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Solution of a nonlinear fractional-order initial value problem via a $\mathscr{C}^*$-algebra-valued $\mathcal{R}$-metric space

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Abstract

In this article, we prove new common fixed-point theorems on a $\mathscr{C}^*$-algebra-valued $\mathcal{R}$-metric space. An example is given based on our obtained results. To enhance our results, a strong application based on the fractional-order initial value problem is provided.

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1 Introduction

The concept of $\mathscr{C}^*$-AVMS was outlined by Ma et al. in 2014, [1] and they proved some fixed-point results with a new contraction type. Many authors and researchers have generalized with a new type of outcome (see [2–5]).

Let $\mathcal{B}$ be the unital algebra with unit $\mathcal{I}$. The conjugate linear map $\delta \mapsto \delta^*$ on $\mathcal{B}$ is such that $\delta^{**} = \delta$ and $(\delta\eta)^* = \eta^*\delta^*$ for all $\delta, \eta \in \mathcal{B}$. The set of all bounded linear operators on a Hilbert space $\mathcal{H}$, under the norm topology $\mathcal{L}(\mathcal{H})$, is a $\mathscr{C}^*$-algebra. The concept of a cone metric space was outlined by Huang and Zhang in 2007 [6] and they replaced the set of real numbers by an ordered Banach space.

The CFP for commuting mappings in metric space was investigated by Jungck in 1966 [7]. Likewise, many fixed and CFP results were obtained in different types like cone metric space [8], uniform space [9], noncommutative Banach space [10], fuzzy metric space [11] and so on. Hussain et al. proved Suzuki–Berinde-type fixed-point theorems and the CFP theorem on a cone b-metric space in these works [12, 13], respectively. Khalehoghli, Rahimi and Gordji introduced the $\mathcal{R}$-metric space to prove the fixed-point theorem [14]. Wardowski proposed a new Banach contraction principle in a complete metric space to prove the fixed-point theorem [15]. Astha, Deepak and Choonkil proposed a $\mathscr{C}^*$ algebra-valued $\mathcal{R}$-metric space to prove a unique fixed-point theorem [16]. Afshari and Khoshvaghti proved a unique fixed-point theorem in an operator equation on the ordered Banach space [17]. Afshari et al. [18], used a fixed-point theorem to study a boundary value problem for a fractional differential equation in a b-metric space. Deuri and Das in [19] proved the fixed-point theorem in a newly constructed contraction operator. Chandra Deuri et al. [20] investigated the existence of a fractional integral equation by using the Darbo fixed-point theorem. Further, Das et al. [21], proved the fixed-point theorem based on the Darbo-type theorem. Researchers in [22], utilized the fixed-point theorem for discussing a generalized proportional fractional integral equation in a Banach space. Das and Deuri [23], proved the fixed-point theorem on a generalization of Darbo's fixed-point theorem in a Banach space. The authors of [24, 25], established the qualitative properties of fractional differential equation in unbounded domains.

In this paper, we prove some CFP theorems on a $\mathscr{C}^*$-algebra-valued $\mathcal{R}$-metric space. Additionally, we established the uniqueness of a common solution for the fractional-order initial value problem. Throughout this paper, $\mathcal{B}$ will denote $\mathscr{C}^*$-algebra with unit $\mathcal{I}$ and $\mathcal{R}$ denotes a nonempty binary relation. $\mathscr{C}^*$-AVMS means a $\mathscr{C}^*$-algebra-valued metric space and $\mathscr{C}^*$-AV$\mathcal{R}$-MS means a $\mathscr{C}^*$-algebra-valued $\mathcal{R}$- metric space. CFP means Common Fixed Point.

2 Preliminaries

Definition 2.1 Let a nonvoid set be $\mathcal{X}$. Let the mapping $\varpi : \mathcal{X} \times \mathcal{X} \to \mathcal{B}$ be such that:

(1) $0_{\mathcal{B}} \leq \varpi(\zeta, \vartheta)$ for all $\zeta, \vartheta \in \mathcal{X}$;

(2) $\varpi(\zeta, \vartheta) = 0_{\mathcal{B}}$ iff $\zeta = \vartheta$;

(3) $\varpi(\zeta, \vartheta) = \varpi(\vartheta, \zeta)$ for all $\zeta, \vartheta \in \mathcal{X}$;

(4) $\varpi(\zeta, \vartheta) \leq \varpi(\zeta, \nu) + \varpi(\nu, \vartheta)$ for all $\zeta, \vartheta, \nu \in \mathcal{X}$.

