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Solving nonlinear and dynamic programming equations on extended b-metric spaces with the fixed-point technique

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In memory of Prof. Fernando Lobo Pereira

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Abstract

In this article, we present an approach to solve a wide range of nonlinear equations formulated in extended b-metric spaces based on a new fixed-point theorem on these spaces. This research effort was motivated by challenges arising in solving pattern problems efficiently that can not be addressed by using standard metric spaces. Our approach relies on a novel common fixed-point theorem for Ćirić-type operators on extended b-metric spaces requiring only very weak assumptions that we present and derive in this article. The proposed approach is illustrated by applications asserting the existence and uniqueness of the solutions to Bellman equations, Volterra integral equations, and fractional differential equations formulated in extended b-metric spaces. Moreover, the obtained results provide general constructive recursive procedures to solve the above types of nonlinear equations.

Keywords: Single-valued mappings; Extended b-metric spaces; Common fixed point; Nonlinear integral equations; Fractional differential equations system; Bellman's equation

1 Introduction

This article concerns generalized contractive mappings of Ćirić type in extended b-metric spaces, and, our main result concerns the existence of a common fixed point in this context. The relevance of this result is that it can be applied to address practical problems whose formulation is not possible in the usual metric spaces and may even involve discontinuous operators.

One important general class of problems consists in, for example, reinforcement learning iterative schemes for optimal control problems with state constraints for which the value function is merely lower semicontinuous. This has been addressed in [7], extending the applicability of well-known dynamic programming results for optimal control [9–12], in which fixed-point theory-based methods were used. Given the wide range of applications, there has been an intense research effort in extending the well-known Banach

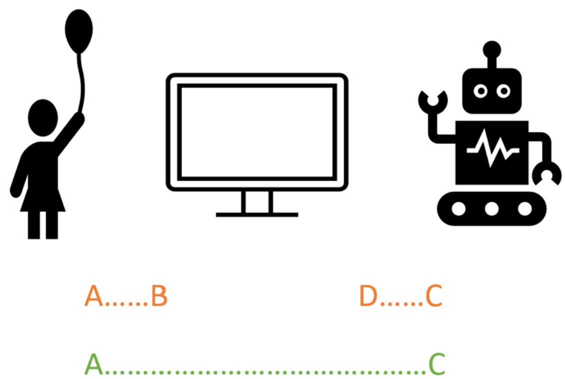

Figure 1 Matching pattern for which the triangle inequality does not hold but the relaxed triangle inequality does

contraction principle to various generalized metric spaces and classes of contractive operators, [1, 3, 14, 16, 23, 41, 47].

A significant application challenge motivating the research reported in this article concerns the problem of optimally managing the storage and retrieval of data, notably, digital images from databases by using the QBIC ("Query by Image Content") system, [37]. The key difficulty is to define a mathematical framework that enables the optimization of the performance of this class of systems. A comparison of methods on how well the human perceptual differences are matched is provided in [45], and, in [19], a nonlinear elastic matching (NEM) distance measure is introduced. However, this measure does not satisfy either the triangle inequality, or the relaxed triangle inequality property. In order to address this difficulty, a revised version of NEM, designated by NEM_r, was designed by [38], and it is currently widely used in the QBIC systems. Essentially, NEM_r consists in stretching r times the NEM distance, and adding it to the distance itself so that the two boundaries are matched. Of crucial relevance, it satisfies the relaxed triangle inequality for any $r > 0$ and bounded set S. For a clearer understanding, just consider the example depicted in Fig. 1. It is clear that while the distance from the Girl to the TV, and the one from the TV to the Robot are small, the distance from the Girl to the Robot is large, and such that $NEM(Girl, TV) + NEM(TV, Robot) < NEM(Girl, Robot)$.

Thus, it makes sense to apply NEM_r measure that is a weaker form of the NEM distance, and satisfies the relaxed triangle inequality: $NEM_r(Girl, Robot) \leq c(NEM_r(Girl, TV) + NEM_r(TV, Robot))$, for some constant c. Clearly, this is a context for which it is critical to consider a b-metric space, which was further exploited in many works. In [25], the authors computed the form dissimilarity ratio of a given image with respect to dataset samples in the IBM QBIC database system, and devised schemes to retrieve the image with the closest shape. Since then, many works concerned the calculation of the ratio for other dissimilarities emerged (see, e.g., [2, 48]), and were further extended to exploit the relaxed triangle inequality in order to increase the efficiency of algorithms solving the traveling-salesman problem (TSP).

In [24], further extension of the relaxed triangle inequality coefficient to address the entire family of dissimilarities was developed by using supplementary information enclosed in the boundary matching between two shapes, i.e., in addition to the NEM_r distance obtained from the boundary matching. Now, c becomes a function, and the measure used for the distance between shapes in QBIC system turning the space of shapes into an extended b-metric space. Thus, the amount of stretching required is the extra information leading to consider the ratio of the triangle inequality as a function that is weaker than the addressed by NEM_r. This measure is denoted by $NEM_{\sigma(x_i,y_j)}$, where $\sigma : X \times X \to [0, +\infty)$. The practical relevance of this measure is that it can be used even in shapes that have variant sample points enabling us to consider distances between two shapes that do not depend on the starting points on the boundaries of two shapes, and it satisfies the relaxed triangle inequality; that is, for all $x, y, z \in X$,

$$NEM_{\sigma(x,z)} \leq \theta(x,z)(NEM_{\sigma(x,y)} + NEM_{\sigma(y,z)}).$$

This means that, if the function $\theta(x_i, y_i)$ depends on the length of the sequences A, B, C, then the number of sample points varies from shape to shape.

Let us consider Fig. 1 again. If the girl starts moving with random velocity, if someone controls the TV remotely, and the robot is made to move in the same direction with random velocity, then:

1 As the three shapes are approaching and moving away at random speeds from one another, then they do not depend on the starting point in the boundaries of two shapes.
2 The relaxed triangle inequality holds here but the ratio this time is a function $\theta(\cdot,\cdot)$ that depends on the element x, y, z of the sequences A, B, C, respectively, and the velocities of the shapes themselves.
3 The supplementary information required to perform the required extent of stretching is very large compared with the previous one, thus we consider a function $\sigma(\cdot,\cdot)$ instead of r.

Moreover, the consideration of extended b-metric spaces raises technical challenges inherent to the fact that, in general, they are non-Hausdorff spaces where examples of convergent sequences with distinct limits and of compact subsets whose intersection is not compact, can be found. Recall that the statements "convergent sequences have single limits" and "the intersection of compact subsets is compact" are both valid in Hausdorff spaces.

