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Fixed-point results for generalized contraction in K -sequentially complete ordered dislocated fuzzy quasimetric spaces

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Abstract

The ambition of this work is to introduce the notion of left (right) K -sequentially complete ordered dislocated fuzzy quasimetric spaces and to define a relevant Hausdorff metric on compact sets. A new approach, given in (Shoaib et al. in Filomat 34(2):323–338, 2020) has been used to obtain fixed-point results for multivalued mappings fulfilling generalized contraction in the latest framework. For the authenticity of our result, an example is formulated.

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Keywords: Common fixed point; Left (right) K -sequentially complete dislocated fuzzy quasimetric space; Open ball; Multivalued mappings; Partial order

1 Introduction and preliminaries

One of the branch of functional analysis is fixed-point theory. Fixed-point theory plays a key role to find solutions of mathematical and engineering problems. The fixed-point results for multivalued mappings generalizes the results for single-valued mappings. Applications of the results for multivalued mappings can be seen in Nash equilibria, engineering, and game theory [6, 9, 10, 20, 27, 28, 30]. With the help of multivalued mapping, many results have been proved, for example, see [7, 19, 25, 26, 29, 36–38].

A solution for matrix equations was obtained by a fixed-point result in an ordered metric space that had been proved by Ran and Reurings [24]. In a complete ordered metric space, Altun et al. [3] proved a common fixed point for the mappings satisfying a new restriction of order. For more results in ordered spaces, see [1, 4, 5, 8, 15, 41]. The idea of fuzzy sets was given by Lotfi Zadeh for the first time in 1965 [43]. This concept has been extended in fuzzy functional analysis, fuzzy topology, fuzzy control theory, and decision making, see [18, 22, 23, 31]. One of the significant developments of fuzzy sets in fuzzy functional analysis is fuzzy quasimetric spaces, see [11, 13, 14, 32]. The symmetric condition was not assumed in the fuzzy quasimetric spaces. Arshad et al. [5] observed that there were mappings that had fixed points but there were no results to ensure the existence of a fixed point of such mappings. They introduced a contraction on a closed ball to achieve common fixed points for such mappings. For further results on a closed ball, see

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[33–37, 40]. Hitzler et al. [16] established a dislocated metric space and obtained some results, see also [17]. Recently, Poovaragavan et al. [21] introduced the concept of right complete dislocated quasi- G -fuzzy metric spaces and gave some results on a closed ball in these spaces. In this paper, we have introduced the concept of ordered dislocated fuzzy quasimetric spaces and dislocated Hausdorff fuzzy quasimetric spaces. We have used a new type of contraction on an intersection of an open ball and a sequence to obtain the common fixed point of multivalued mappings in left(right) K -sequentially complete dislocated fuzzy quasimetric spaces. Our results have extended the results of Altun et al. [3] and Shoaib et al. [39] in many ways. The idea of this manuscript is motived by [39], where an ordered left K -sequentially complete dislocated quasimetric space is replaced by a left K -sequentially complete ordered dislocated fuzzy quasimetric space. Theorem 2.1 is analogous to Theorem 2.2 in [39]. We give the following definitions and results that will be helpful to understand the paper.

Definition 1.1 ([12]) A binary operation $\circledast : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous triangular norm (t -norm) if the following axioms hold: $\mathcal{T}1)$ $a \circledast b = b \circledast a$ and $a \circledast (b \circledast c) = (a \circledast b) \circledast c$; $\mathcal{T}2)$ \circledast is continuous; $\mathcal{T}3)$ $a \circledast 1 = 1$ for all $a \in [0, 1]$; $\mathcal{T}4)$ $a \circledast b < c \circledast d$ whenever $a < c$ and $b < d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.2 ([42]) Let Ψ be the class of all mappings $\mu : [0, 1] \rightarrow [0, 1]$ such that 1) μ is continuous and nondecreasing; 2) $\mu(t) > t$ for all $t \in (0, 1)$.

Lemma 1.3 ([42]) If $\mu \in \Psi$, then 1) $\mu(1) = 1$; 2) $\lim_{k \rightarrow \infty} \mu^k(t) = 1$ for all $t \in (0, 1)$.

Definition 1.4 ([3]) Let \mathcal{Y} be a nonempty set. Then, \preceq is a partial order on \mathcal{Y} if: (i) $x \preceq x$ for all $x \in \mathcal{Y}$; (ii) $x \preceq y$ and $y \preceq x$ implies $x = y$ for all $(x, y) \in \mathcal{Y} \times \mathcal{Y}$; (iii) $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $(x, y, z) \in \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}$. Let $A \neq \phi$ and $A \subseteq \mathcal{Y}$. Then, $x \preceq A$ iff $x \preceq a$, for all $a \in A$ and $x \succeq A$ iff $x \succeq a$, for all $a \in A$.

Definition 1.5 ([2]) Let $\mathcal{Y} \neq \phi$ be an arbitrary set, \circledast a continuous t -norm, and \mathcal{F}_{dq} a fuzzy set on $\mathcal{Y} \times \mathcal{Y} \times [0, \infty)$. The 3-tuple $(\mathcal{Y}, \mathcal{F}_{dq}, \circledast)$ is said to be a dislocated fuzzy quasimetric space, if \mathcal{F}_{dq} satisfies the following constraints for each $x, y, z \in X$ and $u, t > 0$:

$\mathcal{F}1)$ If $\mathcal{F}_{dq}(x, y, u) = \mathcal{F}_{dq}(y, x, u) = 1$, then $x = y$;

$\mathcal{F}2)$ $\mathcal{F}_{dq}(x, y, u) \circledast \mathcal{F}_{dq}(y, z, s) \leq \mathcal{F}_{dq}(x, z, u + s)$.

For $x_o \in \mathcal{Y}$, $u > 0$, $B_{\mathcal{F}_{dq}}(x_o, r, u) = \{y \in \mathcal{Y} : \mathcal{F}_{dq}(x_o, y, u) > 1 - r \wedge \mathcal{F}_{dq}(y, x_o, u) > 1 - r\}$ and $\overline{B_{\mathcal{F}_{dq}}(x_o, r, u)} = \{y \in \mathcal{Y} : \mathcal{F}_{dq}(x_o, y, u) \geq 1 - r \wedge \mathcal{F}_{dq}(y, x_o, u) \geq 1 - r\}$ are open and closed balls, respectively, in $(\mathcal{Y}, \mathcal{F}_{dq}, \circledast)$.

Example 1.6 Let $\mathcal{Y} = [0, \infty)$. Then, $\mathcal{F}_{dq}(x, y, u) = \frac{u}{u+x+2y}$ is a dislocated fuzzy quasimetric with $a \circledast b = \min\{a, b\}$. Note that $(\mathcal{Y}, \mathcal{F}_{dq}, \circledast)$ is neither a dislocated fuzzy metric space nor a fuzzy quasimetric space.

Definition 1.7 Let X be a nonempty set. Then, $(X, \preceq, \mathcal{F}_{dq})$ is called an ordered dislocated fuzzy quasimetric space if: (i) \mathcal{F}_{dq} is a dislocated fuzzy quasimetric on X and (ii) \preceq is a partial order on X .

Definition 1.8 Let $(\mathcal{Y}, \mathcal{F}_{dq}, \circledast)$ be a dislocated fuzzy quasimetric space. A sequence $\{x_k\}$ in $(\mathcal{Y}, \mathcal{F}_{dq}, \circledast)$ is said to be

- 1) dislocated fuzzy quasiconvergent to a point $x \in \mathcal{Y}$, if $\lim_{k \rightarrow \infty} \mathcal{F}_{d_q}(x_k, x, u) = 1 = \lim_{k \rightarrow \infty} \mathcal{F}_{d_q}(x, x_k, u)$ for all $u > 0$ or for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k > k_0$, $\mathcal{F}_{d_q}(x_k, x, u) < \varepsilon$ and $\mathcal{F}_{d_q}(x, x_k, u) < \varepsilon$. In this case x is called a limit of $\{x_k\}$.
- 2) a left (right) K -Cauchy sequence if for $k, m \in \mathbb{N}$ with $k > m$, $\lim_{k, m \rightarrow \infty} \mathcal{F}_{d_q}(x_k, x_m, u) = 1$ ($\lim_{k, m \rightarrow \infty} \mathcal{F}_{d_q}(x_m, x_k, u) = 1$) for all $u > 0$ or for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k, m \in \mathbb{N}$ with $k > m \geq k_0$, $\mathcal{F}_{d_q}(x_k, x_m, u) < \varepsilon$ ($\mathcal{F}_{d_q}(x_m, x_k, u) < \varepsilon$) for all $u > 0$.
- 3) $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ is called left (right) K -sequentially complete if every left (right) K -Cauchy sequence in \mathcal{Y} converges to a point $x \in \mathcal{Y}$.

Definition 1.9 Let $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ be a dislocated fuzzy quasimetric space. Then, for each $a \in \mathcal{Y}$, $B \subseteq \mathcal{Y}$, and $u > 0$, define $\mathcal{F}_{d_q}(a, B, u) = \sup\{\mathcal{F}_{d_q}(a, b, u) : b \in B\}$ and $\mathcal{F}_{d_q}(B, a, u) = \sup\{\mathcal{F}_{d_q}(b, a, u) : b \in B\}$.

For a given fuzzy metric space $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$, $K_0(\mathcal{Y})$ will represent the set of nonempty compact subsets of $(\mathcal{Y}, \tau_{\mathcal{F}})$, where $(\mathcal{Y}, \tau_{\mathcal{F}})$ is a metrizable topological space, generated by a fuzzy metric space $(\mathcal{Y}, \mathcal{F}, \circledast)$.

Lemma 1.10 Let $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ be a dislocated fuzzy quasimetric space. Then, for each $a \in \mathcal{Y}$, $B \in K_0(\mathcal{Y})$ and $u > 0$, there is $b_0 \in B$ such that $\mathcal{F}_{d_q}(a, B, u) = \mathcal{F}_{d_q}(a, b_0, u)$ and $\mathcal{F}_{d_q}(B, a, u) = \mathcal{F}_{d_q}(b_0, a, u)$.

Lemma 1.11 Let $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ be a dislocated fuzzy quasimetric space. Then, for each $A \in K_0(\mathcal{Y})$ and for any nonempty subset B of \mathcal{Y} and $u > 0$, there exists $a_0 \in A$ such that $\inf_{a \in A} \mathcal{F}_{d_q}(a, B, u) = \mathcal{F}_{d_q}(a_0, B, u)$ and $\inf_{a \in A} \mathcal{F}_{d_q}(B, a, u) = \mathcal{F}_{d_q}(B, a_0, u)$.