Then, $(\mathcal{X}, \mathcal{B}, \varpi)$ is called a $\mathscr{C}^*$-AVMS.

Definition 2.2 Let a nonvoid set be $\mathcal{X}$ defined a binary relation on $\mathcal{R}$, a sequence $\{\zeta_\phi\}_{\phi\in\mathbb{N}} \in \mathcal{X}$ is called a $\mathcal{R}$-sequence if $(\zeta_\phi, \zeta_{\phi+1}) \in \mathcal{R}$ for all $\phi \in \mathbb{N}$.

Definition 2.3 A binary relation $\mathcal{R}$ on a metric space $(\mathcal{X}, \varpi)$ is called a $\mathcal{R}$-metric space and it is denoted by $(\mathcal{X}, \varpi, \mathcal{R})$.

Lemma 2.1 [26]

1. *If* $\{\eta_\phi\}_{\phi=1}^{\infty} \subseteq \mathcal{B}$ *and* $\lim_{\phi\to\infty} \eta_\phi = 0_{\mathcal{B}}$, *then for any* $\delta \in \mathcal{B}$, $\lim_{\phi\to\infty} \delta^* \eta_\phi \delta = 0_{\mathcal{B}}$.
2. *If* $\delta, \eta \in \mathcal{B}_{\mathfrak{h}}$ *and* $\mathfrak{c} \in \mathcal{B}'_+$, *then* $\delta \leq \eta$ *deduces* $\mathfrak{c}\delta \leq \mathfrak{c}\eta$, *where* $\mathcal{B}'_+ = \mathcal{B}_+ \cap \mathcal{B}'$.
3. *Let* $\{\zeta_\phi\}_{\phi=1}^{\infty}$ *be a sequence in* $\mathcal{X}$. *If* $\{\zeta_\phi\}$ *converges to* ζ *and* ϑ, *respectively, then* $\zeta = \vartheta$.

Definition 2.4 Let $(\mathcal{X}, \mathcal{B}, \varpi, \mathcal{R})$ be a $\mathscr{C}^*$-AV$\mathcal{R}$-MS, let a $\mathcal{R}$-sequence $\{\zeta_\phi\}_{\phi\in\mathbb{N}} \subset \mathcal{X}$ be said to be $\mathcal{R}$-Cauchy, if $\kappa > 0$, we can find $\phi_0 \in \mathbb{N}$ that satisfies $\|\varpi(\zeta_\phi, \zeta_{\mathfrak{m}})\| \leq \kappa$, $\forall \phi, \mathfrak{m} \geq \phi_0$.

Definition 2.5 Let $(\mathcal{X}, \mathcal{B}, \varpi, \mathcal{R})$ be a $\mathscr{C}^*$-AV$\mathcal{R}$-MS that is called a Complete $\mathscr{C}^*$-AV$\mathcal{R}$-MS, if every $\mathcal{R}$- Cauchy sequence with respect to $\mathcal{B}$ is convergent.

Definition 2.6 Let two mappings Ξ and Φ on $(\mathcal{X}, \mathcal{B}, \varpi)$ be a $\mathscr{C}^*$-AVMS be called compatible, if the sequence $\{\zeta_\phi\}_{\phi=1}^{\infty} \subseteq \mathcal{X}$, such that $\lim_{\phi\to\infty} \Xi\zeta_\phi = \lim_{\phi\to\infty} \Phi\zeta_\phi = \sigma \in \mathcal{X}$, then $\varpi(\Xi\Phi\zeta_\phi, \Phi\Xi\zeta_\phi) \xrightarrow{\|\cdot\|_{\mathcal{B}}} 0_{\mathcal{B}}$ $(\phi \to \infty)$.

3 Main results

We prove our first result.

Theorem 3.1 *Let $(\mathcal{X},\mathcal{B},\varpi,\mathcal{R})$ be a complete $\mathscr{C}^*$-AV$\mathcal{R}$-MS and let the two mappings $\Xi,\Phi: \mathcal{X}\to\mathcal{X}$, such that*

(i) $\Xi(\mathcal{X})\subseteq\mathcal{X}, \quad \Phi(\mathcal{X})\subseteq\mathcal{X}$;

(ii) Ξ,Φ *are $\mathcal{R}$-preserving*;

(iii) *We can find some $\zeta_0\in\mathcal{X}$ satisfying $(\zeta_0,\vartheta)\in\mathcal{R}$ for all $\vartheta\in\Xi(\mathcal{X})$*;

(iv) *For all $\zeta,\vartheta\in\mathcal{X}$ with $(\zeta,\vartheta)\in\mathcal{R}$, there exists $\delta\in\mathcal{B}$, where $\|\delta\|<1$ such that*

$$\varpi(\Xi\zeta,\Phi\vartheta)\le\delta^*\varpi(\zeta,\vartheta)\delta,\quad \text{for any } \zeta,\vartheta\in\mathcal{X}.$$

Then, Ξ and Φ have a unique CFP.