This article is organized as follows. In Section, 2, concepts and results for b-metric, and extended b-metric spaces are presented. Then, in Sect. 3, the main results of this article, the existence of common fixed points for Ćirić operators, are presented and proved. These results will be applied to show the existence of solution to three quite different types of equations. In Sect. 4, Volterra-type integral equations are considered. The existence of a common solution for a system of nonlinear fractional differential equations system is shown in Sect. 5. In Sect. 6, the existence of a solution to a very general Bellman equation is shown. Finally, in Sect. 7 some brief conclusions and prospective research are outlined.

2 Preliminary concepts and results

The well-known Banach contraction principle, states that if (X, d) is a complete metric space, and $T : X \to X$ is a mapping satisfying

$$d\big(T(x), T(y)\big) \leq \sigma d(x, y), \tag{2.1}$$

for some $\sigma \in (0,1)$ and for all $x, y \in X$, then T has a unique fixed point x^*, and the sequence $\{x_n\}$ generated by the iterative process $x_{n+1} = Tx_n$ converges to x^* for some $x^* \in X$.

A generalized class of contractive mappings was first introduced in the context of metric spaces by Ćirić in [18]. A self-map $T : X \to X$ on a metric space (X, d) is said to be a Ćirić mapping if, for some $\sigma \in (0,1)$, it satisfies the following inequality, for all x, and y in X,

$$d(Tx, Ty) \leq \sigma \max\left\{d(x,y), d(x, Tx), d(y, Ty), \frac{1}{2}\big(d(x, Ty) + d(y, Tx)\big)\right\}. \tag{2.2}$$

To see the relevance of this extension, just consider the following very simple example of a Ćirić contractive mapping that it is not a contraction. Let $T : X \to X$, defined by

$$Tx = \begin{cases} \frac{3x}{5} & \text{for } x \in X_1, \\ \frac{x}{8} & \text{for } x \in X_2, \end{cases} \tag{2.3}$$

where:

$$X_1 = \left\{\frac{m}{n} : m = 0, 1, 3, 9, \ldots; n = 1, 4, \ldots 3k + 1, \ldots\right\},$$

$$X_2 = \left\{\frac{m}{n} : m = 1, 3, 9, 27, \ldots; n = 2, 5, \ldots, 3k + 2, \ldots\right\}$$

and $X = X_1 \cup X_2$. The mapping T is Ćirić with $\sigma = \frac{3}{5}$. Indeed, if both x, and y are in X_1 or in X_2. Then,

if $x > \frac{5}{24}y$, then $d(Tx, Ty) = \frac{3}{5}|x - \frac{5}{24}y| \leq \frac{3}{5}|x - \frac{1}{8}y| = \frac{3}{5}d(x, Ty)$, and
if $x > \frac{5}{24}y$, then $d(Tx, Ty) = \frac{3}{5}|\frac{5}{24}y - x| \leq \frac{3}{5}|y - x| = \frac{3}{5}d(x, y)$.

Therefore, T satisfies the condition:

$$d(Tx, Ty) \leq \frac{3}{5}\max\big\{d(x, y), d(x, Ty), d(y, Tx)\big\}$$

and, hence, (2.2).

To show that T is not a Banach contraction on X, we just produce a counter example: Let $x = 1$, and $y = \frac{1}{2}$. Then, we have $d(x, y) = |1 - \frac{1}{2}|$, and also $d(Tx, Ty) = \frac{43}{80}$. Thus, we obtain $d(Tx, Ty) = \frac{43}{80} > \sigma\frac{1}{2}$, for any $\sigma \in (0, 1)$. Hence, the Banach contraction is obviously not satisfied.

A Ćirić mapping does not need to be continuous in general, but it is always continuous at a fixed point.

The notion of b-metric was introduced in [6, 20] to address problems formulated in spaces whose associated notion of metric requires a relaxed version of the triangle inequality. In these, and other articles (see, for example, [6, 17, 21, 27, 33, 36]), fixed-point theorems have been proved, and applications have been considered.

Let us define some notation, and recall definitions that will play a key role in the derivation of our results.

Definition 2.1 ([6, 21]) Let X be a nonempty set, and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied:

1 $d(x,y) = 0$ if and only if $x = y$;
2 $d(x,y) = d(y,x)$;
3 $d(x,z) \leq s[d(x,y) + d(y,z)]$;
for all $x,y,z \in X$. A pair (X,d) is called a b-metric space.

Example 2.2 ([43]) Let (X,d) be a metric space, and $\rho(x,y) = (d(x,y))^p$, where $p \geq 1$ is a real number. Then, (X,ρ) is a b-metric space with $s = 2^{p-1}$.

It is clear that a b-metric space becomes a metric space if we take $s = 1$. This shows clearly that the class of b-metric spaces is larger than that of metric spaces.

In [28], Kamran and coauthors introduced the concept of an extended b-metric space that generalizes the concept of a b-metric space. Later, Kiran et. al. discussed in [32] few problems for the case of multivalued operators in extended b-metric space.

Definition 2.3 Let X be a nonempty, and $\theta : X \times X \to [1,+\infty)$. A function $b_\theta : X \times X \to [0,+\infty)$ is an extended b-metric if, for all $x,y,z \in X$, it satisfies:
1) $b_\theta(x,y) = 0$ if and only if $x = y$;
2) $b_\theta(x,y) = b_\theta(y,x)$;
3) $b_\theta(x,z) \leq \theta(x,z)[b_\theta(x,y) + b_\theta(y,z)]$.
The pair (X,b_θ) is called an extended b-metric space.

Denote the open ball (the closed ball, respectively) of radius $r > 0$ about x as the set:

$$B_r(x) = \{x \in X : b_\theta(x,y) < r\}, \quad \bigl(B_r[x] = \{x \in X : b_\theta(x,y) \leq r\}\bigr).$$

Remark 2.4 If $\theta(x,y) = s$ for $s \geq 1$, then (X,b_θ) satisfies the definition of a b-metric space.

Example 2.5 Let $X = [0,+\infty)$, and mappings b_θ and θ with $\theta : X \times X \to [1,+\infty)$, defined by $b_\theta(x,y) = (x-y)^2$ and $\theta(x,y) = x + y + 2$. Then, (X,b_θ) is an extended b-metric space.

Example 2.6 Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real-valued functions defined on $[a,b]$. Let $b_\theta(x,y) = \sup_{t\in[a,b]}\{|x(t) - y(t)|^2\}$, and $\theta : X \times X \to [1,+\infty)$ defined by $\theta(x,y) := |x(t)| + |y(t)| + 2$, then (X,b_θ) is a complete extended b-metric space.

Definition 2.7 Let (X,b_θ) be an extended b-metric space.