Definition 1.12 Let $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ be a dislocated fuzzy quasimetric space. We define a function H_{d_q} on $K_0(\mathcal{Y}) \times K_0(\mathcal{Y}) \times (0, \infty)$ by

$$H_{d_q}(A, B, u) = \min \left\{ \inf_{a \in A} \mathcal{F}_{d_q}(a, B, u), \inf_{b \in B} \mathcal{F}_{d_q}(A, b, u) \right\}.$$

$(K_0(\mathcal{Y}), H_{d_q}, \circledast)$ is known as a dislocated Hausdorff fuzzy quasimetric space on $K_0(\mathcal{Y})$.

Lemma 1.13 Let $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ be a dislocated fuzzy quasimetric space and $(K_0(\mathcal{Y}), H_{d_q}, \circledast)$ be a Hausdorff metric space on $K_0(\mathcal{Y})$. Then, for arbitrary $A, B \in K_0(\mathcal{Y})$ and for each $a \in A$, there exists $b_a \in B$ such that $H_{d_q}(A, B, u) \leq \mathcal{F}_{d_q}(a, b_a, u)$ and $H_{d_q}(B, A, u) \leq \mathcal{F}_{d_q}(b_a, a, u)$, where $\mathcal{F}_{d_q}(a, B, u) = \mathcal{F}_{d_q}(a, b_a, u)$ and $\mathcal{F}_{d_q}(B, a, u) = \mathcal{F}_{d_q}(b_a, a, u)$.

2 Main result

Let $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ be a dislocated fuzzy quasimetric space, $e_0 \in \mathcal{Y}$ and $\mathcal{T} : \mathcal{Y} \rightarrow K_0(\mathcal{Y})$ be a multivalued mapping on \mathcal{Y} . As $\mathcal{T}e_0$ is a compact set, there exists $e_1 \in \mathcal{T}e_0$ such that $\mathcal{F}_{d_q}(e_0, \mathcal{T}e_0, u) = \mathcal{F}_{d_q}(e_0, e_1, u)$ and $\mathcal{F}_{d_q}(\mathcal{T}e_0, e_0, u) = \mathcal{F}_{d_q}(e_1, e_0, u)$. Now, for $e_1 \in \mathcal{Y}$, there exists $e_2 \in \mathcal{T}e_1$ such that $\mathcal{F}_{d_q}(e_1, \mathcal{T}e_1, u) = \mathcal{F}_{d_q}(e_1, e_2, u)$ and $\mathcal{F}_{d_q}(\mathcal{T}e_1, e_1, u) = \mathcal{F}_{d_q}(e_2, e_1, u)$. Continuing this process, we construct a sequence e_k of points in \mathcal{Y} such that $e_{k+1} \in \mathcal{T}e_k$, $\mathcal{F}_{d_q}(e_k, \mathcal{T}e_k, u) = \mathcal{F}_{d_q}(e_k, e_{k+1}, u)$ and $\mathcal{F}_{d_q}(\mathcal{T}e_k, e_k, u) = \mathcal{F}_{d_q}(e_{k+1}, e_k, u)$. We denote this iterative sequence by $\{\mathcal{Y}\mathcal{T}(e_k)\}$ and say that $\{\mathcal{Y}\mathcal{T}(e_k)\}$ is a sequence in \mathcal{Y} generated by e_0 .

Theorem 2.1 Let $(\mathcal{Y}, \preceq, \mathcal{F}_{d_q}, \circledast)$ be a left K -sequentially complete ordered dislocated fuzzy quasimetric space with $a * b = \min\{a, b\}$. Let $(K_0(\mathcal{Y}), H_{d_q}, \circledast)$ be a dislocated Hausdorff fuzzy

quasimetric space on $K_0(\mathcal{Y})$. Let $S, \mathcal{T} : \mathcal{Y} \rightarrow K_0(\mathcal{Y})$ be multivalued mappings. Assume that the following assertions hold: (i) There exist $\mu \in \Psi$, $\epsilon_o \in \mathcal{Y}$ and $r > 0$ such that for every $\epsilon, f \in B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\}$ with $\epsilon \succeq S\epsilon, f \preceq Sf$, we have

$$\min\{H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} \geq \mu(D(\epsilon, f, u)),$$

for all $u > 0$, where

$$D(\epsilon, f, u) = \min\{\mathcal{F}_{d_q}(\epsilon, f, u), \mathcal{F}_{d_q}(\epsilon, \mathcal{T}\epsilon, u), \mathcal{F}_{d_q}(f, \mathcal{T}f, u)\}.$$

(ii) If $\epsilon \in B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o)$,

$$\mathcal{F}_{d_q}(\epsilon, \mathcal{T}\epsilon, u) = \mathcal{F}_{d_q}(\epsilon, f, u) \text{ and } \mathcal{F}_{d_q}(\mathcal{T}\epsilon, \epsilon, u) = \mathcal{F}_{d_q}(f, \epsilon, u),$$

then (iia) If $\epsilon \preceq S\epsilon$, then $f \succeq Sf$. (iib) If $\epsilon \succeq S\epsilon$, then $f \preceq Sf$. (iii) The set $G(S) = \{\epsilon : \epsilon \preceq S\epsilon \text{ and } \epsilon \in B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o)\}$ is closed and contains ϵ_o . (iv) For $k \in \cup\{0\}$, we have

$$\bigoplus_{p=0}^k \min\left\{\mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2^{p+1}}\right)\right), \mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2^{p+1}}\right)\right)\right\} > 1 - r.$$

Then, the subsequence $\{\epsilon_{2k}\}$ of $\{\mathcal{Y}\mathcal{T}(\epsilon_k)\}$ is a sequence in $G(S)$ and $\epsilon_{2k} \rightarrow \epsilon^* \in G(S)$ and $\mathcal{F}_{d_q}(\epsilon^*, \epsilon^*, u) = 1$. Also, if the inequality (i) holds for ϵ^* , then S and \mathcal{T} have a common fixed point ϵ^* in $B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o)$.

Proof Since $\epsilon_o \in G(S)$, (iii) implies that $\epsilon_o \preceq S\epsilon_o$ and $\epsilon_o \in B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o)$. Consider the sequence $\{\mathcal{Y}\mathcal{T}(\epsilon_k)\}$. Then, there exists $\epsilon_1 \in \mathcal{T}\epsilon_o$ such that

$$\mathcal{F}_{d_q}(\epsilon_o, \mathcal{T}\epsilon_o, u) = \mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u) \quad \text{and} \quad \mathcal{F}_{d_q}(\mathcal{T}\epsilon_o, \epsilon_o, u) = \mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u).$$

Now, (iia) implies that $\epsilon_1 \succeq S\epsilon_1$. By using the property of the t -norm and (iv), we have

$$\begin{aligned} & \min\{\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u_o), \mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u_o)\} \\ &= \min\{\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u_o), \mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u_o)\} \otimes 1 \\ &\geq \min\left\{\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2}\right)\right\} \otimes \min\left\{\mu\left(\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2^2}\right)\right), \right. \\ &\quad \left. \mu\left(\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2^2}\right)\right)\right\} \\ &= \bigoplus_{p=0}^1 \min\left\{\mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2^{p+1}}\right)\right), \mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2^{p+1}}\right)\right)\right\} \\ &> 1 - r. \end{aligned}$$

It follows that $\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u_o) > 1 - r$ and $\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u_o) > 1 - r$. Hence, $\epsilon_1 \in B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o)$. Also,

$$\mathcal{F}_{d_q}(\epsilon_1, \mathcal{T}\epsilon_1, u) = \mathcal{F}_{d_q}(\epsilon_1, \epsilon_2, u) \quad \text{and} \quad \mathcal{F}_{d_q}(\mathcal{T}\epsilon_1, \epsilon_1, u) = \mathcal{F}_{d_q}(\epsilon_2, \epsilon_1, u).$$

As $\epsilon_1 \succeq S\epsilon_1$, (iib) implies $\epsilon_2 \preceq S\epsilon_2$. By the triangle inequality, we have

$$\mathcal{F}_{d_q}(\epsilon_o, \epsilon_2, u_o) \geq \mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2}\right) \circledast \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_2, \frac{u_o}{2}\right). \quad (2.1)$$

By Lemma 1.13, we have

$$\begin{aligned} \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_2, \frac{u}{2}\right) &\geq H_{d_q}\left(\mathcal{T}\epsilon_o, \mathcal{T}\epsilon_1, \frac{u}{2}\right) \\ &\geq \min\left\{H_{d_q}\left(\mathcal{T}\epsilon_o, \mathcal{T}\epsilon_1, \frac{u}{2}\right), H_{d_q}\left(\mathcal{T}\epsilon_1, \mathcal{T}\epsilon_o, \frac{u}{2}\right)\right\}. \end{aligned}$$

As $\epsilon_o, \epsilon_1 \in B_{\mathcal{F}_{d_q}}(\epsilon_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\}$, $\epsilon_1 \succeq S\epsilon_1$ and $\epsilon_o \preceq S\epsilon_o$, by (i), we have

$$\begin{aligned} \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_2, \frac{u}{2}\right) &\geq \mu\left(D\left(\epsilon_1, \epsilon_o, \frac{u}{2}\right)\right) \\ &= \mu\left(\min\left\{\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_o, \mathcal{T}\epsilon_o, \frac{u}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_1, \mathcal{T}\epsilon_1, \frac{u}{2}\right)\right\}\right) \\ &= \mu\left(\min\left\{\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_2, \frac{u}{2}\right)\right\}\right). \end{aligned}$$

If $\min\{\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, \frac{u}{2}), \mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, \frac{u}{2}), \mathcal{F}_{d_q}(\epsilon_1, \epsilon_2, \frac{u}{2})\} = \mathcal{F}_{d_q}(\epsilon_1, \epsilon_2, \frac{u}{2})$, then a contradiction arises due to the fact that $\mu(u) > u$. Hence, we have

$$\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_2, \frac{u}{2}\right) \geq \mu\left(\min\left\{\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u}{2}\right)\right\}\right). \quad (2.2)$$