Proof Let $\zeta_0\in\mathcal{X}$ and consider a $\mathcal{R}$-sequence $\{\zeta_\phi\}_{\phi=0}^{\infty}\subseteq\mathcal{X}$, such that $\zeta_\phi=\Phi\zeta_{\phi-1}$, $\zeta_{\phi+1}=\Xi\zeta_\phi$, $\zeta_{\phi-1}=\Xi\zeta_{\phi-2}$. From condition (iv),

$$\begin{aligned}\varpi(\zeta_{\phi+1},\zeta_\phi)&=\varpi(\Xi\zeta_\phi,\Phi\zeta_{\phi-1})\\&\le\delta^*\varpi(\zeta_\phi,\zeta_{\phi-1})\delta\\&\le(\delta^*)^2\varpi(\zeta_{\phi-1},\zeta_{\phi-2})(\delta)^2\\&\vdots\\&\le(\delta^*)^\phi\varpi(\zeta_1,\zeta_0)(\delta)^\phi.\end{aligned}$$

Since, $\eta,\mathfrak{c}\in\mathcal{B}_{\mathfrak{h}}$, then $\eta\le\mathfrak{c}$, which implies $\delta^*\eta\delta\le\delta^*\mathfrak{c}\delta$.

Similarly,

$$\begin{aligned}\varpi(\zeta_\phi,\zeta_{\phi-1})&=\varpi(\Phi\zeta_{\phi-1},\Xi\zeta_{\phi-2})\\&\le\delta^*\varpi(\zeta_{\phi-1},\zeta_{\phi-2})\delta\\&\vdots\\&\le(\delta^*)\varpi(\zeta_1,\zeta_0)(\delta)^{\phi-1},\end{aligned}$$

for any $\mathfrak{p}\in\mathbb{N}$, then by the triangle inequality,

$$\begin{aligned}\varpi(\zeta_{\phi+\mathfrak{p}})&\le\varpi(\zeta_{\phi+\mathfrak{p}},\zeta_{\phi+\mathfrak{p}-1})+\varpi(\zeta_{\phi+\mathfrak{p}-1},\zeta_{\phi+\mathfrak{p}-2})+\cdots+\varpi(\zeta_{\phi+1},\zeta_\phi)\\&\le\sum_{\upsilon=\phi}^{\phi+\mathfrak{p}-1}(\delta^*)^\upsilon\varpi(\zeta_1,\zeta_0)(\delta)^\upsilon\\&\le\sum_{\upsilon=\phi}^{\phi+\mathfrak{p}-1}(\delta^*)^\upsilon\eta^2(\delta)^\upsilon\\&\le\sum_{\upsilon=\phi}^{\phi+\mathfrak{p}-1}(\delta^*)^\upsilon\eta\cdot\eta(\delta)^\upsilon\\&\le\sum_{\upsilon=\phi}^{\phi+\mathfrak{p}-1}(\eta\delta^\upsilon)^*\cdot(\eta\delta^\upsilon)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{\upsilon=\phi}^{\phi+\mathfrak{p}-1} \left|\eta\delta^{\upsilon}\right|^2 \\
&\le \sum_{\upsilon=\phi}^{\phi+\mathfrak{p}-1} \left\|\left|\eta\delta^{\upsilon}\right|^2\right\| 1_{\mathcal{B}} \\
&\le \|\eta\|^2 1_{\mathcal{B}} \sum_{\upsilon=\phi}^{\phi+\mathfrak{p}-1} \left\|\delta^{\upsilon}\right\| \to 0_{\mathcal{B}} \quad \text{as } \phi \to \infty,
\end{aligned}
$$

where $1_{\mathcal{B}}$ is a unit element in $\mathcal{B}$ and $\varpi(\zeta_1, \zeta_0) = \eta^2$ for some $\eta \in \mathcal{B}$. From definition 2.5, we obtain that $\{\zeta_\phi\}_{\phi=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$. We can find $\zeta \in \mathcal{X}$ satisfying $\lim_{\phi\to\infty} \zeta_\phi = \zeta$.