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to $x \in X$ if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$b_\theta(x_n,x) < \varepsilon$$

for all $n \geq N$. Alternatively, we may write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is Cauchy, if for every $\epsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$b_\theta(x_m,x_n) < \varepsilon,$$

for all $m,n \geq N$.

Definition 2.8 An extended b-metric space (X,b_θ) is complete if every Cauchy sequence in X is convergent.

An extended b-metric space is not necessarily Hausdorff. Indeed, let us consider the following example:

Example 2.9 Let $X = [0,+\infty)$, $\theta : X \times X \to [1,+\infty)$, $b_\theta(x,y) = (x-y)^2$, and $\theta(x,y) = x+y+2$. Clearly, (X, b_θ) is an extended b-metric space. Moreover, it is not Hausdorff since there does not exist $r, r' > 0$ such that:

$$B_r(x) \cap B_{r'}(y) = \emptyset.$$

Indeed, we have $B_r(x) \subset [0, \sqrt{r} + |z|)$, and $B_{r'}(x) \subset [0, \sqrt{r'} + |z'|)$. Thus,

$$B_r(x) \cap B_{r'}(y) \neq \emptyset \quad \forall z, z' \in X.$$

3 Common fixed points for Ćirić-type operators

First, let us recall some lemmas given in [36] for the case of extended b-metric spaces, which will be useful in proving our first common fixed-point result.

Lemma 3.1 ([36]) *For every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements from an extended b-metric space (X, b_θ), the inequality*

$$b_\theta(x_0, x_k) \leq \sum_{i=0}^{k-1} b_\theta(x_i, x_{i+1}) \prod_{l=0}^{i} \theta(x_l, x_k)$$

holds for every $k \in \mathbb{N}$.

Lemma 3.2 ([36]) *Every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements from an extended b-metric space (X, b_θ), satisfying the property*

$$\exists \gamma \in [0,1) \text{ such that } b_\theta(x_{n+1}, x_n) \leq \gamma b_\theta(x_n, x_{n-1}) \text{ for every } n \in \mathbb{N},$$

is a Cauchy sequence.

Now, we present the main result of our paper, a common fixed-point result for Ćirić-type operators in the case of extended b-metric spaces.

Theorem 3.3 *Let (X, b_θ) be a complete extended b-metric space such that b_θ is continuous and let $F_1, F_2 : X \to X$ be two self-operators such that*

$$\begin{aligned} & b_\theta(F_1x, F_2y) \\ & \quad \leq \gamma \max\left\{ b_\theta(x,y), b_\theta(x, F_1x), b_\theta(y, F_2y), \frac{1}{2}\big(b_\theta(x, F_2y) + b_\theta(y, F_1x)\big) \right\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$, where $0 < \gamma < 1$, is such that, for each $x_0 \in X$, and any convergent sequence $\{x_n\}$,

$$\gamma \lim_{n,m\to+\infty} \theta(x_n, x_m) < 1.$$

Then, the operators F_1, and F_2 have a unique common fixed point.

Remark 3.4 This result is very general since it applies to a very large class of operators as continuity is not required. Examples of such operators appear, for example, in optimal control problem with state constraints for which the value function is only lower semi-continuous.

Remark 3.5 This result may, with a simple recursive argument, be easily generalized for any finite set of operators $\{F_i : i = 1\dots,N\}$, with $N \in \mathbb{N}$ of operators.

The proof is organized in two stages. In the first stage we assume that the operators F_1, and F_2 are continuous, while, in a second stage, we lift this assumption. Given the interest in its own right of the result in the first stage, we explicitly formalize it in this article.

Theorem 3.6 *Let the space* (X, b_θ), *and the operators* F_1, *and* F_2 *be as in Theorem* 3.3. *Moreover, assume that* F_1 *and* F_2 *are continuous.*

Then, the operators F_1, *and* F_2 *have a unique common fixed point.*

Proof Let x_0 be an arbitrary point in X, and define a sequence $\{x_n\}$ as follows

$$x_{2n+1} = F_1x_{2n}, \quad \text{and} \quad x_{2n+2} = F_2x_{2n+1}, \quad n = 0,1,2,\dots. \tag{3.2}$$

Then, by (3.1) and (3.2), we obtain

$$\begin{aligned} b_\theta(x_{2n+1},x_{2n+2}) &= b_\theta(F_1x_{2n},F_2x_{2n+1}) \\ &\leq \gamma \max\biggl\{b_\theta(x_{2n},x_{2n+1}), b_\theta(x_{2n},F_1x_{2n}), b_\theta(x_{2n+1},F_2x_{2n+1}), \\ &\quad \frac{1}{2}\bigl(b_\theta(x_{2n},F_2x_{2n+1}) + b_\theta(x_{2n+1},F_1x_{2n})\bigr)\biggr\} \\ &\leq \gamma \max\biggl\{b_\theta(x_{2n},x_{2n+1}), b_\theta(x_{2n},x_{2n+1}), b_\theta(x_{2n+1},x_{2n+2}), \\ &\quad \frac{1}{2}\bigl(b_\theta(x_{2n},x_{2n+2}) + b_\theta(x_{2n+1},x_{2n+1})\bigr)\biggr\} \\ &= \gamma \max\biggl\{b_\theta(x_{2n},x_{2n+1}), b_\theta(x_{2n+1},x_{2n+2}), \frac{1}{2}b_\theta(x_{2n},x_{2n+2})\biggr\}. \end{aligned}$$

The following cases can be considered:

- Case I.

  If $\max\{b_\theta(x_{2n},x_{2n+1}), b_\theta(x_{2n+1},x_{2n+2}), \frac{1}{2}b_\theta(x_{2n},x_{2n+2})\} = b_\theta(x_{2n+1},x_{2n+2})$, then we have

  $$b_\theta(x_{2n+1},x_{2n+2}) \leq \gamma b_\theta(x_{2n+1},x_{2n+2}).$$

  This entails that $\gamma \geq 1$ and, hence, a contradiction.

- *Case II.*

  If $\max\{b_\theta(x_{2n},x_{2n+1}), b_\theta(x_{2n+1},x_{2n+2}), \frac{1}{2}b_\theta(x_{2n},x_{2n+2})\} = b_\theta(x_{2n},x_{2n+1})$, then we have

  $$b_\theta(x_{2n+1},x_{2n+2}) \leq \gamma b_\theta(x_{2n},x_{2n+1}). \tag{3.3}$$

For the next step we have

$$\begin{aligned} & b_\theta(x_{2n+2}, x_{2n+3}) \\ & \quad \leq \gamma \max\left\{ b_\theta(x_{2n+1}, x_{2n+2}), b_\theta(x_{2n+2}, x_{2n+3}), \frac{1}{2} b_\theta(x_{2n+1}, x_{2n+3}) \right\}. \end{aligned}$$

Then, we have to consider the following cases:

– Case IIa.

$$b_\theta(x_{2n+2}, x_{2n+3}) \leq \gamma b_\theta(x_{2n+2}, x_{2n+3}),$$

which implies $\gamma \geq 1$, and, thus, a contradiction.

