hence, $\|w^k - x^*\| \le \eta(k)^{-1}(1 + L'(k))\|r\|$. This implies that

$$\operatorname{dist}(w^k, \operatorname{Sol}(S^n_+, F_k)) \le c(k)\|(F_k)^{\mathrm{nat}}_{S^n_+}(w^k)\|, \tag{5.9}$$

where $c(k) = \eta(k)^{-1}(1 + L'(k))$. This completes the proof. □

Consequently, the computation of w^k can be accomplished by obtaining an inexact solution of $VI(S^n_+, F_k)$ satisfying the residual condition

$$\|(F_k)^{\mathrm{nat}}_{S^n_+}(w^k)\| \le \frac{1}{c(k)}\varepsilon_k. \tag{5.10}$$

We note that the operator $\prod_{S^n_+}(\cdot)$ is directionally differentiable and strongly semismooth everywhere (see, e.g., [12]). If $F_k(\cdot)$ is continuously differentiable, then we get that

$$(F_k)^{\mathrm{nat}}_{S^n_+}(w) = w - \prod_{S^n_+}(w - F_k(w)) \tag{5.11}$$

is directionally differentiable.

In what follows, we present the following path Newton method for solving the equation $(F_k)^{\mathrm{nat}}_{S^n_+}(w^k) = 0$.

Algorithm 5.2

Data. $w^0 \in S^n$, $\gamma \in (0,1)$, *and* $\rho \in (0,1)$.

Step 1. Set $j = 0$.

Step 2. If $(F_k)^{\mathrm{nat}}_{S^n_+}(w^j) = 0$, *stop.*

Step 3. Select an element $V_j \in \partial[(F_k)^{\mathrm{nat}}_{S^n_+}(w^j)]$ *and consider the corresponding path* $p^j(\cdot) = w^j - (\cdot)V_j^{-1}(F_k)^{\mathrm{nat}}_{S^n_+}(w^j)$ *with domain* $I_j = [0, \bar{\tau}_j)$ *for some* $\bar{\tau}_j \in (0,1]$. *Find the smallest nonnegative integer* i_j *such that with* $i = i_j$, $\rho^i\bar{\tau}_j \in I_j$ *and*

$$\|(F_k)^{\mathrm{nat}}_{S^n_+}(p^j(\rho^i\bar{\tau}_j))\| \le (1 - \gamma\rho^i\bar{\tau}_j)\|(F_k)^{\mathrm{nat}}_{S^n_+}(w^j)\|. \tag{5.12}$$

Step 4. Set $\tau_j = \rho^{i_j}\bar{\tau}_j$, $w^{j+1} = p^j(\tau_j)$, *and* $j \leftarrow j + 1$; *go to Step 2.*

Theorem 5.3. *Let F, G, M, A, and B satisfy all the conditions of Assumption 4.3. If for all* $w \in S^n_+$ *every matrix in* $\partial[(F_k)^{\mathrm{nat}}_{S^n_+}(w)]$ *is nonsingular, then the sequence* $\{w^j\}$ *generated by Algorithm 5.2 has at least one accumulation point and every accumulation point of the sequence* $\{w^j\}$ *is the zero point of* $(F_k)^{\mathrm{nat}}_{S^n_+}$.

Proof. The nonnegative sequence $\{\|(F_k)_{S_+^n}^{\text{nat}}(w^j)\|\}$ is monotonically decreasing; thus it is bounded. It follows from Lemma 5.1 that the sequence $\{w^j\}$ is bounded. That implies that $\{w^j\}$ has at least one accumulation point.

Assume that there exists a subsequence $\{w^{j_m}\}$ of $\{w^j\}$ converging to w^* such that

$$(F_k)_{S_+^n}^{\text{nat}}(w^*) \neq 0, \tag{5.13}$$

that is, there exist positive constants δ and η satisfying

$$||(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})|| \geq \eta, \quad \forall w^{j_m} \in \mathbb{B}(w^*,\delta). \tag{5.14}$$

By the strong semismoothness of $(F_k)_{S_+^n}^{\text{nat}}$, we have that

$$(F_k)_{S_+^n}^{\text{nat}}(w+h) - (F_k)_{S_+^n}^{\text{nat}}(w) - V(h) = O(\|h\|^2), \quad \forall w \in \mathbb{B}(w^*,\delta),\ h \in S_+^n, \tag{5.15}$$

where $V \in \partial(F_k)_{S_+^n}^{\text{nat}}(w)$ is nonsingular and there is a positive constant $\hat{c}$ such that

$$\sup_{\substack{w\in\mathbb{B}(w^*,\delta)\\ V\in\partial(F_k)_{S_+^n}^{\text{nat}}(w)}} \max\{\|V\|,||V^{-1}||\} \leq \hat{c}. \tag{5.16}$$