Now, using the triangle inequality

$$
\begin{aligned}
\varpi(\zeta, \Phi\zeta) &\le \varpi(\zeta, \zeta_\phi) + \varpi(\zeta_\phi, \Phi\zeta) \\
&\le \varpi(\zeta, \zeta_\phi) + \varpi(\Phi\zeta_{\phi-1}, \Phi\zeta) \\
&\le \varpi(\zeta, \zeta_\phi) + \delta^* \varpi(\zeta_{\phi-1}, \zeta)\delta.
\end{aligned}
$$

Taking $\phi \to \infty$, the right-hand side approaches $0_{\mathcal{B}}$, by lemma 2.1 (condition 1), we obtain $\Phi\zeta = \zeta$.

Similarly,

$$
\begin{aligned}
\varpi(\Xi\zeta, \zeta) &= \varpi(\Xi\zeta, \Phi\zeta) \\
&\le \delta^* \varpi(\zeta, \zeta)\delta \\
&= 0_{\mathcal{B}}.
\end{aligned}
$$

We have,

$$
\varpi(\Xi\zeta, \zeta) = 0_{\mathcal{B}},
$$

which means, $\Xi\zeta = \zeta$.

Let us take another fixed point $\vartheta \in \mathcal{X}$ such that $\Xi\vartheta = \Phi\vartheta = \vartheta$, From condition (iv) of Theorem 3.1:

$$
\varpi(\zeta, \vartheta) = \varpi(\Xi\zeta, \Phi\vartheta) \le \delta^* \varpi(\zeta, \vartheta)\delta,
$$

with $\|\delta\| < 1$, such that

$$
\begin{aligned}
0 &\le \left\|\varpi(\zeta, \vartheta)\right\| \le \|\delta\|^2 \left\|\varpi(\zeta, \vartheta)\right\| \\
&\le \left\|\varpi(\zeta, \vartheta)\right\|.
\end{aligned}
$$

Thus, $\|\varpi(\zeta, \vartheta)\| = 0$ and $\varpi(\zeta, \vartheta) = 0_{\mathcal{B}}$, which gives $\zeta = \vartheta$. Hence, Ξ and Φ have a unique CFP in $\mathcal{X}$. □

Here, we prove our second result.

Theorem 3.2 *Let $(\mathcal{X},\mathcal{B},\varpi,\mathcal{R})$ be a complete $\mathscr{C}^*$-AV$\mathcal{R}$-MS and let the two mapping $\Xi,\Phi:\mathcal{X}\to\mathcal{X}$ such that*

(i) $\Xi(\mathcal{X})\subseteq\mathcal{X}, \quad \Phi(\mathcal{X})\subseteq\mathcal{X}$;

(ii) *Ξ,Φ is $\mathcal{R}$-preserving*;

(iii) *We can find some $\zeta_0\in\mathcal{X}$ satisfying $(\zeta_0,\vartheta)\in\mathcal{R}$ for all $\vartheta\in\Xi(\mathcal{X})$*;

(iv) *For all $\zeta,\vartheta\in\mathcal{R}$ with $(\zeta,\vartheta)\in\mathcal{R}$, there exist $\delta\in\mathcal{B}$, where $\|\delta\|<1$ such that*

$$\varpi(\Xi\zeta,\Xi\vartheta)\le\delta\varpi(\Xi\zeta,\Phi\zeta)+\delta\varpi(\Xi\vartheta,\Phi\vartheta).$$

Then, Ξ and Φ have a unique CFP.

Proof Let $\zeta_0\in\mathcal{X}$ and consider a $\mathcal{R}$-sequence $\{\zeta_\phi\}_{\phi=0}^\infty\subseteq\mathcal{X}$ such that $\Phi\zeta_\phi=\zeta_{\phi+1}$, and $\Phi\zeta_{\phi+1}=\zeta_{\phi+2}$, then