Using (2.2) in (2.1), we have

$$\begin{aligned} \mathcal{F}_{d_q}(\epsilon_o, \epsilon_2, u_o) &\geq \mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2}\right) \circledast \mu\left(\min\left\{\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2}\right)\right\}\right) \\ &\geq \min\left\{\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2}\right), \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2}\right)\right\} \\ &\circledast \min\left\{\mu\left(\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2^2}\right)\right), \mu\left(\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2^2}\right)\right)\right\} \\ &= \bigcircledast_{p=0}^1 \min\left\{\mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2^{p+1}}\right)\right), \mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2^{p+1}}\right)\right)\right\}. \end{aligned} \quad (2.3)$$

$$\mathcal{F}_{d_q}(\epsilon_o, \epsilon_2, u_o) > 1 - r.$$

Now, by the triangular inequality, we have

$$\mathcal{F}_{d_q}(\epsilon_2, \epsilon_o, u_o) \geq \mathcal{F}_{d_q}\left(\epsilon_2, \epsilon_1, \frac{u_o}{2}\right) \circledast \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{u_o}{2}\right). \quad (2.4)$$

By Lemma 1.13, we have

$$\begin{aligned} \mathcal{F}_{d_q}\left(\epsilon_2, \epsilon_1, \frac{u}{2}\right) &\geq H_{d_q}\left(\mathcal{T}\epsilon_1, \mathcal{T}\epsilon_o, \frac{u}{2}\right) \\ &\geq \min\left\{H_{d_q}\left(\mathcal{T}\epsilon_1, \mathcal{T}\epsilon_o, \frac{u}{2}\right), H_{d_q}\left(\mathcal{T}\epsilon_o, \mathcal{T}\epsilon_1, \frac{u}{2}\right)\right\}. \end{aligned}$$

As $\mathbf{e}_o, \mathbf{e}_1 \in B_{\mathcal{F}_{dq}}(\mathbf{e}_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$, $\mathbf{e}_1 \succeq S\mathbf{e}_1$ and $\mathbf{e}_o \preceq S\mathbf{e}_o$, by (i), we have

$$\begin{aligned} F_{dq}\left(e_2, e_1, \frac{u}{2}\right) &\geq \mu\left(D\left(e_1, e_o, \frac{u}{2}\right)\right) \\ &= \mu\left(\min\left\{\mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2}\right), \mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_2, \frac{u}{2}\right), \mathcal{F}_{dq}\left(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2}\right)\right\}\right). \end{aligned}$$

If $\min\{\mathcal{F}_{dq}(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2}), \mathcal{F}_{dq}(\mathbf{e}_1, \mathbf{e}_2, \frac{u}{2}), \mathcal{F}_{dq}(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2})\} = \mathcal{F}_{dq}(\mathbf{e}_1, \mathbf{e}_2, \frac{u}{2})$, then by (2.2)

$$\mathcal{F}_{dq}\left(\mathbf{e}_2, \mathbf{e}_1, \frac{u}{2}\right) \geq \mu\left(\min\left\{\mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2}\right), \mathcal{F}_{dq}\left(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2}\right)\right\}\right).$$

If $\min\{\mathcal{F}_{dq}(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2}), \mathcal{F}_{dq}(\mathbf{e}_1, \mathbf{e}_2, \frac{u}{2}), \mathcal{F}_{dq}(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2})\} = \min\{\mathcal{F}_{dq}(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2}), \mathcal{F}_{dq}(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2})\}$, then in both cases, we have

$$\mathcal{F}_{dq}\left(\mathbf{e}_2, \mathbf{e}_1, \frac{u}{2}\right) \geq \mu\left(\min\left\{\mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2}\right), \mathcal{F}_{dq}\left(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2}\right)\right\}\right). \quad (2.5)$$

By inserting (2.5) into (2.4), we have

$$\begin{aligned} \mathcal{F}_{dq}(\mathbf{e}_2, \mathbf{e}_o, u_o) &\geq \mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2}\right) \circledast \mu\left(\min\left\{\mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2}\right), \mathcal{F}_{dq}\left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2}\right)\right\}\right) \\ &\geq \min\left\{\mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2}\right), \mathcal{F}_{dq}\left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2}\right)\right\} \\ &\circledast \min\left\{\mu\left(\mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2^2}\right)\right), \mu\left(\mathcal{F}_{dq}\left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2^2}\right)\right)\right\} \\ &= \bigcircledast_{p=0}^1 \min\left\{\mu^p\left(\mathcal{F}_{dq}\left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2^{p+1}}\right)\right), \mu^p\left(\mathcal{F}_{dq}\left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2^{p+1}}\right)\right)\right\}, \end{aligned} \quad (2.6)$$

$$\mathcal{F}_{dq}(\mathbf{e}_2, \mathbf{e}_o, u_o) > 1 - r.$$

From (2.3) and (2.6), it follows that $\mathcal{F}_{dq}(\mathbf{e}_o, \mathbf{e}_2, u_o) > 1 - r$ and $\mathcal{F}_{dq}(\mathbf{e}_2, \mathbf{e}_o, u_o) > 1 - r$. Hence, $\mathbf{e}_2 \in B_{\mathcal{F}_{dq}}(\mathbf{e}_o, r, u_o)$. Also,

$$\mathcal{F}_{dq}(\mathbf{e}_2, \mathcal{T}\mathbf{e}_2, u) = \mathcal{F}_{dq}(\mathbf{e}_2, \mathbf{e}_3, u) \quad \text{and} \quad \mathcal{F}_{dq}(\mathcal{T}\mathbf{e}_2, \mathbf{e}_2, u) = \mathcal{F}_{dq}(\mathbf{e}_3, \mathbf{e}_2, u).$$

As $\mathbf{e}_2 \preceq S\mathbf{e}_2$, by (iib), we have $\mathbf{e}_3 \succeq S\mathbf{e}_3$. Let $\mathbf{e}_3, \mathbf{e}_4, \dots, \mathbf{e}_s \in B_{\mathcal{F}_{dq}}(\mathbf{e}_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$, $\mathbf{e}_s \preceq S\mathbf{e}_s$ and $\mathbf{e}_{s-1} \succeq S\mathbf{e}_{s-1}$ for some $s \in \mathbb{N}$, where $s = 2p$ and $p = 1, 2, 3, \dots, \frac{s}{2}$. By using Lemma 1.13, we have

$$\begin{aligned} \mathcal{F}_{dq}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u) &\geq H_{dq}(\mathcal{T}\mathbf{e}_{2p-1}, \mathcal{T}\mathbf{e}_{2p}, u) \\ &\geq \min\{H_{dq}(\mathcal{T}\mathbf{e}_{2p-1}, \mathcal{T}\mathbf{e}_{2p}, u), H_{dq}(\mathcal{T}\mathbf{e}_{2p}, \mathcal{T}\mathbf{e}_{2p-1}, u)\}. \end{aligned}$$

As $\mathbf{e}_{2p-1}, \mathbf{e}_{2p} \in B_{\mathcal{F}_{dq}}(\mathbf{e}_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$, $\mathbf{e}_{2p-1} \succeq S\mathbf{e}_{2p-1}$ and $\mathbf{e}_{2p} \preceq S\mathbf{e}_{2p}$, by (i), we have

$$\begin{aligned} \mathcal{F}_{dq}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u) &\geq \mu(D(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u)) \\ &= \mu\left(\min\{\mathcal{F}_{dq}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u), \mathcal{F}_{dq}(\mathbf{e}_{2p-1}, \mathcal{T}\mathbf{e}_{2p-1}, u), \right. \\ &\quad \left.\mathcal{F}_{dq}(\mathbf{e}_{2p}, \mathcal{T}\mathbf{e}_{2p-1}, u)\}\right). \end{aligned}$$

$$\begin{aligned} & \mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathcal{T}\mathbf{e}_{2p}, u) \} \\ &= \mu(\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u)\}). \end{aligned}$$

If $\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u)\} = \mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u)$, then a contradiction arises due to the fact that $\mu(u) > u$. Therefore,

$$\mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u) \geq \mu(\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u)), \quad (2.7)$$

which implies that

$$\mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u) \geq \min\{\mu(\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u)), \mu(\mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathbf{e}_{2p-1}, u))\}. \quad (2.8)$$

Now, by Lemma 1.13, we have

$$\begin{aligned} \mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u) &\geq H_{d_q}(\mathcal{T}\mathbf{e}_{2p-2}, \mathcal{T}\mathbf{e}_{2p-1}, u) \\ &\geq \min\{H_{d_q}(\mathcal{T}\mathbf{e}_{2p-2}, \mathcal{T}\mathbf{e}_{2p-1}, u), H_{d_q}(\mathcal{T}\mathbf{e}_{2p-1}, \mathcal{T}\mathbf{e}_{2p-2}, u)\}. \end{aligned}$$

As $\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2} \in B_{\mathcal{F}_{d_q}}(\mathbf{e}_\circ, r, u_\circ) \cap \{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$, $\mathbf{e}_{2p-1} \succeq S\mathbf{e}_{2p-1}$ and $\mathbf{e}_{2p-2} \preceq S\mathbf{e}_{2p-2}$, by (i), we have

$$\begin{aligned} \mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u) &\geq \mu(D(\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2}, u)) \\ &= \mu(\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u), \\ &\quad \mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u)\}). \end{aligned}$$

If $\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u)\} = \mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u)$, then a contradiction arises due to the fact that $\mu(u) > u$, so

$$\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u) \geq \mu(\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u)\}).$$

Apply μ on both sides. Since μ is a nondecreasing function,

$$\begin{aligned} \mu(\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u)) &\geq \mu^2(\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u)\}), \\ \mu(\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p}, u)) &\geq \min\{\mu^2(\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2}, u)), \mu^2(\mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u))\}. \end{aligned} \quad (2.9)$$

By inserting (2.9) into (2.7), we obtain

$$\mathcal{F}_{d_q}(\mathbf{e}_{2p}, \mathbf{e}_{2p+1}, u) \geq \min\{\mu^2(\mathcal{F}_{d_q}(\mathbf{e}_{2p-1}, \mathbf{e}_{2p-2}, u)), \mu^2(\mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u))\}. \quad (2.10)$$

Now, by Lemma 1.13, we have

$$\begin{aligned} \mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u) &\geq H_{d_q}(\mathcal{T}\mathbf{e}_{2p-3}, \mathcal{T}\mathbf{e}_{2p-2}, u) \\ &\geq \min\{H_{d_q}(\mathcal{T}\mathbf{e}_{2p-3}, \mathcal{T}\mathbf{e}_{2p-2}, u), H_{d_q}(\mathcal{T}\mathbf{e}_{2p-2}, \mathcal{T}\mathbf{e}_{2p-3}, u)\}. \end{aligned}$$