Letting $\tau \in (0,\bar{\tau}_{j_m})$, $h = p^{j_m}(\tau) - w^{j_m}$, and $V_{j_m} \in \partial[(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})]$, we have that

$$V_{j_m}(p^{j_m}(\tau) - w^{j_m}) = (F_k)_{S_+^n}^{\text{nat}}(p^{j_m}(\tau)) - (F_k)_{S_+^n}^{\text{nat}}(w^{j_m}) - O(||p^{j_m}(\tau) - w^{j_m}||^2). \tag{5.17}$$

It follows that

$$\frac{||(F_k)_{S_+^n}^{\text{nat}}(p^{j_m}(\tau)) - V_{j_m}(p^{j_m}(\tau) - w^{j_m}) - (F_k)_{S_+^n}^{\text{nat}}(w^{j_m})||}{||p^{j_m}(\tau) - w^{j_m}||} = \frac{O(||p^{j_m}(\tau) - w^{j_m}||^2)}{||p^{j_m}(\tau) - w^{j_m}||}. \tag{5.18}$$

So we can choose a positive t_* small enough so that

$$\frac{o(t)}{t} \leq \frac{1-\gamma}{\hat{c}}, \quad \forall t \in (0,t_*]. \tag{5.19}$$

From the definition of p^{j_m}, we know that there exists a constant $\tau^* \in (0,\bar{\tau}_{j_m}]$ small enough so that $\|p^{j_m}(\tau) - w^{j_m}\| \leq t_*$, for all $\tau \in (0,\tau^*]$, which implies that

$$\frac{o(||p^{j_m}(\tau) - w^{j_m}||)}{||p^{j_m}(\tau) - w^{j_m}||} \leq \frac{1-\gamma}{\hat{c}}, \quad \forall \tau \in (0,\tau^*]. \tag{5.20}$$

It follows from (5.16), (5.18), (5.19), and (5.20) that

$$\begin{aligned} ||(F_k)_{S_+^n}^{\text{nat}}(p^{j_m}(\tau))|| \leq& ||V_{j_m}(p^{j_m}(\tau) - w^{j_m}) + (F_k)_{S_+^n}^{\text{nat}}(w^{j_m})|| \\ &+||p^{j_m}(\tau) - w^{j_m}||\frac{o(||p^{j_m}(\tau) - w^{j_m}||)}{||p^{j_m}(\tau) - w^{j_m}||} \\ \leq&(1-\tau)||(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})|| + \tau||V_{j_m}^{-1}||\,||(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})||\frac{1-\gamma}{\hat{c}} \\ \leq& (1-\tau\gamma)||(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})||, \quad \forall \tau \in (0,\tau^*]. \end{aligned} \tag{5.21}$$

Consequently,

$$||(F_k)_{S_+^n}^{\text{nat}}(p^{j_m}(\tau))|| \le (1-\tau\gamma)||(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})||, \quad \forall \tau \in (0,\tau^*]. \tag{5.22}$$

By the definition of the step-size τ_{j_m}, it follows that there exists $\xi \in (0,\tau^*)$ such that $\tau_{j_m} \ge \xi$ for all j_m. Indeed, if no such ξ exists, then $\{\tau_{j_m}\}$ converges to zero. This implies that the sequence of integers $\{i_{j_m}\}$ is unbounded. Consequently, by the definition of i_{j_m}, we have, for all j_m sufficiently large,

$$||(F_k)_{S_+^n}^{\text{nat}}(p^k(\rho^{i_{jm}-1}\bar{\tau}_k))|| > (1-\gamma\rho^{i_{jm}-1}\bar{\tau}_k)||(F_k)_{S_+^n}^{\text{nat}}(w^k)||; \tag{5.23}$$

but this contradicts (5.22) with $\tau \equiv \rho^{i_{jm}-1}\bar{\tau}_{j_m}$. Consequently, the desired ξ exists. The inequality (5.22) implies that

$$||(F_k)_{S_+^n}^{\text{nat}}(w^{j_{m+1}})|| \le (1-\tau_{j_m}\gamma)||(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})||. \tag{5.24}$$