$$\begin{aligned}
\varpi(\zeta_{\phi+2},\zeta_{\phi+1}) &= \varpi(\Phi\zeta_{\phi+1},\Phi\zeta_\phi)\\
&\le \delta\varpi(\Xi\zeta_{\phi+1},\Phi\zeta_{\phi+1})+\delta\varpi(\Xi\zeta_\phi,\Phi\zeta_\phi)\\
&\le \delta\varpi(\zeta_{\phi+1},\zeta_{\phi+2})+\delta\varpi(\zeta_\phi,\zeta_{\phi+1})\\
&\le \delta\varpi(\zeta_{\phi+2},\zeta_{\phi+1})+\delta\varpi(\zeta_{\phi+1},\zeta_\phi),\\
\varpi(\zeta_{\phi+2},\zeta_{\phi+1})-\delta\varpi(\zeta_{\phi+2},\zeta_{\phi+1}) &= \delta\varpi(\zeta_{\phi+1},\zeta_\phi),\\
(1_{\mathcal{B}}-\delta)\varpi(\zeta_{\phi+2},\zeta_{\phi+1}) &= \delta\varpi(\zeta_{\phi+1},\zeta_\phi),\\
\varpi(\zeta_{\phi+2},\zeta_{\phi+1}) &\le \frac{\delta}{(1_{\mathcal{B}}-\delta)}\varpi(\zeta_{\phi+1},\zeta_\phi),\\
\varpi(\zeta_{\phi+2},\zeta_{\phi+1}) &\le \eta\varpi(\zeta_{\phi+1},\zeta_\phi), \quad \text{where } \eta=\frac{\delta}{(1_{\mathcal{B}}-\delta)}.
\end{aligned}$$

By induction,

$$\varpi(\zeta_{\phi+2},\zeta_{\phi+1})\le\eta^\phi\varpi(\zeta_1,\zeta_0).$$

For $\phi>\mathfrak{m}$,

$$\begin{aligned}
\varpi(\zeta_{\phi+1},\zeta_{\mathfrak{m}}) &\le \varpi(\zeta_{\phi+1},\zeta_\phi)+\varpi(\zeta_\phi,\zeta_{\phi-1})+\cdots+\varpi(\zeta_{\mathfrak{m}+1},\zeta_{\mathfrak{m}})\\
&\le \left(\eta^\phi+\eta^{\phi-1}+\cdots+\eta^{\mathfrak{m}}\right)\varpi(\zeta_1,\zeta_0)\\
&\le \left\|\eta^\phi+\eta^{\phi-1}+\cdots+\eta^{\mathfrak{m}}\right\|\left\|\varpi(\zeta_1,\zeta_0)\right\|1_{\mathcal{B}}\\
&\le \left\|\eta^\phi\right\|+\left\|\eta^{\phi-1}\right\|+\cdots+\left\|\eta^{\mathfrak{m}}\right\|\left\|\varpi(\zeta_1,\zeta_0)\right\|1_{\mathcal{B}}\\
&\le \frac{\|\eta\|^{\mathfrak{m}}}{1-\|\eta\|}\left\|\varpi(\zeta_1,\zeta_0)\right\|1_{\mathcal{B}}.
\end{aligned}$$

Hence, $\{\zeta_\phi\}_{\phi=0}^\infty$ is a Cauchy sequence in $\mathcal{R}$-sequence. We can find $\mathfrak{q}\in\mathcal{X}$ satisfying $\lim_{\phi\to\infty}\zeta_\phi=\mathfrak{q}$. By condition (iv),

$$\begin{aligned}
\varpi(\zeta_{\phi+1},\mathfrak{q}) &= \varpi(\Phi\zeta_\phi,\Xi\mathfrak{q})\\
&\le \delta\varpi(\Phi\zeta_\phi,\Xi\zeta_\phi)+\delta\varpi(\Xi\mathfrak{q},\Phi\mathfrak{q})
\end{aligned}$$

$$\leq \delta\varpi(\Phi\zeta_\phi, \Xi\mathfrak{q}) + \delta\varpi(\Xi\mathfrak{q}, \Xi\zeta_\phi) + \delta\varpi(\Xi\mathfrak{q}, \Phi\zeta_\phi) + \delta\varpi(\Phi\zeta_\phi, \Phi\mathfrak{q})$$
$$\leq 2\delta\varpi(\Phi\zeta_\phi, \Xi\mathfrak{q}) + \delta\varpi(\Xi\mathfrak{q}, \Xi\zeta_\phi) + \delta\varpi(\Phi\zeta_\phi, \Phi\mathfrak{q}),$$
$$(1_{\mathcal{B}} - 2\delta)\varpi(\zeta_{\phi+1}, \mathfrak{q}) \leq \delta\varpi(\Xi\mathfrak{q}, \Xi\zeta_\phi) + \delta\varpi(\Phi\zeta_\phi, \Phi\mathfrak{q}).$$