As $\mathbf{e}_{2p-3}, \mathbf{e}_{2p-2} \in B_{\mathcal{F}_{d_q}}(\mathbf{e}_\circ, r, u_\circ) \cap \{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$, $\mathbf{e}_{2p-3} \succeq S\mathbf{e}_{2p-3}$ and $\mathbf{e}_{2p-2} \preceq S\mathbf{e}_{2p-2}$, by (i), we have

$$\mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u) \geq \mu(\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2p-3}, \mathbf{e}_{2p-2}, u), \mathcal{F}_{d_q}(\mathbf{e}_{2p-2}, \mathbf{e}_{2p-1}, u)\}).$$

If $\min\{\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u), \mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-1}, u)\} = \mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-1}, u)$, then a contradiction arises. Therefore,

$$\mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-1}, u) \geq \mu(\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u)), \quad (2.11)$$

$$\mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-1}, u) \geq \mu(\min\{\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u), \mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-3}, u)\}),$$

$$\mu^2 \mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-1}, u) \geq \mu^3 (\min\{\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u), \mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-3}, u)\}). \quad (2.12)$$

Now, by Lemma 1.13, we have

$$\begin{aligned} \mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p-2}, u) &\geq H_{dq}(\mathcal{T}\epsilon_{2p-2}, \mathcal{T}\epsilon_{2p-3}, u) \\ &\geq \min\{H_{dq}(\mathcal{T}\epsilon_{2p-3}, \mathcal{T}\epsilon_{2p-2}, u), H_{dq}(\mathcal{T}\epsilon_{2p-2}, \mathcal{T}\epsilon_{2p-3}, u)\}. \end{aligned}$$

As $\epsilon_{2p-3}, \epsilon_{2p-2} \in B_{\mathcal{F}_{d_q}}(\epsilon_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\}$, $\epsilon_{2p-3} \succeq S\epsilon_{2p-3}$ and $\epsilon_{2p-2} \preceq S\epsilon_{2p-2}$, by (i), we have

$$\mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p-2}, u) \geq \mu(\min\{\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u), \mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-1}, u)\}).$$

By using inequality (2.11), we have

$$\begin{aligned} \mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p-2}, u) &\geq \mu(\min\{\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u), \mu(\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u))\}) \\ &= \mu(\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u)). \end{aligned}$$

This implies that

$$\mu^2 \mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p-2}, u) \geq \mu^2 (\mu(\min\{\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u), \mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-3}, u)\})). \quad (2.13)$$

Combining inequalities (2.10), (2.12), and (2.13), we have

$$\mathcal{F}_{d_q}(\epsilon_{2p}, \epsilon_{2p+1}, u) \geq \min\{\mu^3(\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u)), \mu^3(\mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-3}, u))\}. \quad (2.14)$$

Following the patterns of inequalities (2.8), (2.10), and (2.14), we have

$$\mathcal{F}_{d_q}(\epsilon_{2p}, \epsilon_{2p+1}, u) \geq \min\{\mu^{2p}(\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u)), \mu^{2p}(\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u))\}. \quad (2.15)$$

Now, by Lemma 1.13, we have

$$\begin{aligned} \mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) &\geq H_{dq}(\mathcal{T}\epsilon_{2p}, \mathcal{T}\epsilon_{2p-1}, u) \\ &\geq \min\{H_{dq}(\mathcal{T}\epsilon_{2p-1}, \mathcal{T}\epsilon_{2p}, u), H_{dq}(\mathcal{T}\epsilon_{2p}, \mathcal{T}\epsilon_{2p-1}, u)\}. \end{aligned}$$

As $\epsilon_{2p-1}, \epsilon_{2p} \in B_{\mathcal{F}_{d_q}}(\epsilon_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\}$, $\epsilon_{2p-1} \succeq S\epsilon_{2p-1}$ and $\epsilon_{2p} \preceq S\epsilon_{2p}$, by (i), we have

$$\begin{aligned} \mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) &\geq \mu(\min\{\mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p}, u), \mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p}, u), \\ &\quad \mathcal{F}_{d_q}(\epsilon_{2p}, \epsilon_{2p+1}, u)\}) \\ &= \mu(\min\{\mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p}, u), \mathcal{F}_{d_q}(\epsilon_{2p}, \epsilon_{2p+1}, u)\}). \end{aligned}$$

Using (2.7), we have

$$\begin{aligned}\mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) &\geq \mu \left(\min \{ \mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p}, u), \mu(\mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p}, u)) \} \right) \\ \mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) &\geq \mu(\mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p}, u)),\end{aligned}\tag{2.16}$$

which implies that

$$\mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) \geq \min \{ \mu(\mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p}, u)), \mu(\mathcal{F}_{d_q}(\epsilon_{2p}, \epsilon_{2p-1}, u)) \}.\tag{2.17}$$

By (2.9) and (2.16), we obtain

$$\mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) \geq \min \{ \mu^2(\mathcal{F}_{d_q}(\epsilon_{2p-1}, \epsilon_{2p-2}, u)), \mu^2(\mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-1}, u)) \}.\tag{2.18}$$

Combining (2.12), (2.13), and (2.18), we have

$$\mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) \geq \min \{ \mu^3(\mathcal{F}_{d_q}(\epsilon_{2p-2}, \epsilon_{2p-3}, u)), \mu^3(\mathcal{F}_{d_q}(\epsilon_{2p-3}, \epsilon_{2p-2}, u)) \}.$$

Continuing in this way, we obtain

$$\mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p}, u) \geq \min \{ \mu^{2p}(\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u)), \mu^{2p}(\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u)) \}.\tag{2.19}$$

If $s = 2p + 1$, $p = 1, 2, \dots, \frac{s-1}{2}$, $\mathcal{F}_{d_q}(\epsilon_{2p+1}, \mathcal{T}\epsilon_{2p+1}, u) = \mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p+2}, u)$ and $\mathcal{F}_{d_q}(\mathcal{T}\epsilon_{2p+1}, \epsilon_{2p+1}, u) = \mathcal{F}_{d_q}(\epsilon_{2p+2}, \epsilon_{2p+1}, u)$. By using the same procedure as above, we have

$$\mathcal{F}_{d_q}(\epsilon_{2p+1}, \epsilon_{2p+2}, u) \geq \min \{ \mu^{2p+1}(\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u)), \mu^{2p+1}(\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u)) \}\tag{2.20}$$

and

$$\mathcal{F}_{d_q}(\epsilon_{2p+2}, \epsilon_{2p+1}, u) \geq \min \{ \mu^{2p+1}(\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u)), \mu^{2p+1}(\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u)) \}.\tag{2.21}$$

By combining (2.15) and (2.20), we have

$$\mathcal{F}_{d_q}(\epsilon_s, \epsilon_{s+1}, u) \geq \min \{ \mu^s(\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u)), \mu^s(\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u)) \}, \quad \text{for some } s \in .\tag{2.22}$$

and by combining (2.19) and (2.21), we obtain

$$\mathcal{F}_{d_q}(\epsilon_{s+1}, \epsilon_s, u) \geq \min \{ \mu^s(\mathcal{F}_{d_q}(\epsilon_1, \epsilon_o, u)), \mu^s(\mathcal{F}_{d_q}(\epsilon_o, \epsilon_1, u)) \}, \quad \text{for some } s \in .\tag{2.23}$$

By using the triangular inequality, (iv) and (2.22), we have

$$\begin{aligned}\mathcal{F}_{d_q}(\epsilon_o, \epsilon_{s+1}, u_o) &\geq \mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2}\right) \circledast \mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_2, \frac{u_o}{2^2}\right) \circledast \cdots \circledast \mathcal{F}_{d_q}\left(\epsilon_{s-1}, \epsilon_s, \frac{u_o}{2^s}\right) \\ &\quad \circledast \mathcal{F}_{d_q}\left(\epsilon_s, \epsilon_{s+1}, \frac{u_o}{2^s}\right) \\ &\geq \mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{u_o}{2}\right) \circledast \cdots \circledast \mathcal{F}_{d_q}\left(\epsilon_{s-1}, \epsilon_s, \frac{u_o}{2^s}\right) \circledast \mathcal{F}_{d_q}\left(\epsilon_s, \epsilon_{s+1}, \frac{u_o}{2^{s+1}}\right)\end{aligned}$$

$$\begin{aligned}
&\geq \min \left\{ \mathcal{F}_{d_q} \left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2} \right), \mathcal{F}_{d_q} \left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2} \right) \right\} \circledast \\
&\quad \min \left\{ \mu \left(\mathcal{F}_{d_q} \left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2^2} \right) \right), \mu \left(\mathcal{F}_{d_q} \left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2^2} \right) \right) \right\} \circledast \cdots \circledast \\
&\quad \min \left\{ \mu^s \left(\mathcal{F}_{d_q} \left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2^{s+1}} \right) \right), \mu^s \left(\mathcal{F}_{d_q} \left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2^{s+1}} \right) \right) \right\} \\
&= \bigcircledast_{p=0}^s \min \left\{ \mu^p \left(\mathcal{F}_{d_q} \left(\mathbf{e}_o, \mathbf{e}_1, \frac{u_o}{2^{p+1}} \right) \right), \mu^p \left(\mathcal{F}_{d_q} \left(\mathbf{e}_1, \mathbf{e}_o, \frac{u_o}{2^{p+1}} \right) \right) \right\}.
\end{aligned}$$