Passing to the limit $m \to \infty$, we deduce a contradiction because $\lim_{m\to\infty} ||(F_k)_{S_+^n}^{\text{nat}}(w^{j_m})|| \ge \eta > 0$ and the sequence $\{\tau_{j_m}\}$ is bounded away from zero. This yields that $(F_k)_{S_+^n}^{\text{nat}}(w^*) = 0$. This completes the proof. □

Remark 5.4. As stated above, Algorithm 5.2 generates a sequence converging to the zero point of $(F_k)_{S_+^n}^{\text{nat}}$, Step 3 in Algorithm 4.5 is implementable. Obviously, Algorithm 5.2 stops within a finite number of iterations at a w^k such that (5.10) holds.

Example 5.5. Assume that there exists a positive constant $\bar{c}$ such that

$$\sup_{w\in S_+^n} \sup_{V\in\partial\prod_{S_+^n}(x^k-c_k(F(w)+G(x^k)))} ||VJF(w)|| \le \bar{c}. \tag{5.25}$$

Let $c_k \in (0,1/\bar{c})$. Suppose that $M(Aw,Bw) = w$, for all $w \in S_+^n$ and F is Lipschitz-continuous and strongly monotone. We have $(F_k)_{S_+^n}^{\text{nat}}(w) = w - \prod_{S_+^n}(x^k - c_k(F(w)+G(x^k)))$, for all $w \in S_+^n$. Then $\partial[(F_k)_{S_+^n}^{\text{nat}}(w)] \subset \{I - c_k VJF(w) \mid V \in \partial\prod_{S_+^n}(x^k - c_k(F(w)+G(x^k)))\}$, for all $w \in S_+^n$. We easily get that every matrix in $\partial[(F_k)_{S_+^n}^{\text{nat}}(w)]$ is nonsingular for all $w \in S_+^n$. It follows from Theorem 5.3 that every accumulation point of $\{w^k\}$ generated by Algorithm 5.2 is the zero point of $(F_k)_{S_+^n}^{\text{nat}}$.

At first sight, the M-monotonicity of $T = F + \mathcal{N}(\cdot,S_+^n)$ seems having little use because the algorithm based on maximal monotonicity can also solve the $VI(S_+^n,F+G)$ directly. However, we will see that in some practical cases the variational inequality using Algorithm 4.5, which is based on M-monotone operator, is actually much simpler to solve and easier to analyze than using algorithm based on maximal monotone map. We illustrate this by the following example.

Example 5.6. Let $F: S_+^n \to S^n$ be defined by

$$F(x) = S(x) + \frac{1}{16}x, \quad G(x) = \frac{1}{8}x \quad \forall x \in S_+^n, \tag{5.26}$$

where $S: S_+^n \to S^n$ is s-Lipschitz-continuous and monotone with $\langle S(x),x\rangle \ge -\infty$.

We have F is $(s+(1/16))$-Lipschitz-continuous, $(1/16)$-strongly monotone, and G is $(1/8)$-Lipschitz-continuous.

Now, we take $M(Ax,Bx) = Ax + Bx$, where $Ax = (1+(c_k/16))x$ and $Bx = -c_k S(x) - (c_k/8)x$ for all $x \in S^n$ and $0 < c_k < 16$. Then, we can easily prove that $M(\cdot,\cdot)$ is Lipschitz-continuous with first and second arguments, $M(A,B)$ is bounded and semicontinuous; and A and B are both Lipschitz-continuous. It is also easy to see that

$$\lim_{\|x\|\to\infty} \frac{\langle M(Ax,Bx),x\rangle}{\|x\|} = \lim_{\|x\|\to\infty} \frac{\langle -c_k S(x) + (1-(c_k/16))x, x\rangle}{\|x\|} = +\infty, \tag{5.27}$$

which implies that $M(A,B)$ is coercive. Also, we can deduce that $M(A,B)$ is $(1+(c_k/16))$-strongly monotone with respect to A and $c_k(s+(1/8))$-relaxed monotone with respect to B and $(1+(c_k/16)) > c_k(s+(1/8))$, if we let $s < (1/c_k) - (1/16)$. Also, we can prove that G is strongly monotone with respect to $M(A,B)$.