Since $\|\delta\| < 1$, then $1_{\mathcal{B}} - 2\delta$ is invertible:

$$\varpi(\zeta_{\phi+1}, \mathfrak{q}) \leq \frac{\delta}{(1_{\mathcal{B}} - 2\delta)}\varpi(\Xi\mathfrak{q}, \Xi\zeta_\phi) + \frac{\delta}{(1_{\mathcal{B}} - 2\delta)}\varpi(\Phi\zeta_\phi, \Phi\mathfrak{q}),$$

then $\lim_{\phi\to\infty}\zeta = \mathfrak{q}$. Let us choose $\Xi\mathfrak{q} = \Phi\mathfrak{q}$. Hence, Ξ and Φ have a coincidence point in $\mathcal{X}$.

Assume $\mathfrak{p} \in \mathcal{X}$ such that $\Xi\mathfrak{p} = \Phi\mathfrak{p}$, and by using condition (iv), we obtain

$$\varpi(\Phi\mathfrak{p}, \Phi\mathfrak{q}) = \varpi(\Xi\mathfrak{p}, \Xi\mathfrak{q}) \leq \delta\varpi(\Xi\mathfrak{p}, \Phi\mathfrak{p}) + \delta\varpi(\Xi\mathfrak{q}, \Phi\mathfrak{q}),$$

which shows that $\|\varpi(\Phi\mathfrak{p}, \Phi\mathfrak{q})\| = 0$, then

$$\Phi\mathfrak{p} = \Phi\mathfrak{q}.$$

Similarly,

$$\Xi\mathfrak{p} = \Xi\mathfrak{q}.$$

Hence, Ξ and Φ have a unique CFP in $\mathcal{X}$. □

Example 3.3 Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{B} = \mathcal{M}_2(\mathbb{R})$. Define relation $\mathcal{R}$ on $\mathcal{X}$ as $(\zeta, \vartheta) \in \mathbb{R}$ iff $\zeta, \vartheta \geq 0$ and $\varpi(\zeta, \vartheta) = \begin{bmatrix} |\zeta-\vartheta|^2 & 0 \\ 0 & \upsilon|\zeta-\vartheta|^2 \end{bmatrix}$, where $\zeta, \vartheta \in \mathbb{R}$ and $\upsilon \geq 0$ is a constant. Then, $(\mathcal{X}, \mathcal{B}, \varpi, \mathcal{R})$ is a complete $\mathscr{C}^*$-AV$\mathcal{R}$-MS:

$$\Xi\zeta = \begin{cases} 2 - \frac{1}{\zeta}, & \zeta \in [0, \frac{5}{4}), \\ 2, & \zeta \in (\frac{5}{4}, 3], \end{cases} \qquad \Phi\zeta = \begin{cases} \frac{2}{\zeta^2}, & \zeta \in [0, 1), \\ \zeta, & \zeta \in (1, 3]. \end{cases}$$

Clearly, Ξ and Φ are $\mathcal{R}$-preserving. First, the set of their coincidence points is singleton $\{2\}$, and then we have Ξ and Φ commute at this point. Thereby, Ξ and Φ are weak compatible.

Let the sequence $\{\zeta_\phi\} \subseteq \mathcal{X}$ such that $\zeta_\phi = 1 - \phi \in \mathcal{X}$, hence,

$$\Xi\zeta_\phi = 2 - \frac{1}{1-\phi} = \frac{1-2\phi}{1-\phi}, \qquad \Phi\zeta_\phi = \frac{2}{(1-\phi)^2}.$$

Then, $\lim_{\phi\to\infty}\Xi\zeta_\phi = \lim_{\phi\to\infty}\Phi\zeta_\phi = 3$,

$$\varpi(\Xi\zeta_\phi, 3) = \varpi\left(\frac{1-2\phi}{1-\phi}, 3\right) = \begin{bmatrix} |\frac{\phi-2}{1-\phi}|^2 & 0 \\ 0 & \upsilon|\frac{\phi-2}{1-\phi}|^2 \end{bmatrix} \xrightarrow{\|\cdot\|_{\mathcal{B}}} 0_{\mathcal{B}}, \quad \text{as } \phi \to \infty,$$
$$\varpi(\Phi\zeta_\phi, 3) = \varpi\left(\frac{2}{(1-\phi)^2}, 3\right) = \begin{bmatrix} |\frac{3\phi-1}{1-\phi}|^2 & 0 \\ 0 & \upsilon|\frac{3\phi-1}{1-\phi}| \end{bmatrix} \xrightarrow{\|\cdot\|_{\mathcal{B}}} 0_{\mathcal{B}}, \quad \text{as } \phi \to \infty.$$