Hence, by (iv), we have

$$\mathcal{F}_{d_q}(\mathbf{e}_o, \mathbf{e}_{s+1}, u_o) > 1 - r. \quad (2.24)$$

Similarly, by the triangular inequality, (iv) and (2.23), we have

$$\mathcal{F}_{d_q}(\mathbf{e}_{s+1}, \mathbf{e}_o, u_o) > 1 - r. \quad (2.25)$$

From (2.24) and (2.25), $\mathbf{e}_{s+1} \in B_{\mathcal{F}_{d_q}}(\mathbf{e}_o, r, u_o)$. Hence, by mathematical induction, we have $\mathbf{e}_k \in B_{\mathcal{F}_{d_q}}(\mathbf{e}_o, r, u_o) \cap \{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$, $\mathbf{e}_{2k} \preceq S\mathbf{e}_{2k}$ and $\mathbf{e}_{2k+1} \succeq S\mathbf{e}_{2k+1}$ for all $k \in \mathbb{N}$. Also, from (iii) $\mathbf{e}_{2k} \in G(S)$. Now, inequalities (2.22) and (2.23) will be held for all $s \in \mathbb{N}$. Now, for $k, m \in \mathbb{N}$ with $k < m$, we have

$$\begin{aligned}
\mathcal{F}_{d_q}(\mathbf{e}_k, \mathbf{e}_{k+m}, u) &\geq \mathcal{F}_{d_q} \left(\mathbf{e}_k, \mathbf{e}_{k+1}, \frac{u}{2} \right) \circledast \mathcal{F}_{d_q} \left(\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \frac{u}{2^2} \right) \\
&\quad \circledast \cdots \circledast \mathcal{F}_{d_q} \left(\mathbf{e}_{k+m-2}, \mathbf{e}_{k+m-1}, \frac{u}{2^{m-1}} \right) \circledast \mathcal{F}_{d_q} \left(\mathbf{e}_{k+m-1}, \mathbf{e}_{k+m}, \frac{u}{2^{m-1}} \right) \\
&\geq \mathcal{F}_{d_q} \left(\mathbf{e}_k, \mathbf{e}_{k+1}, \frac{u}{2^m} \right) \circledast \mathcal{F}_{d_q} \left(\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \frac{u}{2^m} \right) \\
&\quad \circledast \cdots \circledast \mathcal{F}_{d_q} \left(\mathbf{e}_{k+m-2}, \mathbf{e}_{k+m-1}, \frac{u}{2^m} \right) \circledast \mathcal{F}_{d_q} \left(\mathbf{e}_{k+m-1}, \mathbf{e}_{k+m}, \frac{u}{2^m} \right) \\
&\geq \bigcircledast_{p=0}^{m-1} \left\{ \mu^{k+p} \left(\mathcal{F}_{d_q} \left(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2^m} \right) \right), \mu^{k+p} \left(\mathcal{F}_{d_q} \left(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2^m} \right) \right) \right\}.
\end{aligned}$$

By Lemma 1.3,

$$\lim_{k \rightarrow \infty} \mu^{k+p} \left(\mathcal{F}_{d_q} \left(\mathbf{e}_o, \mathbf{e}_1, \frac{u}{2^m} \right) \right) = 1, \quad \text{for every } p \in \{0, 1, \dots, m-1\}$$

and

$$\lim_{k \rightarrow \infty} \mu^{k+p} \left(\mathcal{F}_{d_q} \left(\mathbf{e}_1, \mathbf{e}_o, \frac{u}{2^m} \right) \right) = 1, \quad \text{for every } p \in \{0, 1, \dots, m-1\}.$$

Hence, $\mathcal{F}_{d_q}(\mathbf{e}_k, \mathbf{e}_{k+m}, u) \rightarrow 1$ as $k \rightarrow \infty$. Thus, we proved that $\{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$ is a left K -Cauchy sequence in $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$. As $(\mathcal{Y}, \mathcal{F}_{d_q}, \circledast)$ is left K -sequentially complete, so $\{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\} \rightarrow \mathbf{e}^* \in \mathcal{Y}$. As $\{\mathbf{e}_{2k}\}$ is a subsequence of $\{\mathcal{Y}\mathcal{T}(\mathbf{e}_k)\}$, so $\mathbf{e}_{2k} \rightarrow \mathbf{e}^*$. Also, $\{\mathbf{e}_{2k}\}$ is a sequence in $G(S)$ and $G(S)$ is closed, so $\mathbf{e}^* \in G(S)$ and therefore $\mathbf{e}^* \preceq S\mathbf{e}^*$. Also,

$$\lim_{k \rightarrow \infty} \mathcal{F}_{d_q}(\mathbf{e}_{2k}, \mathbf{e}^*, u) = 1 = \lim_{k \rightarrow \infty} \mathcal{F}_{d_q}(\mathbf{e}^*, \mathbf{e}_{2k}, u). \quad (2.26)$$

Now,

$$\mathcal{F}_{d_q}(\mathbf{e}^*, \mathbf{e}^*, u) \geq \mathcal{F}_{d_q}\left(\mathbf{e}^*, \mathbf{e}_{2k}, \frac{u}{2}\right) \circledast \mathcal{F}_{d_q}\left(\mathbf{e}_{2k}, \mathbf{e}^*, \frac{u}{2}\right)$$

$$\lim_{k \rightarrow \infty} \mathcal{F}_{d_q}(\mathbf{e}^*, \mathbf{e}^*, u) \geq 1 \circledast 1 = 1.$$

Hence,

$$\mathcal{F}_{d_q}(\mathbf{e}^*, \mathbf{e}^*, u) = 1. \quad (2.27)$$

Now,

$$\begin{aligned} \mathcal{F}_{d_q}(\mathbf{e}_{2k+2}, \mathcal{T}\mathbf{e}^*, u) &\geq H_{d_q}(\mathcal{T}\mathbf{e}_{2k+1}, \mathcal{T}\mathbf{e}^*, u) \\ &\geq \min\{H_{d_q}(\mathcal{T}\mathbf{e}_{2k+1}, \mathcal{T}\mathbf{e}^*, u), H_{d_q}(\mathcal{T}\mathbf{e}^*, \mathcal{T}\mathbf{e}_{2k+1}, u)\}. \end{aligned}$$

By assumption, inequality (i) holds for \mathbf{e}^* , also $\mathbf{e} \succeq S\mathbf{e}_{2k+1}$ and $\mathbf{e}^* \preceq S\mathbf{e}^*$. Hence,

$$\begin{aligned} \mathcal{F}_{d_q}(\mathbf{e}_{2k+2}, \mathcal{T}\mathbf{e}^*, u) &\geq \mu(D(\mathbf{e}_{2k+1}, \mathbf{e}^*, u)) \\ &= \mu(\min\{\mathcal{F}_{d_q}(\mathbf{e}_{2k+1}, \mathbf{e}^*, u), \mathcal{F}_{d_q}(\mathbf{e}_{2k+1}, \mathbf{e}_{2k+2}, u), \mathcal{F}_{d_q}(\mathbf{e}^*, \mathcal{T}\mathbf{e}^*, u)\}). \end{aligned}$$

Letting $k \rightarrow \infty$ and by using (2.27), we obtain

$$\mathcal{F}_{d_q}(\mathbf{e}^*, \mathcal{T}\mathbf{e}^*, u) \geq \mu(\mathcal{F}_{d_q}(\mathbf{e}^*, \mathcal{T}\mathbf{e}^*, u)).$$

By definition of μ , we obtain

$$\mathcal{F}_{d_q}(\mathbf{e}^*, \mathcal{T}\mathbf{e}^*, u) = 1. \quad (2.28)$$

$$\begin{aligned} \mathcal{F}_{d_q}(\mathcal{T}\mathbf{e}^*, \mathbf{e}^*, u) &\geq \mathcal{F}_{d_q}\left(\mathcal{T}\mathbf{e}^*, \mathbf{e}_{2k+2}, \frac{u}{2}\right) \circledast \mathcal{F}_{d_q}\left(\mathbf{e}_{2k+2}, \mathbf{e}^*, \frac{u}{2}\right) \\ &\geq H_{d_q}\left(\mathcal{T}\mathbf{e}^*, \mathcal{T}\mathbf{e}_{2k+1}, \frac{u}{2}\right) \circledast \mathcal{F}_{d_q}\left(\mathbf{e}_{2k+2}, \mathbf{e}^*, \frac{u}{2}\right) \\ &\geq \min\left\{H_{d_q}\left(\mathcal{T}\mathbf{e}^*, \mathcal{T}\mathbf{e}_{2k+1}, \frac{u}{2}\right), H_{d_q}\left(\mathcal{T}\mathbf{e}_{2k+1}, \mathcal{T}\mathbf{e}^*, \frac{u}{2}\right)\right\} \\ &\quad \circledast \mathcal{F}_{d_q}\left(\mathbf{e}_{2k+2}, \mathbf{e}^*, \frac{u}{2}\right). \end{aligned}$$

By assumption, inequality (i) holds for \mathbf{e}^* , also $\mathbf{e}_{2k+1} \succeq S\mathbf{e}_{2k+1}$ and $\mathbf{e}^* \preceq S\mathbf{e}^*$. Hence,

$$\begin{aligned} \mathcal{F}_{d_q}(\mathcal{T}\mathbf{e}^*, \mathbf{e}^*, u) &\geq \mu\left(D\left(\mathbf{e}_{2k+1}, \mathbf{e}^*, \frac{u}{2}\right)\right) \circledast \mathcal{F}_{d_q}\left(\mathbf{e}_{2k+2}, \mathbf{e}^*, \frac{u}{2}\right) \\ &= \mu\left(\min\left\{\mathcal{F}_{d_q}\left(\mathbf{e}_{2k+1}, \mathbf{e}^*, \frac{u}{2}\right), \mathcal{F}_{d_q}\left(\mathbf{e}_{2k+1}, \mathbf{e}_{2k+2}, \frac{u}{2}\right)\right.\right. \\ &\quad \left.\left.\mathcal{F}_{d_q}\left(\mathbf{e}^*, \mathcal{T}\mathbf{e}^*, \frac{u}{2}\right)\right\}\right) \circledast \mathcal{F}_{d_q}\left(\mathbf{e}_{2k+2}, \mathbf{e}^*, \frac{u}{2}\right). \end{aligned}$$

Letting $k \rightarrow \infty$, and by using (2.26) and (2.28), we obtain

$$\mathcal{F}_{d_q}(\mathcal{T}\epsilon^*, \epsilon^*, u) \geq \mu\left(\mathcal{F}_{d_q}\left(\epsilon^*, \mathcal{T}\epsilon^*, \frac{u}{2}\right)\right) = \mu(1).$$

This implies that $\mathcal{F}_{d_q}(\mathcal{T}\epsilon^*, \epsilon^*, u) = 1$. Hence, $\epsilon^* \in \mathcal{T}\epsilon^*$. Also,

$$\epsilon^* \preceq S\epsilon^*, \quad (2.29)$$

and

$$\mathcal{F}_{d_q}(\mathcal{T}\epsilon^*, \epsilon^*, u) = \mathcal{F}_{d_q}(\epsilon^*, \mathcal{T}\epsilon^*, u) = \mathcal{F}_{d_q}(\epsilon^*, \epsilon^*, u) = 1.$$

Then, from (ii)

$$\epsilon^* \succeq S\epsilon^*. \quad (2.30)$$

From (2.29) and (2.30), we have $\epsilon^* \preceq S\epsilon^* \preceq \epsilon^*$. This implies

$$\epsilon^* \preceq f \preceq \epsilon^* \quad \text{for all } f \in S\epsilon^*.$$