– Case IIb.

$$b_\theta(x_{2n+2}, x_{2n+3}) \leq \gamma b_\theta(x_{2n+1}, x_{2n+2}). \tag{3.4}$$

Then, from (3.3) and (3.4), for all $n \in \mathbb{N}$, we obtain

$$b_\theta(x_{n+1}, x_{n+2}) \leq \gamma b_\theta(x_n, x_{n+1}).$$

Thus, the conditions of Lemma 3.2 hold for all terms of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and, hence, the generated sequence is Cauchy.

– Case IIc

$$\begin{aligned} & b_\theta(x_{2n+2}, x_{2n+3}) \leq \gamma \frac{1}{2} b_\theta(x_{2n+1}, x_{2n+3}), \\ & \frac{1}{2} b_\theta(x_{2n+1}, x_{2n+3}) \leq \frac{1}{2} \theta(x_{2n+1}, x_{2n+3}) \big(b_\theta(x_{2n+1}, x_{2n+2}) \\ & \qquad + b_\theta(x_{2n+2}, x_{2n+3}) \big). \end{aligned}$$

In this case, we obtain

$$\begin{aligned} b_\theta(x_{2n+2}, x_{2n+3}) \leq {} & \frac{\theta(x_{2n+1}, x_{2n+3})\gamma}{2} \big(b_\theta(x_{2n+1}, x_{2n+2}) \\ & + b_\theta(x_{2n+2}, x_{2n+3}) \big), \end{aligned}$$

and, hence,

$$\begin{aligned} & \left(1 - \frac{\theta(x_{2n+1}, x_{2n+3})\gamma}{2} \right) b_\theta(x_{2n+2}, x_{2n+3}) \\ & \quad \leq \frac{\gamma \theta(x_{2n+1}, x_{2n+3})}{2} b_\theta(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Thus, we conclude that

$$b_\theta(x_{2n+2}, x_{2n+3}) \leq \frac{\gamma \theta(x_{2n+1}, x_{2n+3})}{2 - \gamma \theta(x_{2n+1}, x_{2n+3})} b_\theta(x_{2n+1}, x_{2n+2}). \tag{3.5}$$

Thus, from (3.3) and (3.5), it follows that $b_\theta(x_{n+1},x_{n+2}) \leq \eta b_\theta(x_n,x_{n+1})$, where $\eta := \max\{\frac{\gamma\theta(x_n,x_{n+2})}{2-\gamma\theta(x_n,x_{n+2})},\gamma\}$. Now, we show that there exists $N_\eta \in \mathbb{N}$ such that $\eta = \eta(N_\eta) < 1$, for all $n > N_\eta$.

Since $\gamma \lim_{n,m\to+\infty}\theta(x_n,x_m) < 1$, we have $2-\gamma\lim_{n,m\to\infty}\theta(x_n,x_m) > 1$. From this, it follows that

$$\gamma \lim_{n,m\to+\infty}\theta(x_n,x_m) \leq 2-\gamma\lim_{n,m\to+\infty}\theta(x_n,x_m),$$

and, thus, $\eta < 1$. By applying Lemma 3.2 we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

- Case III.

If $\max\{b_\theta(x_{2n},x_{2n+1}),b_\theta(x_{2n+1},x_{2n+2}),\frac{1}{2}b_\theta(x_{2n},x_{2n+2})\} = \frac{1}{2}b_\theta(x_{2n},x_{2n+2})$, then we have

$$\begin{aligned}\frac{1}{2}b_\theta(x_{2n},x_{2n+2}) &\leq \frac{1}{2}b_\theta(x_{2n},x_{2n+2}) \\ &\leq \frac{1}{2}\theta(x_{2n},x_{2n+2})\big(b_\theta(x_{2n},x_{2n+1})+b_\theta(x_{2n+1},x_{2n+2})\big).\end{aligned} \tag{3.6}$$

In this case, we obtain

$$b_\theta(x_{2n+1},x_{2n+2}) \leq \frac{\theta(x_{2n},x_{2n+2})\gamma}{2}\big(b_\theta(x_{2n},x_{2n+1})+b_\theta(x_{2n+1},x_{2n+2})\big),$$

and, hence,

$$\left(1-\frac{\theta(x_{2n},x_{2n+2})\gamma}{2}\right)b_\theta(x_{2n+1},x_{2n+2}) \leq \frac{\gamma\theta(x_{2n},x_{2n+2})}{2}b_\theta(x_{2n},x_{2n+1}).$$

Thus, we conclude that

$$b_\theta(x_{2n+1},x_{2n+2}) \leq \frac{\gamma\theta(x_{2n},x_{2n+2})}{2-\gamma\theta(x_{2n},x_{2n+2})}b_\theta(x_{2n},x_{2n+1}). \tag{3.7}$$

For the next step, we obtain

$$\begin{aligned}&b_\theta(x_{2n+2},x_{2n+3}) \\ &\quad\leq \gamma\max\left\{b_\theta(x_{2n+1},x_{2n+2}),b_\theta(x_{2n+2},x_{2n+3}),\frac{1}{2}b_\theta(x_{2n+1},x_{2n+3})\right\}.\end{aligned}$$

Then, we have three cases:

– *Case IIIa.*

$$b_\theta(x_{2n+2},x_{2n+3}) \leq \gamma b_\theta(x_{2n+2},x_{2n+3}).$$

This leads to $\gamma \geq 1$ and, thus, a contradiction.

– *Case IIIb.*

$$b_\theta(x_{2n+2},x_{2n+3}) \leq \gamma b_\theta(x_{2n+1},x_{2n+2}). \tag{3.8}$$

Then, by (3.7) and (3.3) it follows that

$$b_\theta(x_{n+1},x_{n+2}) \le \eta(n) b_\theta(x_n,x_{n+1}),$$

where $\eta(n)$ is defined by $\eta(n) := \frac{\gamma\theta(x_n,x_{n+2})}{2-\gamma\theta(x_n,x_{n+2})}$. Thus, the conditions of Lemma 3.2 hold for all terms of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and, hence, the generated sequence is Cauchy.