However,

$$\begin{aligned}\varpi(\Xi\Phi\zeta_\phi,\Phi\Xi\zeta_\phi) &= \varpi\left(\Xi\left(\frac{1-2\phi}{1-\phi}\right),\Phi\left(\frac{2}{(1-\phi)^2}\right)\right)\\ &= \varpi(3,2)\\ &= \begin{bmatrix}1 & 0\\ 0 & \upsilon\end{bmatrix},\end{aligned}$$

which means $\varpi(\Xi\Phi\zeta_\phi,\Phi\Xi\zeta_\phi)\nrightarrow 0_{\mathcal{B}}$. Hence, Ξ and Φ have a unique CFP.

4 Application

Consider the nonlinear fractional-order initial value problem (FIVP) of the form

$$\begin{aligned}&\mathcal{D}_0^\alpha\zeta(\sigma)=\kappa\zeta(\varrho)+\mathfrak{g}\big(\varrho,\zeta(\varrho)\big),\quad \sigma\geq 0,\\ &\zeta(0)=\mu,\end{aligned}\tag{4.1}$$

where $0<\alpha\leq 1$ is the fractional order, κ is a nonnegative real constant, and μ is a real constant. The nonlinear term is $\mathfrak{g}$ and it is continuous for every $\sigma\in\mathbb{R}^{\mathfrak{n}}$. (For more details see [27]).

The solution of equation (4.1) is

$$\zeta(\sigma)=\mu+\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}\big[\kappa\cdot\zeta(\varrho)+\mathfrak{g}\big(\varrho,\zeta(\varrho)\big)\big]\,d\varrho.$$

Let $\mathcal{X}=\{e\in\mathcal{C}(\mathcal{I},\mathbb{R}): e(\sigma)>0,\forall\sigma\in\mathcal{I}\}$ and $\mathcal{B}=\mathcal{M}_2(\mathbb{R})$. Define relation $\mathcal{R}$ on $\mathcal{X}$ as $(\zeta,\vartheta)\in\mathcal{R}$ iff $\zeta,\vartheta\geq 0$ and $\varpi(\zeta,\vartheta)=\left[\begin{smallmatrix}|\zeta-\vartheta & 0\\ 0 & \upsilon|\zeta-\vartheta|\end{smallmatrix}\right]$, where $\zeta,\vartheta\in\mathcal{R}$ and $\upsilon\geq 0$ is a constant. Then, $(\mathcal{X},\mathcal{B},\varpi,\mathcal{R})$ is a complete $\mathscr{C}^*$-AV$\mathcal{R}$-MS.

Theorem 4.1 *Assume the nonlinear fractional-order initial value problem as given in* (4.1). *Suppose that the following condition is satisfied*:

(i) *Consider that the solutions of the nonlinear fractional-order initial value problem* (4.1) *are*

$$\begin{aligned}\zeta(\sigma)&=\mu+\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}\big[\kappa\cdot\zeta(\varrho)+\mathfrak{g}_1\big(\varrho,\zeta(\varrho)\big)\big]\,d\varrho,\\ \zeta(\sigma)&=\mu+\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}\big[\kappa\cdot\zeta(\varrho)+\mathfrak{g}_2\big(\varrho,\zeta(\varrho)\big)\big]\,d\varrho,\end{aligned}$$

where $\mathfrak{g}_1$, $\mathfrak{g}_2$ *are nonnegative real constants.*

(ii) *There exist a constant* $\mathcal{L}\in\mathbb{R}^+$ *and* $\kappa>0$ *such that* $|\mathfrak{g}(\sigma,e)-\mathfrak{g}(\sigma,l)|\leq\frac{\mathcal{L}}{\kappa}|e-l|$,

(iii) *There exists* $0<\alpha\leq 1$ *such that* $\frac{\sigma^\alpha}{\Gamma(\alpha)\mathcal{L}}<1$.