Therefore,

$$\epsilon^* = f \quad \text{for all } f \in S\epsilon^*.$$

Hence, ϵ^* is a common fixed point of \mathcal{T} and S . \square

Example 2.2 Let $\mathcal{Y} = [0, +\infty)$ and let $\mathcal{F}_{d_q}(\epsilon, f, u) = \frac{u}{u+\epsilon+2f}$ for all $\epsilon, f, u \in \mathcal{Y}$. Let \mathcal{R} be the binary relation on \mathcal{Y} defined by

$$\begin{aligned} \mathcal{R} = \{(\epsilon, \epsilon) : \epsilon \in \mathcal{Y}\} &\cup \left\{ \left(\epsilon, \frac{\epsilon}{5}\right) : \epsilon \in \left\{0, \frac{3}{7}, \frac{3}{7 \times 25}, \frac{3}{7 \times 25^2}, \frac{3}{7 \times 25^3}, \dots\right\} \right\} \\ &\cup \left\{ \left(\frac{\epsilon}{5}, \epsilon\right) : \epsilon \in \left\{0, \frac{3}{35}, \frac{3}{35 \times 25}, \frac{3}{35 \times 25^2}, \dots\right\} \right\}. \end{aligned}$$

Consider the partial order on \mathcal{Y} defined by

$$(\epsilon, f) \in \mathcal{Y} \times \mathcal{Y}, \epsilon \preceq f \text{ if and only } (\epsilon, f) \in R.$$

Then, $(\mathcal{Y}, \preceq, \mathcal{F}_{d_q}, \otimes)$ is a left K -sequentially complete ordered dislocated fuzzy quasimetric space with $a \otimes b = \min\{a, b\}$. Define the pair of mappings $\mathcal{T}, S : \mathcal{Y} \rightarrow \mathcal{Y}$ by

$$\mathcal{T}(\epsilon) = \begin{cases} [\frac{\epsilon}{7}, \frac{\epsilon}{5}] & \text{if } \epsilon \in B_{\mathcal{F}_{d_q}}(\epsilon_0, r, u_0) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\}, \\ [3\epsilon, 6\epsilon] & \text{if } \epsilon \notin B_{\mathcal{F}_{d_q}}(\epsilon_0, r, u_0) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\} \end{cases}$$

and

$$S(\epsilon) = \begin{cases} \{\frac{\epsilon}{5}\} & \text{if } \epsilon \in B_{\mathcal{F}_{d_q}}(\epsilon_0, r, u_0) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\}, \\ [7\epsilon + 5, 8\epsilon + 9] & \text{if } \epsilon \notin B_{\mathcal{F}_{d_q}}(\epsilon_0, r, u_0) \cap \{\mathcal{Y}\mathcal{T}(\epsilon_k)\}. \end{cases}$$

Define a nondecreasing mapping $\mu : [0, 1] \rightarrow [0, 1]$ by

$$\mu(u) = \sqrt{u}.$$

Observe that in this case, we have

$$A = \{\epsilon : \epsilon \leq S\epsilon\} = \left\{0, \frac{3}{7}, \frac{3}{7 \times 25}, \frac{3}{7 \times 25^2}, \frac{3}{7 \times 25^3}, \dots\right\},$$

$$B = \{f : f \geq Sf\} = \left\{0, \frac{3}{35}, \frac{3}{35 \times 25}, \frac{3}{35 \times 25^2}, \dots\right\}.$$

Let $\epsilon_0 = \frac{3}{7}$, $r = \frac{3}{4}$ and $u = 1$

$$B_{\mathcal{F}_{dq}}\left(\frac{3}{7}, \frac{3}{4}, 1\right) = \left\{f : \mathcal{F}_{dq}\left(\frac{3}{7}, f, 1\right) > 1 - \frac{3}{4} \wedge \mathcal{F}_{dq}\left(f, \frac{3}{7}, 1\right) > 1 - \frac{3}{4}\right\}$$

$$= \left\{f : \frac{1}{1 + \frac{3}{7} + 2f} > \frac{1}{4} \wedge \frac{1}{1 + f + 2(\frac{3}{7})} > \frac{1}{4}\right\}$$

$$= \{f : 28 > 10 + 14f \wedge 28 > 13 + 7f\}$$

$$= \left\{f : \frac{9}{7} > f \wedge \frac{15}{7} > f\right\} = \left[0, \frac{9}{7}\right).$$

Then,

$$G(S) = \{\epsilon : \epsilon \leq S\epsilon \text{ and } \epsilon \in B_{\mathcal{F}_{dq}}(\epsilon_0, r, u_0)\}$$

$$= \left\{0, \frac{3}{7}, \frac{3}{7 \times 25}, \frac{3}{7 \times 25^2}, \frac{3}{7 \times 25^3}, \dots\right\} \cap \left[0, \frac{9}{7}\right)$$

$$= \left\{0, \frac{3}{7}, \frac{3}{7 \times 25}, \frac{3}{7 \times 25^2}, \frac{3}{7 \times 25^3}, \dots\right\}.$$

$G(S)$ contains $\frac{3}{7}$ and is also a closed set. Now, $\frac{3}{7 \times 25^{k-1}} \in B_{\mathcal{F}_{dq}}(\epsilon_0, r, u_0)$, where $k \in \mathbb{N}$.

$$\begin{aligned} \mathcal{F}_{dq}(\epsilon, T\epsilon, u) &= \mathcal{F}_{dq}\left(\frac{3}{7 \times 25^{k-1}}, T\left(\frac{3}{7 \times 25^{k-1}}\right), u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{7 \times 25^{k-1}}, \frac{3}{5 \times 7 \times 25^{k-1}}, u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{7 \times 25^{k-1}}, \frac{3}{35 \times 25^{k-1}}, u\right), \\ \mathcal{F}_{dq}(T\epsilon, \epsilon, u) &= \mathcal{F}_{dq}\left(T\left(\frac{3}{7 \times 25^{k-1}}\right), \frac{3}{7 \times 25^{k-1}}, u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{5 \times 7 \times 25^{k-1}}, \frac{3}{7 \times 25^{k-1}}, u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{35 \times 25^{k-1}}, \frac{3}{7 \times 25^{k-1}}, u\right). \end{aligned}$$

Also, $(\frac{3}{7 \times 25^{k-1}}, \frac{3}{5 \times 7 \times 25^{k-1}}) \in \mathcal{R}$, so $\frac{3}{7 \times 25^{k-1}} \leq S(\frac{3}{7 \times 25^{k-1}})$. As $(\frac{3}{5 \times 35 \times 25^{k-1}}, \frac{3}{35 \times 25^{k-1}}) \in \mathcal{R}$ so $\frac{3}{35 \times 25^{k-1}} \geq S(\frac{3}{35 \times 25^{k-1}})$. Hence, condition (iiia) is satisfied. Now $\frac{3}{35 \times 25^{k-1}} \in B_{\mathcal{F}_{dq}}(\epsilon_0, r, u_0)$,

where $k \in$.

$$\begin{aligned}\mathcal{F}_{dq}\left(\frac{3}{35 \times 25^{k-1}}, \mathcal{T}\left(\frac{3}{35 \times 25^{k-1}}\right), u\right) &= \mathcal{F}_{dq}\left(\frac{3}{35 \times 25^{k-1}}, \frac{3}{5 \times 35 \times 25^{k-1}}, u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{35 \times 25^{k-1}}, \frac{3}{25 \times 7 \times 25^{k-1}}, u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{35 \times 25^{k-1}}, \frac{3}{7 \times 25^k}, u\right), \\ \mathcal{F}_{dq}\left(\mathcal{T}\left(\frac{3}{35 \times 25^{k-1}}\right), \left(\frac{3}{35 \times 25^{k-1}}\right), u\right) &= \mathcal{F}_{dq}\left(\frac{3}{5 \times 35 \times 25^{k-1}}, \frac{3}{35 \times 25^{k-1}}, u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{25 \times 7 \times 25^{k-1}}, \frac{3}{35 \times 25^{k-1}}, u\right) \\ &= \mathcal{F}_{dq}\left(\frac{3}{7 \times 25^k}, \frac{3}{35 \times 25^{k-1}}, u\right).\end{aligned}$$

Also, $(\frac{3}{5 \times 35 \times 25^{k-1}}, \frac{3}{35 \times 25^{k-1}}) \in \mathcal{R}$ so $\frac{3}{35 \times 25^{k-1}} \succeq S(\frac{3}{35 \times 25^{k-1}})$. As $(\frac{3}{7 \times 25^k}, \frac{3}{5 \times 7 \times 25^k}) \in \mathcal{R}$ so $\frac{3}{7 \times 25^k} \preceq S(\frac{3}{7 \times 25^k})$. Hence, condition (iiia) is satisfied.

$$\begin{aligned}B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o) \cap \mathcal{Y}\mathcal{T}(\epsilon_k) &= \left\{ \frac{3}{7}, \frac{3}{7 \times 5}, \frac{3}{7 \times 5^2}, \frac{3}{7 \times 5^3}, \dots \right\} \\ &= \left\{ \frac{3}{7}, \frac{3}{35}, \frac{3}{7 \times 25}, \frac{3}{35 \times 25}, \frac{3}{7 \times 25^2}, \frac{3}{35 \times 25^2}, \dots \right\}.\end{aligned}$$

Now, for $\epsilon, f \notin B_{\mathcal{F}_{dq}}(\epsilon_o, r, u_o) \cap \mathcal{Y}\mathcal{T}(\epsilon_k)$. Let $\epsilon = 5, f = 6$ and $u = \frac{1}{2}$.

$$\begin{aligned}H_{dq}(\mathcal{T}\epsilon, \mathcal{T}f, u) &= H_{dq}\left(\mathcal{T}(5), \mathcal{T}(6), \frac{1}{2}\right) \\ &= H_{dq}\left([15, 30], [18, 36], \frac{1}{2}\right) \\ &= \min\left\{\mathcal{F}_{dq}\left(15, 36, \frac{1}{2}\right), \mathcal{F}_{dq}\left(30, 18, \frac{1}{2}\right)\right\} \\ &= \min\left\{\frac{1}{175}, \frac{1}{133}\right\} = 0.005714,\end{aligned}$$

$$\begin{aligned}H_{dq}(\mathcal{T}f, \mathcal{T}\epsilon, u) &= H_{dq}\left(\mathcal{T}(6), \mathcal{T}(5), \frac{1}{2}\right) \\ &= \min\{\mathcal{F}_{dq}\left(18, 30, \frac{1}{2}\right), \mathcal{F}_{dq}\left(36, 15, \frac{1}{2}\right)\} \\ &= \min\left\{\frac{1}{181}, \frac{1}{151}\right\} = 0.00552486.\end{aligned}$$

$$\min\{H_{dq}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{dq}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} = \min\{0.005714, 0.00552486\} = 0.00552486.$$

$$\begin{aligned}\mu(D(\epsilon, f, u)) &= \mu\left\{\mathcal{F}_{dq}\left(5, 6, \frac{1}{2}\right), \mathcal{F}_{dq}\left(5, 30, \frac{1}{2}\right), \mathcal{F}_{dq}\left(6, 36, \frac{1}{2}\right)\right\} \\ &= \mu\left\{\frac{1}{35}, \frac{1}{131}, \frac{1}{157}\right\} = \mu\left\{\frac{1}{157}\right\} = 0.079749,\end{aligned}$$

$$\min\{H_{d_q}(\mathcal{T}\mathbf{e}, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\mathbf{e}, u)\} \not\geq \mu(D(\mathbf{e}, f, u)).$$

Hence, contraction does not hold on the whole space \mathcal{Y} . Now, for $\mathbf{e}, f \in B_{\mathcal{F}_{d_q}}(\mathbf{e}_o, r, u_o) \cap \mathcal{Y}\mathcal{T}(\mathbf{e}_k)$ with $\mathbf{e} \succeq S\mathbf{e}$ and $f \preceq Sf$, $\mathbf{e} \in B$ and $f \in A$. In general, for some $k, m \in$.