– *Case IIIc*

$$b_\theta(x_{2n+2},x_{2n+3}) \le \gamma \frac{1}{2} b_\theta(x_{2n+1},x_{2n+3}).$$

After simple calculations we obtain:

$$b_\theta(x_{2n+2},x_{2n+3}) \le \frac{\gamma\theta(x_{2n+1},x_{2n+3})}{2-\gamma\theta(x_{2n+1},x_{2n+3})} b_\theta(x_{2n+1},x_{2n+2}). \tag{3.9}$$

By (3.7) and (3.9), it follows that $b_\theta(x_{n+1},x_{n+2}) \le \eta b_\theta(x_n,x_{n+1})$, where

$$0 < \eta(n) := \frac{\gamma\theta(x_n,x_{n+2})}{2-\gamma\theta(x_n,x_{n+2})} < 1.$$

By applying Lemma 3.2, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

From the above, we have that, for all the three cases, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $b_\theta(x_n,x^*) \to 0$ as $n \to \infty$. Then, it follows that $b_\theta(x_{2n},x^*) \to 0$ as $n \to \infty$.

From the continuity of F_1, we have that $x_{2n+1} = F_1x_{2n} \to F_1x^*$ as $n \to \infty$ and, from the uniqueness of the limit, we conclude that $x^* = F_1x^*$.

At the same time, we have $b_\theta(x_{2n+1},x^*) \to 0$ as $n \to \infty$. From the continuity of F_2, it follows that $x_{2n+2} = F_2x_{2n+1} \to F_2x^*$ as $n \to \infty$ and, from the uniqueness of the limit, we obtain $x^* = F_2x^*$. Thus, we conclude that x^* is a common fixed point of the pair (F_1,F_2).

It remains to show the uniqueness of x^*. Assume that $y^* \in X$ is another common fixed point for the pair (F_1,F_2). Then,

$$\begin{aligned} & b_\theta(x^*,y^*) \\ & \quad = b_\theta(F_1x^*,F_2y^*) \\ & \quad \le \gamma \max\left\{ b_\theta(x^*,y^*), b_\theta(x^*,F_1x^*), b_\theta(y^*,F_2y^*), \frac{1}{2}\big(b_\theta(x^*,F_2y^*) + b_\theta(y^*,F_1x^*)\big)\right\} \\ & \quad \le \gamma \max\left\{ b_\theta(x^*,y^*), b_\theta(x^*,x^*), b_\theta(y^*,y^*), \frac{1}{2}\big(b_\theta(x^*,y^*) + b_\theta(y^*,x^*)\big)\right\} \\ & \quad = \gamma b_\theta(x^*,y^*). \end{aligned}$$

This implies that $x^* = y^*$. The proof is complete. □

Now, by using Theorem 3.6, we proceed to complete the proof of Theorem 3.3 by dropping the continuity assumption of the operators F_1 and F_2.

Proof Let the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ be constructed as in Theorem 3.3. Since X is complete, there exists $x^* \in X$ such that $b_\theta(x_n, x^*) \to 0$ as $n \to \infty$.

Since F_1 and F_2 are not continuous let us assume that $b_\theta(x^*, F_1x^*) = r > 0$. Then, we may write the following estimates

$$\begin{aligned}
r &= b_\theta\bigl(x^*, F_1x^*\bigr) \\
&\le \theta\bigl(x^*, F_1x^*\bigr)\bigl(b_\theta\bigl(x^*, x_{2k+2}\bigr) + b_\theta\bigl(x_{2k+2}, F_1x^*\bigr)\bigr) \\
&\le \theta\bigl(x^*, F_1x^*\bigr)b_\theta\bigl(x^*, x_{2k+2}\bigr) + \theta\bigl(x^*, Sx^*\bigr)b_\theta\bigl(F_2x_{2k+1}, F_1x^*\bigr) \\
&\le \theta\bigl(x^*, F_1x^*\bigr)b_\theta\bigl(x^*, x_{2k+2}\bigr) + \theta\bigl(x^*, F_1x^*\bigr)\gamma \max\biggl\{b_\theta\bigl(x_{2k+1}, x^*\bigr), b_\theta(x_{2k+1}, F_2x_{2k+1}), \\
&\quad b_\theta\bigl(x^*, F_1x^*\bigr), \frac{1}{2}\bigl(b_\theta\bigl(x_{2k+1}, F_1x^*\bigr) + b_\theta\bigl(x^*, F_2x_{2k+1}\bigr)\bigr)\biggr\} \\
&\le \theta\bigl(x^*, F_1x^*\bigr)b_\theta\bigl(x^*, x_{2k+2}\bigr) + \theta\bigl(x^*, F_1x^*\bigr)\gamma \max\biggl\{b_\theta\bigl(x_{2k+1}, x^*\bigr), b_\theta(x_{2k+1}, x_{2k+2}), \\
&\quad b_\theta\bigl(x^*, F_1x^*\bigr), \frac{1}{2}\bigl(b_\theta\bigl(x_{2k+1}, F_1x^*\bigr) + b_\theta\bigl(x^*, x_{2k+2}\bigr)\bigr)\biggr\} \\
&\le \theta\bigl(x^*, F_1x^*\bigr)b_\theta\bigl(x^*, x_{2k+2}\bigr) + \gamma\theta\bigl(x^*, F_1x^*\bigr)b_\theta\bigl(x^*, F_1x^*\bigr) \\
&\le \theta\bigl(x^*, F_1x^*\bigr)b_\theta\bigl(x^*, x_{k+2}\bigr) + \theta\bigl(x^*, F_1x^*\bigr)\gamma r.
\end{aligned}$$

From the last inequality we obtain

$$r \le \theta\bigl(x^*, F_1x^*\bigr)\bigl(b_\theta\bigl(x^*, x_{2k+2}\bigr) + \gamma r\bigr).$$

Since this inequality has to hold for all situations, by considering $\theta(x^*, F_1x^*) = 1$ and $\lim_{k\to\infty} b_\theta(x^*, x_{2k+2}) = 0$, it follows that $\gamma \ge 1$ and, hence, a contradiction. Then, we have $x^* = F_1x^*$.

In the same way, we obtain $x^* = F_2x^*$. Hence, x^* is a common fixed point for the pair (F_1, F_2). For the uniqueness of the common fixed point x^*, we use arguments similar to those in the proof of Theorem 3.3. □

If we take $F_1 = F_2 = F$ we obtain the following generalization of Ćirić operators in an extended b-metric space.