Then, the nonlinear fractional-order initial value value problem (4.1), *has a unique common solution.*

Proof Define $\Xi,\Phi:\mathcal{X}\to\mathcal{X}$ by

$$\Xi\zeta(\sigma)=\mu+\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}\big[\kappa\cdot\zeta(\varrho)+\mathfrak{g}_1\big(\varrho,\zeta(\varrho)\big)\big]\,d\varrho,$$

$$\Phi\zeta(\sigma) = \mu + \frac{1}{\Gamma(\alpha)}\int_0^\sigma (\sigma-\varrho)^{\alpha-1}\big[\kappa\cdot\zeta(\varrho) + \mathfrak{g}_2\big(\varrho,\zeta(\varrho)\big)\big]\,d\varrho.$$

Clearly, $\varXi$ and Φ are $\mathcal{R}$-preserving. For all $(\zeta,\vartheta)\in\mathcal{R}$, one has

$$\begin{aligned}
&\varpi(\varXi\zeta,\varPhi\vartheta)\\
&= \begin{bmatrix} |\varXi\zeta-\varPhi\vartheta| & 0\\ 0 & \upsilon|\varXi\zeta-\varPhi\vartheta|\end{bmatrix}\\
&= \begin{bmatrix} \begin{matrix}|\mu+\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\zeta(\varrho)\\ +\mathfrak{g}_1(\varrho,\zeta(\varrho))]\,d\varrho\\ -\mu-\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\vartheta(\varrho)\\ +\mathfrak{g}_2(\varrho,\vartheta(\varrho))]\,d\varrho|\end{matrix} & 0\\ 0 & \begin{matrix}\upsilon|\mu+\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\zeta(\varrho)\\ +\mathfrak{g}_1(\varrho,\zeta(\varrho))]\,d\varrho\\ -\mu-\frac{1}{\Gamma(\alpha)}\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\vartheta(\varrho)\\ +\mathfrak{g}_2(\varrho,\vartheta(\varrho))]\,d\varrho|\end{matrix}\end{bmatrix}\\
&= \begin{bmatrix} \begin{matrix}|\frac{1}{\Gamma(\alpha)}[\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\zeta(\varrho)+\mathfrak{g}_1(\varrho,\zeta(\varrho))]\,d\varrho\\ -\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\vartheta(\varrho)+\mathfrak{g}_2(\varrho,\vartheta(\varrho))]\,d\varrho]|\end{matrix} & 0\\ 0 & \begin{matrix}\upsilon|\frac{1}{\Gamma(\alpha)}[\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\zeta(\varrho)+\mathfrak{g}_1(\varrho,\zeta(\varrho))]\,d\varrho\\ -\int_0^\sigma(\sigma-\varrho)^{\alpha-1}[\kappa\cdot\vartheta(\varrho)+\mathfrak{g}_2(\varrho,\vartheta(\varrho))]\,d\varrho]|\end{matrix}\end{bmatrix}\\
&\le \begin{bmatrix} \frac{1}{\Gamma(\alpha)}\mathcal{L}\|\zeta-\vartheta\||\int_0^\sigma(\sigma-\varrho)^{\alpha-1}\,d\varrho| & 0\\ 0 & \frac{\upsilon}{\Gamma(\alpha)}\mathcal{L}\|\zeta-\vartheta\||\int_0^\sigma(\sigma-\varrho)^{\alpha-1}\,d\varrho|\end{bmatrix}\\
&= \begin{bmatrix} \frac{1}{\Gamma(\alpha)}\mathcal{L}\|\zeta-\vartheta\|\frac{\sigma^\alpha}{\varrho} & 0\\ 0 & \frac{\upsilon}{\Gamma(\alpha)}\mathcal{L}\|\zeta-\vartheta\|\frac{\sigma^\alpha}{\varrho}\end{bmatrix}\\
&\le \left(\frac{\sigma^\varrho}{\Gamma(\alpha)\varrho}\right)\mathcal{L}\begin{bmatrix}\|\zeta-\vartheta\| & 0\\ 0 & \upsilon\|\zeta-\vartheta\|\end{bmatrix},
\end{aligned}$$

which implies that

$$\varpi(\varXi\zeta,\varPhi\vartheta)\le\mathcal{P}\varpi(\zeta,\vartheta),\quad \text{where } \mathcal{P} = \left(\frac{\sigma^\alpha}{\Gamma(\alpha)\varrho}\right)\mathcal{L}<1.$$

Therefore, all the hypothesis of Theorem 3.1 are satisfied. Hence, $\varXi$ and $\varPhi$ have a unique common solution. □

5 Conclusion

In this paper, we proved some CFP theorems on $\mathscr{C}^*$-AV$\mathcal{R}$-MS. In addition, based on our obtained results an example was provided. Specifically, an application of a fractional-order initial value problem was presented.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
All authors have equal contributions. All authors read and approved the final manuscript.

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