Case (i): Let $k \leq m$, $\mathbf{e} = \frac{3}{35 \times 25^{m-1}}, f = \frac{3}{7 \times 25^{k-1}} u > 0$. We have

$$\begin{aligned} H_{d_q}(\mathcal{T}\mathbf{e}, \mathcal{T}f, u) &= H_{d_q}\left(\mathcal{T}\left(\frac{3}{35 \times 25^{m-1}}\right), \mathcal{T}\left(\frac{3}{7 \times 25^{k-1}}\right), u\right) \\ &= H_{d_q}\left(\left[\frac{3}{7 \times 35 \times 25^{m-1}}, \frac{3}{5 \times 35 \times 25^{m-1}}\right], \right. \\ &\quad \left.\left[\frac{3}{7 \times 7 \times 25^{k-1}}, \frac{3}{5 \times 7 \times 25^{k-1}}\right], u\right) \\ &= \min\left\{\mathcal{F}_{d_q}\left(\frac{3}{7 \times 35 \times 25^{m-1}}, \frac{3}{5 \times 7 \times 25^{k-1}}, u\right), \right. \\ &\quad \left.\mathcal{F}_{d_q}\left(\frac{3}{5 \times 35 \times 25^{m-1}}, \frac{3}{7 \times 7 \times 25^{k-1}}, u\right)\right\}, \\ H_{d_q}(\mathcal{T}\mathbf{e}, \mathcal{T}f, u) &= \min\left\{\frac{u}{u + \frac{3}{7 \times 35 \times 25^{m-1}} + \frac{6}{35 \times 25^{k-1}}}, \frac{u}{u + \frac{3}{5 \times 35 \times 25^{m-1}} + \frac{6}{7 \times 7 \times 25^{k-1}}}\right\}, \quad (2.31) \\ H_{d_q}(\mathcal{T}\mathbf{e}, \mathcal{T}f, u) &= \min\left\{\frac{245 \times 25^{m-1} u}{245 \times 25^{m-1} u + 3 + 42 \times 25^{m-k}}, \right. \\ &\quad \left.\frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 21 + 150 \times 25^{m-k}}\right\} \\ &= \min\left\{\frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 3(5 + 70 \times 25^{m-k})}, \right. \\ &\quad \left.\frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 3(7 + 50 \times 25^{m-k})}\right\} \\ &= \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 15 + 210 \times 25^{m-k}} \\ &= \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 15(1 + 14 \times 25^{m-k})}. \end{aligned}$$

Now,

$$\begin{aligned} H_{d_q}(\mathcal{T}f, \mathcal{T}\mathbf{e}, u) &= H_{d_q}\left(\mathcal{T}\left(\frac{3}{7 \times 25^{k-1}}\right), \mathcal{T}\left(\frac{3}{35 \times 25^{m-1}}\right), u\right) \\ &= \min\left\{\mathcal{F}_{d_q}\left(\frac{3}{49 \times 25^{k-1}}, \frac{3}{175 \times 25^{m-1}}, u\right), \right. \\ &\quad \left.\mathcal{F}_{d_q}\left(\frac{3}{35 \times 25^{k-1}}, \frac{3}{245 \times 25^{m-1}}, u\right)\right\}, \\ H_{d_q}(\mathcal{T}f, \mathcal{T}\mathbf{e}, u) &= \min\left\{\frac{u}{u + \frac{3}{49 \times 25^{k-1}} + \frac{6}{175 \times 25^{m-1}}}, \frac{u}{u + \frac{3}{35 \times 25^{k-1}} + \frac{6}{245 \times 25^{m-1}}}\right\}, \quad (2.32) \end{aligned}$$

$$\begin{aligned}
H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u) &= \min \left\{ \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 3(25 \times 25^{m-k} + 14)}, \right. \\
&\quad \left. \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 3(35 \times 25^{m-k} + 10)} \right\} \\
&= \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 105 \times 25^{m-k} + 30} \\
&= \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 15(2 + 7 \times 25^{m-k})}.
\end{aligned}$$

Also,

$$\begin{aligned}
\min \{H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} &= \min \left\{ \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 15(1 + 14 \times 25^{m-k})}, \right. \\
&\quad \left. \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 15(2 + 7 \times 25^{m-k})} \right\} \\
&= \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 15 + 210 \times 25^{m-k}}.
\end{aligned}$$

Now, for $\epsilon \succeq S\epsilon, f \preceq Sf$, we have

$$\begin{aligned}
D(\epsilon, f, u) &= \min \left\{ \mathcal{F}_{d_q} \left(\frac{3}{35 \times 25^{m-1}}, \frac{3}{7 \times 25^{k-1}}, u \right), \mathcal{F}_{d_q} \left(\frac{3}{35 \times 25^{m-1}}, \right. \right. \\
&\quad \left. \left[\frac{3}{7 \times 35 \times 25^{m-1}}, \frac{3}{5 \times 35 \times 25^{m-1}} \right], u \right), \\
&\quad \mathcal{F}_{d_q} \left(\frac{3}{7 \times 25^{k-1}}, \left[\frac{3}{7 \times 7 \times 25^{k-1}}, \frac{3}{5 \times 7 \times 25^{k-1}} \right], u \right) \Big\}, \\
D(\epsilon, f, u) &= \min \left\{ \mathcal{F}_{d_q} \left(\frac{3}{35 \times 25^{m-1}}, \frac{3}{7 \times 25^{k-1}}, u \right), \mathcal{F}_{d_q} \left(\frac{3}{35 \times 25^{m-1}}, \frac{3}{175 \times 25^{m-1}}, u \right), \right. \\
&\quad \left. \mathcal{F}_{d_q} \left(\frac{3}{7 \times 25^{k-1}}, \frac{3}{35 \times 25^{k-1}}, u \right) \right\}, \\
D(\epsilon, f, u) &= \min \left\{ \frac{35 \times 25^{m-1}u}{35 \times 25^{m-1}u + 3 + 30 \times 25^{m-k}}, \frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 21}, \right. \\
&\quad \left. \frac{35 \times 25^{k-1}u}{35 \times 25^{k-1}u + 21} \right\}, \\
&= \frac{35 \times 25^{m-1}u}{35 \times 25^{m-1}u + 3 + 30 \times 25^{m-k}}, \\
\frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 15 + 210 \times 25^{m-k}} &\geq \sqrt{\frac{35 \times 25^{m-1}u}{35 \times 25^{m-1}u + 3 + 30 \times 25^{m-k}}}, \\
\min \{H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} &\geq \mu(D(\epsilon, f, u)). \tag{2.33}
\end{aligned}$$

Case (ii): For $k > m$, $\epsilon = \frac{3}{35 \times 25^{m-1}}, f = \frac{3}{7 \times 25^{k-1}}$ and $u > 0$. From (2.31), we have

$$\begin{aligned}
H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u) &= \min \left\{ \frac{u}{u + \frac{3}{7 \times 35 \times 25^{m-1}} + \frac{6}{5 \times 7 \times 25^{k-1}}}, \right. \\
&\quad \left. \frac{u}{u + \frac{3}{5 \times 35 \times 25^{m-1}} + \frac{6}{7 \times 7 \times 25^{k-1}}} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \min \left\{ \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 15 \times 25^{k-m} + 210}, \right. \\
&\quad \left. \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 21 \times 25^{k-m} + 150} \right\} \\
&= \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 21 \times 25^{k-m} + 150}.
\end{aligned}$$

From (2.32)

$$\begin{aligned}
H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u) &= \min \left\{ \frac{u}{u + \frac{3}{7 \times 7 \times 25^{k-1}} + \frac{6}{5 \times 35 \times 25^{m-1}}}, \frac{u}{u + \frac{3}{5 \times 7 \times 25^{k-1}} + \frac{6}{7 \times 35 \times 25^{m-1}}} \right\} \\
&= \min \left\{ \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 75 + 42 \times 25^{k-m}}, \right. \\
&\quad \left. \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 105 + 30 \times 25^{k-m}} \right\} \\
&= \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 75 + 42 \times 25^{k-m}}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\min \{ H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u) \} \\
&= \min \left\{ \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 21 \times 25^{k-m} + 150}, \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 75 + 42 \times 25^{k-m}} \right\} \\
&= \frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 75 + 42 \times 25^{k-m}}.
\end{aligned}$$

Now, from (2.33), we have

$$\begin{aligned}
D(\epsilon, f, u) &= \min \left\{ \mathcal{F}_{d_q} \left(\frac{3}{35 \times 25^{m-1}}, \frac{3}{7 \times 25^{k-1}}, u \right), \right. \\
&\quad \mathcal{F}_{d_q} \left(\frac{3}{35 \times 25^{m-1}}, \frac{3}{5 \times 35 \times 25^{m-1}}, u \right), \\
&\quad \left. \mathcal{F}_{d_q} \left(\frac{3}{7 \times 25^{k-1}}, \frac{3}{5 \times 7 \times 25^{k-1}}, u \right) \right\} \\
&= \min \left\{ \frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 3 \times 25^{k-m} + 30}, \frac{175 \times 25^{m-1} u}{175 \times 25^{m-1} u + 21}, \right. \\
&\quad \left. \frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 21} \right\}.
\end{aligned}$$