Theorem 3.7 *Let (X, b_θ) be a complete extended b-metric space such that b_θ is continuous and $F : X \to X$ is a continuous mapping satisfying*

$$b_\theta(Fx, Fy) \le \gamma \max\biggl\{b_\theta(x, y), b_\theta(x, Fx), b_\theta(y, Fy), \frac{1}{2}\bigl(b_\theta(x, Fy) + b_\theta(y, Fx)\bigr)\biggr\}, \tag{3.10}$$

for all $x, y \in X$, where $0 < \gamma < 1$, and, for each $x_0 \in X$, $\gamma \lim_{n,m\to+\infty} \theta(x_n, x_m) < 1$.

Then, F has a unique fixed point.

Let us give now an illustrative example for our results.

Example 3.8 Let $X = [0, \infty)$, and define $b_\theta : X \times X \to \mathbb{R}$, and $\theta : X \times X \to [1, \infty)$ by:

$$b_\theta(x, y) := (x - y)^2, \qquad \theta(x, y) := x + y + 1.$$

Then, (X, b_θ) is a complete extended b-metric space.

Define F_1 and $F_2 : X \to X$ by $F_1 x = \frac{x}{2}$, $F_2 x = \frac{x}{4}$. We have

$$b_\theta(F_1 x, F_2 y) = b_\theta\left(\frac{x}{2}, \frac{y}{4}\right) = \left(\frac{x}{2} - \frac{y}{4}\right)^2 = \frac{x^2}{4} + \frac{y^2}{16} - \frac{xy}{4}.$$

Define $M(x, y) := \max\{b_\theta(x, y), b_\theta(x, F_1 x), b_\theta(y, F_2 y), \frac{1}{2}(b_\theta(x, F_2 y) + b_\theta(y, F_1 x))\}$. Since we have $\frac{1}{2}(b_\theta(x, F_2 y) + b_\theta(y, F_1 x)) = \frac{5x^2}{8} + \frac{17y^2}{32} - \frac{3xy}{4}$, we may write

$$\begin{aligned} b_\theta(F_1 x, F_2 y) &= \frac{x^2}{4} + \frac{y^2}{16} - \frac{xy}{4} = \frac{1}{2}\left(\frac{x^2}{2} + \frac{y^2}{8} - \frac{xy}{2}\right) \\ &\leq \frac{1}{2}\left(\frac{1}{2}\big(b_\theta(x, F_2 y) + b_\theta(y, F_1 x)\big)\right) \leq \frac{1}{2} M(x, y). \end{aligned}$$

Since, we also have $\frac{1}{2^{3n}} x_0 = x_{2n}$, and $\frac{1}{2^{3n+1}} x_0 = x_{2n+1}$, we obtain

$$\lim_{n,m \to \infty} \theta(x_n, x_m) = \lim_{n,m \to \infty} \theta\left(\frac{x_0}{2^{3n}} + \frac{x_0}{2^{3m}} + 1\right) = 1.$$

Therefore, all conditions of Theorem 3.3 are satisfied. Hence, 0 is a common fixed point for F_1 and F_2.

4 Existence of a solution for Volterra-type integral equations

The theory of integral equations has an important place in Applied Mathematics. Initiated in the nineteenth century, it underwent a rapid expansion in the last century in whose developments a variety of methods from Fixed-Point Theory, Variational Analysis, Approximation Theory, and Numeric Analysis consolidated their role. Vito Volterra introduced the notion of Volterra integral equations at the end of the nineteenth century, which were further investigated by Traian Lalescu, in 1912, who published the first book about integral equations, [34].

Volterra integral equations have applications in many physical domains such as demography, viscoelastic materials, actuarial sciences, potential theory and Dirichlet problems, electrostatics, mathematical problems of radiative equilibrium, particle-transport problems of astrophysics and reactor theory, radiative heat-transfer problems, [22, 35, 42, 49, 50], among others. Recently, some interesting methods for solving Volterra integral equations have been introduced, for example: the power-series method [46], Adomain's decomposition method [5], the homotopy perturbation method [15, 51], the block by block method [30], and the expansion method [40].

Let us consider the following Volterra-type integral equation

$$x(t) = \int_0^t P\big(t, s, x(s)\big)\, ds + g(t), \quad t \in [0, 1]. \tag{4.1}$$

The purpose of this section is to prove the existence of a solution to the equation (4.1) by applying Theorem 3.3.

Let us define the operator $F : C([0,1],\mathbb{R}^n) \to C([0,1],\mathbb{R}^n)$ as follows:

$$Fx(t) = \int_0^t P\bigl(t,s,x(s)\bigr)\,ds + g(t), \quad t \in [0,1].$$

Theorem 4.1 *Assume that the data of the integral equation* (4.1) *satisfies the following conditions*:

i) $P : [0,1] \times [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [0,1] \to \mathbb{R}^n$ *are continuous*;

ii) $P(t,s,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is increasing for each* t, *and* $s \in [0,1]$;

iii) *For all* t, *and* $s \in [0,1]$, *there exists* $\gamma \in (0,1)$ *such that*

$$P(t,s,u) - P(t,s,v) \le \gamma Q(u,v),$$

where $Q(a,b) = \max\{|a-b|, |a-Fa|, |b-Fb|, \frac{1}{2}(|a-Fb| + |b-Fa|)\}$ *for each* $t,s \in [0,1]$.

Then, *the integral equation* (4.1) *has a unique solution in* $C([0,1],\mathbb{R}^n)$.

Proof Let $X = C([0,1],\mathbb{R}^n)$ be an extended b-metric space endowed with the extended b-metric $b_\theta(x,y) = \|x-y\|_C = \sup_{t\in[0,1]} \|x(t)-y(t)\|^2$. Note that X is a complete extended b-metric space with $\theta : X \times X \to [1,\infty)$ defined by $\theta(x,y) := 2\|x(t)\| + 3\|y(t)\| + 1$.

First, note that from the common inequality $(a-b)^2 \ge 0$, we readily obtain

$$\frac{1}{4}\bigl(a^2+b^2\bigr) \ge \frac{1}{2}ab. \tag{4.2}$$

We shall prove that the operator S satisfies all the conditions of Theorem 3.7. We have the following estimate:

$$\begin{aligned}
\bigl|Fx(t)-Fy(t)\bigr|^2 &\le \int_0^t \bigl|P\bigl(t,s,x(s)\bigr) - P\bigl(t,s,y(s)\bigr)\bigr|^2\,ds \\
&\le \gamma^2 \int_0^t \max\biggl\{\bigl|x(s)-y(s)\bigr|^2, \bigl|x(s)-Fx(s)\bigr|^2, \bigl|y(s)-Fy(s)\bigr|^2, \\
&\quad \frac{1}{4}\bigl[\bigl|x(s)-Fy(s)\bigr| + \bigl|y(s)-Fx(s)\bigr|\bigr]^2\biggr\}\,ds \\
&\le \gamma^2 \int_0^t \max\biggl\{\bigl|x(s)-y(s)\bigr|^2, \bigl|x(s)-Fx(s)\bigr|^2, \bigl|y(s)-Fy(s)\bigr|^2, \\
&\quad \frac{1}{4}\bigl(x(s)-Fy(s)\bigr)^2 + \frac{1}{2}\bigl(\bigl(x(s)-Fy(s)\bigr)\bigl(y(s)-Fx(s)\bigr) \\
&\quad \frac{1}{4}\bigl(y(s)-Fx(s)\bigr)^2\biggr\}\,ds \\
&\le \gamma^2 \int_0^t \max\biggl\{\bigl|x(s)-y(s)\bigr|^2, \bigl|x(s)-Fx(s)\bigr|^2, \bigl|y(s)-Fy(s)\bigr|^2, \\
&\quad \frac{1}{2}\bigl(\bigl|x(s)-Fy(s)\bigr|^2 + \bigl|y(s)-Fx(s)\bigr|^2\bigr)\biggr\}\,ds,
\end{aligned}$$

where the last inequality follows by using (4.2). By denoting $\|x\|_C = \sup_{t\in[0,1]}\{|x(t)|\}$, we have

$$\begin{aligned}
&\left|Fx(t) - Fy(t)\right|^2 \\
&\quad\le \gamma^2 \int_0^t \max\left\{\|x-y\|_C^2, \|x-Fx\|_C^2, \|y-Fy\|_C^2, \frac{1}{2}\|x-Fy\|_C^2 + \|y-Fx\|_C^2\right\} \\
&\quad\le \gamma^2 \max\left(b_\theta(x,y), b_\theta(x,Fx), b_\theta(y,Fy), \frac{1}{2}\big(b_\theta(x,Fy) + b_\theta(y,Fx)\big)\right) \\
&\quad\le \gamma^2 M(x,y).
\end{aligned}$$

For α such that $0 < \alpha = \gamma^2 < 1$, we have

$$b_\theta(Fx,Fy) = \|Fx - Fy\|_C \le \alpha M(x,y)$$

for each $x, y \in X$. Since $\lim_{n,m\to\infty} \theta(x_n, x_m)1$, the conclusion follows from Theorem 3.7. □

Now, we consider a numerical example illustrating the use of our results to compute the solution of a simple instance of a Volterra equation.

Let us consider the following Volterra integral equation

$$x(t) = t + \int_0^t (s-t)x(s)\,ds. \tag{4.3}$$

First, note that, since we have $(s-t)x(s) - (s-t)y(s) \le |t-s||x(s) - y(s)|$ for $t, s \in [0,1)$, then there exist $\alpha \in (0,1)$ satisfying $|t-s||x(s) - y(s)| \le \alpha M(x,y)$. Thus, the Ćirić operator F defined by:

$$Fx(t) = t + \int_0^t (s-t)x(s)\,ds. \tag{4.4}$$

It can be easily checked that $x(t) = \sin(t)$ is the exact solution to Equation (4.3). The iteration method to compute the value of the integral is

$$x_{n+1}(t) = Fx_n(t) = t + \int_0^t (s-t)x_n(s)\,ds = x_n(t) + \prod_{k=2}^{n}\left(\frac{-1}{(2k-1)(2k-2)}\right)t^{2n-1}.$$

In Table 1 we illustrate the approximations of the exact solution of the operator F.

Table 1 For $t = 0.2$ rad, the exact solution is $x(0.2) = 0.198669331$

n	$x_n(0.2)$	Approximate Solution	Absolute Error
0	$x_0(0.2)$	0	$1.986693 * 10^{-1}$
1	$x_1(0.2)$	0.2	$1.330669 * 10^{-3}$
2	$x_2(0.2)$	0.198666667	$2.66413 * 10^{-6}$
3	$x_3(0.2)$	0.198669333	$2.53827 * 10^{-9}$

5 Existence of a common solution for a system of nonlinear fractional differential equations

Fractional differential calculus is a strong tool in the world of mathematics due to the requirements of many real-world applications. In the last century, fractional calculus found its way to address many challenging applications in modeling, control and optimization in a wide variety of domains, comprising, but not limited to, fluid flow, heat transfer, electromagnetism, biology, engineering, and economics. Fractional calculus is still continuously expanding. At the same time, the fixed-point theory is also used to prove the existence and uniqueness of a solution of integral equations, ordinary differential equations, partial differential equations, and functional equations.

In this section, we prove an existence and uniqueness theorem for a nonlinear fractional differential equation system, of the Caputo type, by using our main common fixed-point result, Theorem 3.3.

For a continuous function $g : [0,\infty) \to \mathbb{R}$ we recall the Caputo derivative of order $\beta > 0$ of the functional g as follows (see [31, 44])

$$ {}^{C}D^{\beta}\big(g(t)\big) := \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\, ds \quad (n-1 < \beta < n, n = [\beta]+1), \tag{5.1}$$

where $[\beta]$, Γ denote the integer part of the positive real number and gamma function, respectively.

In this section, we present the application of Theorem 3.6 to prove the existence of at least a common solution for the nonlinear fractional differential equation system

$$\begin{cases} {}^{C}D^{\beta}(x(t)) + f_1(t,x(t)) = 0, \\ {}^{C}D^{\beta}(y(t)) + f_2(t,y(t)) = 0 \end{cases} \tag{5.2}$$

for $0 \leq t \leq 1$, $\beta < 1$, with the boundary conditions

$$\begin{cases} x(0) = 0 = x(1), \\ y(0) = 0 = y(1), \end{cases} \tag{5.3}$$

where $x \in C([0,1],\mathbb{R})$ and $C([0,1],\mathbb{R})$ is the set of all continuous functions from $[0,1]$ to $\mathbb{R}$, $f_1, f_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions (see [39]), and ${}^{C}D^{\beta}$ is the Caputo derivative of order β. Further, we present the Green function associated with the system (5.2) as follows

$$G(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

Let us state the main existence result of this section.