We choose $x^0 \in S^n_+$, $\{\varepsilon_k\}$, $\{c_k\}$, and $\{\rho_k\}$ satisfying Theorem 4.6 and compute $\{w^k\}$ by the residual rule

$$\begin{aligned}
\|(F_k)^{\mathrm{nat}}_{S_+}(w^k)\| &= \left\| w^k - \prod_{S^n_+}\left(w^k - M(Aw^k,Bw^k) + M(Ax_k,Bx_k) - c_k\left(F(w^k)+G(x_k)\right)\right)\right\| \\
&= \left\| w^k - \prod_{S^n_+}\left(-c_k S(x_k) + \left(1 - 3\frac{c_k}{16}\right)x_k\right)\right\| \\
&\le \frac{\eta(k)}{1+L'(k)}\varepsilon_k,
\end{aligned} \tag{5.28}$$

that is, w^k can be computed as follows:

$$\left\| w^k - \prod_{S^n_+}\left(-c_k S(x_k) + \left(1 - 3\frac{c_k}{16}\right)x_k\right)\right\| \le \frac{\eta(k)}{1+L'(k)}\varepsilon_k. \tag{5.29}$$

It follows from Theorem 4.6 that the sequence $\{x_k\}$ generated by Algorithm 4.5 converges to a solution of

$$\left\langle S(x) + \frac{1}{16}x + \frac{1}{8}x, y - x\right\rangle \ge 0, \quad \forall y \in S^n_+. \tag{5.30}$$

Note that the core of proximal point algorithm is the calculation of w^k. As we have seen, if we use [10, Algorithm 12.3.8], which is based on the maximal monotonicity of $T = F + \mathcal{N}(\cdot, S^n_+)$, w^k will be computed as

$$\left\| w^k - \prod_{S^n_+}\left(x_k - c_k S(w^k) - \frac{c_k}{8}x^k\right)\right\| \le \frac{\eta(k)}{1+L'(k)}\varepsilon_k, \tag{5.31}$$

which is more complicated to solve than (5.29). This example verifies the above comments.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under project Grant no. 10771026 and by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

References

[1] W. B. Liu and J. E. Rubio, "Optimal shape design for systems governed by variational inequalities—I: existence theory for the elliptic case," *Journal of Optimization Theory and Applications*, vol. 69, no. 2, pp. 351–371, 1991.

[2] W. B. Liu and J. E. Rubio, "Optimal shape design for systems governed by variational inequalities—II: existence theory for the evolution case," *Journal of Optimization Theory and Applications*, vol. 69, no. 2, pp. 373–396, 1991.

[3] R. Ahmad, Q. H. Ansari, and S. S. Irfan, "Generalized variational inclusions and generalized resolvent equations in Banach spaces," *Computers & Mathematics with Applications*, vol. 49, no. 11-12, pp. 1825–1835, 2005.

[4] X. P. Ding, "Perturbed proximal point algorithms for generalized quasivariational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 210, no. 1, pp. 88–101, 1997.

[5] Y.-P. Fang and N.-J. Huang, "H-monotone operator and resolvent operator technique for variational inclusions," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 795–803, 2003.

[6] A. Hassouni and A. Moudafi, "A perturbed algorithm for variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 185, no. 3, pp. 706–712, 1994.

[7] C.-H. Lee, Q. H. Ansari, and J.-C. Yao, "A perturbed algorithm for strongly nonlinear variational-like inclusions," *Bulletin of the Australian Mathematical Society*, vol. 62, no. 3, pp. 417–426, 2000.

[8] M. A. Noor, "Implicit resolvent dynamical systems for quasi variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 269, no. 1, pp. 216–226, 2002.

[9] G. X.-Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, vol. 218 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1999.

[10] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. 2, Springer, New York, NY, USA, 2003.

[11] Z. Liu, J. S. Ume, and S. M. Kang, "Resolvent equations technique for general variational inclusions," *Proceedings of the Japan Academy. Series A*, vol. 78, no. 10, pp. 188–193, 2002.

[12] D. Sun and J. Sun, "Semismooth matrix-valued functions," *Mathematics of Operations Research*, vol. 27, no. 1, pp. 150–169, 2002.

Juhe Sun: Department of Applied Mathematics, Dalian University of Technology,
Dalian, Liaoning 116024, China
Email address: juhesun@163.com

Shaowu Zhang: Department of Applied Mathematics, Dalian University of Technology,
Dalian, Liaoning 116024, China
Email address: zhangsw@dlut.edu.cn

Liwei Zhang: Department of Applied Mathematics, Dalian University of Technology,
Dalian, Liaoning 116024, China
Email address: lwzhang@dlut.edu.cn