Assume that

$$D(\epsilon, f, u) = \frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 3 \times 25^{k-m} + 30}.$$

Then,

$$\frac{1225 \times 25^{k-1} u}{1225 \times 25^{k-1} u + 75 + 42 \times 25^{k-m}} \geq \sqrt{\frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 3 \times 25^{k-m} + 30}},$$

$$\min\{H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} \geq \mu(D(\epsilon, f, u)).$$

Assume that

$$D(\epsilon, f, u) = \frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 21}.$$

Then,

$$\frac{1225 \times 25^{k-1}u}{1225 \times 25^{k-1}u + 75 + 42 \times 25^{k-m}} \geq \sqrt{\frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 21}},$$

$$\min\{H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} \geq \mu(D(\epsilon, f, u)).$$

Case (iii): For $\epsilon = 0, f = \frac{3}{7 \times 25^{k-1}}, u > 0$

$$\begin{aligned} H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u) &= H_{d_q}\left(\mathcal{T}(0), \mathcal{T}\left(\frac{3}{7 \times 25^{k-1}}\right), u\right) \\ &= \min\left\{\mathcal{F}_{d_q}\left(0, \frac{3}{5 \times 7 \times 25^{k-1}}, u\right), \mathcal{F}_{d_q}\left(0, \frac{3}{7 \times 7 \times 25^{k-1}}, u\right)\right\} \\ &= \min\left\{\frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 42}, \frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 30}\right\} \\ &= \frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 42}. \end{aligned}$$

Also,

$$\begin{aligned} H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u) &= H_{d_q}\left(\mathcal{T}\left(\frac{3}{7 \times 25^{k-1}}\right), \mathcal{T}(0), u\right) \\ &= H_{d_q}\left\{\left[\frac{3}{7 \times 7 \times 25^{k-1}}, \frac{3}{5 \times 7 \times 25^{k-1}}\right], [0, 0], u\right\} \\ &= \min\left\{\mathcal{F}_{d_q}\left(\frac{3}{7 \times 7 \times 25^{k-1}}, 0, u\right), \mathcal{F}_{d_q}\left(\frac{3}{5 \times 7 \times 25^{k-1}}, 0, u\right)\right\} \\ &= \min\left\{\frac{49 \times 25^{k-1}u}{49 \times 25^{k-1}u + 3}, \frac{35 \times 25^{k-1}u}{35 \times 25^{k-1}u + 3}\right\} \\ &= \min\left\{\frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 15}, \frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 21}\right\} \\ &= \frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 21}. \end{aligned}$$

Now, we simplify the left-hand side of inequality (i).

$$\begin{aligned} \min\{H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} &= \min\left\{\frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 42}, \frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 21}\right\} \\ &= \frac{245 \times 25^{k-1}u}{245 \times 25^{k-1}u + 42}. \end{aligned}$$

Also,

$$\begin{aligned}
 D(\epsilon, f, u) &= \min \left\{ \mathcal{F}_{d_q} \left(0, \frac{3}{7 \times 25^{k-1}}, u \right), \mathcal{F}_{d_q} (0, 0, u), \right. \\
 &\quad \left. \mathcal{F}_{d_q} \left(\frac{3}{7 \times 25^{k-1}}, \frac{3}{5 \times 7 \times 25^{k-1}}, u \right) \right\} \\
 &= \min \left\{ \frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 30}, 1, \frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 21} \right\} \\
 &= \frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 30}.
 \end{aligned}$$

Hence,

$$\frac{245 \times 25^{k-1} u}{245 \times 25^{k-1} u + 42} \geq \mu \left(\frac{35 \times 25^{k-1} u}{35 \times 25^{k-1} u + 30} \right).$$

Case (iv): $\epsilon = \frac{3}{35 \times 25^{m-1}}$, $f = 0$

$$\begin{aligned}
 H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u) &= H_{d_q} \left(\mathcal{T} \left(\frac{3}{35 \times 25^{m-1}} \right), \mathcal{T}(0), u \right) \\
 &= H_{d_q} \left(\left[\frac{3}{7 \times 35 \times 25^{m-1}}, \frac{3}{5 \times 35 \times 25^{m-1}} \right], [0, 0], u \right) \\
 &= \min \left\{ \mathcal{F}_{d_q} \left(\frac{3}{7 \times 35 \times 25^{m-1}}, 0, u \right), \mathcal{F}_{d_q} \left(\frac{3}{5 \times 35 \times 25^{m-1}}, 0, u \right) \right\} \\
 &= \min \left\{ \frac{245 \times 25^{m-1} u}{245 \times 25^{m-1} u + 3}, \frac{175 \times 25^{m-1} u}{175 \times 25^{m-1} u + 3} \right\} \\
 &= \min \left\{ \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 15}, \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 21} \right\} \\
 &= \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 21},
 \end{aligned}$$

$$\begin{aligned}
 H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u) &= H_{d_q} \left(\mathcal{T}(0), \mathcal{T} \left(\frac{3}{35 \times 25^{m-1}} \right), u \right) \\
 &= H_{d_q} \left([0, 0], \left[\frac{3}{7 \times 35 \times 25^{m-1}}, \frac{3}{5 \times 35 \times 25^{m-1}} \right], u \right) \\
 &= \min \left\{ \mathcal{F}_{d_q} \left(0, \frac{3}{5 \times 35 \times 25^{m-1}}, u \right), \mathcal{F}_{d_q} \left(0, \frac{3}{7 \times 35 \times 25^{m-1}}, u \right) \right\} \\
 &= \min \left\{ \frac{175 \times 25^{m-1} u}{175 \times 25^{m-1} u + 6}, \frac{245 \times 25^{m-1} u}{245 \times 25^{m-1} u + 6} \right\} \\
 &= \min \left\{ \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 42}, \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 30} \right\} \\
 &= \frac{1225 \times 25^{m-1} u}{1225 \times 25^{m-1} u + 42}.
 \end{aligned}$$

Now, we simplify the left-hand side of inequality (i).

$$\begin{aligned} & \min\{H_{d_q}(\mathcal{T}\epsilon, \mathcal{T}f, u), H_{d_q}(\mathcal{T}f, \mathcal{T}\epsilon, u)\} \\ &= \min\left\{\frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 21}, \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 42}\right\} \\ &= \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 42}. \end{aligned}$$

Also,

$$\begin{aligned} D(\epsilon, f, u) &= \min\left\{\mathcal{F}_{d_q}\left(\frac{3}{35 \times 25^{m-1}}, 0, u\right), \mathcal{F}_{d_q}\left(\frac{3}{35 \times 25^{m-1}}, \frac{3}{5 \times 35 \times 25^{m-1}}, u\right), \right. \\ &\quad \left.\mathcal{F}_{d_q}(0, 0, u)\right\} \\ &= \min\left\{\frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 15}, \frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 21}, 1\right\} \\ &= \frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 21}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1225 \times 25^{m-1}u}{1225 \times 25^{m-1}u + 42} &\geq \sqrt{\frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 21}} \\ &= \mu\left(\frac{175 \times 25^{m-1}u}{175 \times 25^{m-1}u + 21}\right). \end{aligned}$$

Case (v) The contraction is trivially held for $\epsilon = 0$ and $f = 0$. Now,

$$\begin{aligned} & \bigcirc_{p=0}^k \min\left\{\mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_1, \epsilon_o, \frac{1}{2^{p+1}}\right)\right), \mu^p\left(\mathcal{F}_{d_q}\left(\epsilon_o, \epsilon_1, \frac{1}{2^{p+1}}\right)\right)\right\} \\ &= \bigcirc_{p=0}^k \min\left\{\mu^p\left(\mathcal{F}_{d_q}\left(\frac{3}{35}, \frac{3}{7}, \frac{1}{2^{p+1}}\right)\right), \mu^p\left(\mathcal{F}_{d_q}\left(\frac{3}{7}, \frac{3}{35}, \frac{1}{2^{p+1}}\right)\right)\right\} \\ &= \bigcirc_{p=0}^k \min\left\{\mu^p\left(\frac{\frac{1}{2^{p+1}}}{\frac{1}{2^{p+1}} + \frac{3}{35} + \frac{6}{7}}\right), \mu^p\left(\frac{\frac{1}{2^{p+1}}}{\frac{1}{2^{p+1}} + \frac{3}{7} + \frac{6}{35}}\right)\right\} \\ &= \bigcirc_{p=0}^k \min\left\{\mu^p\left(\frac{35}{35 + 3 \times 2^{p+1} + 30 \times 2^{p+1}}\right), \mu^p\left(\frac{35}{35 + 15 \times 2^{p+1} + 6 \times 2^{p+1}}\right)\right\} \\ &= \min\left\{\left(\frac{35}{101}\right), \left(\frac{35}{77}\right)\right\} \bigcirc \min\left\{\mu\left(\frac{35}{167}\right), \mu\left(\frac{35}{119}\right)\right\} \bigcirc \left\{\mu^2\left(\frac{35}{299}\right), \mu^2\left(\frac{35}{203}\right)\right\} \\ &\quad \bigcirc \cdots \bigcirc \min\left\{\mu^k\left(\frac{35}{35 + 3 \times 2^{k+1} + 30 \times 2^{k+1}}\right), \mu^k\left(\frac{35}{35 + 15 \times 2^{k+1} + 6 \times 2^{k+1}}\right)\right\} \\ &= \frac{35}{101} \bigcirc \mu\left(\frac{35}{167}\right) \bigcirc \mu^2\left(\frac{35}{299}\right) \bigcirc \cdots \bigcirc \mu^k\left(\frac{35}{35 + 3 \times 2^{k+1} + 30 \times 2^{k+1}}\right) = \frac{35}{101} \\ &> \frac{1}{4} = 1 - \frac{3}{4} = 1 - r. \end{aligned}$$

Hence, all the constraints of Theorem 2.1 are satisfied. Hence, S and \mathcal{T} have a common fixed point and it is 0.

Remark 2.3 By taking six proper subsets of $D(\epsilon, f, u)$ instead of $D(\epsilon, f, u)$, we can obtain six new theorems as corollaries of Theorem 2.1.

Remark 2.4 Fixed-point results in right K -sequentially quasimetric spaces can be obtained in a similar way.

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