Theorem 5.1 *Given the nonlinear fractional differential equation* (5.2), *and a function* $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *such that the following assumptions hold*:

(*i*) *There exists* $x_0 \in C([0,1],\mathbb{R})$ *such that* $\mu(x_0(t), \int_0^1 F_1 x_0(t)) \geq 0$, *and* $\mu(x_0(t), \int_0^1 F_2 x_0(t)) \geq 0$ *for all* $t \in [0,1]$, *where* $F_1, F_2 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ *are defined* *as follows*:

$$\begin{cases} F_1x = \int_0^1 G(t,s)f_1(s,x(s))\,ds, \\ F_2y = \int_0^1 G(t,s)f_2(s,y(s))\,ds; \end{cases} \tag{5.4}$$

(*ii*) $|f(t,a) - f(t,b)| \leq \frac{1}{\tau}M(a,b)$ $\forall t \in [0,1]$, $\tau > 1$, *and* $a, b \in \mathbb{R}$ *with* $\mu(a,b) \geq 0$, *where*

$$M(a,b) = \max\left\{b_\theta(a,b), b_\theta(a,F_1a), b_\theta(b,F_2b), \frac{1}{2}\big(b_\theta(a,F_2b) + b_\theta(b,F_1a)\big)\right\};$$

(*iii*) *if* $\{x_n\}$ *is a sequence in* $C([0,1],\mathbb{R})$ *such that* $x_n \to x$ *in* $C([0,1],\mathbb{R})$, *and, for each* $t \in [0,1]$, $\mu(x_n(t),x_{n+1}(t)) \geq 0$ *for all* $n \in \mathbb{N}$, *then* $\mu(x_n(t),x(t)) \geq 0$ *for all* $n \in \mathbb{N}$.

Then, the system (5.2) *has at least one common solution.*

Proof Let $X = C([0,1],\mathbb{R})$ be endowed with the Bielecki norm

$$b_\theta(x,y) = \|x\|_B = \sup_{t\in[0,1]}\left\{\big|x(t)\big|e^{-\tau t}\right\} \quad \text{with } \tau > 1$$

and $\theta : X \times X \to [1,\infty)$ is given by $\theta(x,y) = |x(t)| + 2|y(t)| + 1$. It is straightforward to conclude that (X, b_θ) is a complete extended b-metric space.

It is obvious that $x^* \in X$ is a common solution for the system (5.2) if and only if $x^* \in X$ is a common solution of the system (5.3), for all $t \in [0,1]$. Then, the problem (5.2) can be reduced to finding an element $x^* \in X$ that is a common fixed point for the operators F_1 and F_2.

Let x and $y \in X$ such that $\mu(x(t),y(t)) \geq 0$ for all $t \in [0,1]$. By (*i*) and (*ii*) we obtain the following estimate

$$\begin{aligned} \big|F_1x(t) - F_2y(t)\big|^2 &= \left|\int_0^1 G(t,s)\big[f_1\big(t,s,x(s)\big) - f_2\big(t,s,y(s)\big)\big]\,ds\right|^2 \\ &\leq \left(\int_0^1 G(t,s)\,ds\right)^2 \int_0^1 \big|f_1\big(t,s,x(s)\big) - f_2\big(t,s,y(s)\big)\big|^2\,ds \\ &\leq \frac{1}{\tau^2}\big|M(x,y)e^{-\tau t}\big|^2 e^{2\tau t}\left(\int_0^1 G(t,s)\,ds\right)^2. \end{aligned}$$

Then, we obtain

$$\big|\big[F_1x(t) - F_2y(t)\big]e^{-\tau t}\big|^2 \leq \frac{1}{\tau^2}\big|M(x,y)e^{-\tau t}\big|^2\left(\int_0^1 G(t,s)\,ds\right)^2. \tag{5.5}$$

By taking the supremum over time of the above inequality, we obtain

$$\begin{aligned} \left|\sup_{t\in[0,1]}\left\{\big(F_1x(t) - F_2y(t)\big)e^{-\tau t}\right\}\right|^2 &\leq \frac{1}{\tau^2}\sup_{t\in[0,1]}\big|M(x,y)e^{-\tau t}\big|^2 \sup_{t\in[0,1]}\left\{\left(\int_0^1 G(t,s)\,ds\right)^2\right\} \\ &\leq \frac{1}{\tau^2}\sup_{t\in[0,1]}\left\{\big|M(x,y)e^{-\tau t}\big|^2\right\}. \end{aligned}$$

Then, we have

$$\|F_1x - F_2y\|_B \leq \frac{1}{\tau}\big\|M(x,y)\big\|_B. \tag{5.6}$$

Since $\lim_{n,m\to\infty}\theta(x_n,x_m) = 1 < \tau$, for $0 < \gamma = \frac{1}{\tau} < 1$, we may apply Theorem 3.3 that yields the existence of $x^* \in X$ as a common fixed point of the operators F_1 and F_2. Then, the system (5.2) has at least one common solution. □

6 Application to dynamic programming

In this section we investigate the application of t́he Cirić fixed-point theorem to prove the existence and uniqueness results of the solution of the dynamic programming Bellman's equation under assumptions that are significantly weaker than the ones generally considered in the literature, [10]. The relevance of this application is that it allows us to weaken the continuity assumption on the "reward" operator required by the Banach fixed-point theorem, to merely lower semicontinuity.

Our results enable us to further extend the power of an already extremely relevant class of approaches to solve a vast array of optimization problems, notably optimal control, that features a long history. Indeed, in [8], Bellman introduced Dynamic Programming (DP) whose key idea consists in solving a large decision problem by organizing it into simpler nested subproblems that are solved recursively over time. The effectiveness of dynamic programming techniques in optimization connected with feedback control has been extremely relevant, and, since then, expanded to a large variety of control problems, notably, impulsive control (see [4, 26], and references therein) [29]. A disruptive development emerged in [13] with the introduction of the class of Reinforcement Learning algorithms of which Value Iteration (VI) and Policy Iteration (PI) are well known. More recently, Bertsekas in [12] shows that the Bellman equation, and the optimality condition stated in terms of the well posedness of the compact operator that plays a central role in the algorithm that is that DP theory is intimately connected with the theory of abstract mappings and their fixed points, as well as a more unified, economical, and streamlined analysis.

We consider the state space X and the set of control values $U(x) \subset U$. We denote by $\mathcal{M}$ the set of all functions $\mu : X \to U$ with $\mu(x) \in U(x)$ for all $x \in X$, by $\mathcal{M}$ that we refer to as "stationary policy". Let $\mathcal{R}(X)$ be the set of real-valued functions $J : X \to \mathbb{R}$. We have a mapping $H : X \times U \times \mathcal{R}(X) \to \mathbb{R}$ and each policy $\mu \in \mathbb{M}$, we consider the mapping $F_\mu : \mathcal{R}(X) \to \mathcal{R}(X)$ defined by

$$F_\mu J(x) = H\big(x, \mu(x), J\big) \quad \forall x \in X.$$

We also consider the mapping $F : \mathcal{R}(X) \to \mathcal{R}(X)$ given by

$$FJ(x) = \inf_{u\in U(x)}\big\{H\big(x,\mu(x),J\big)\big\} = \min_{\mu\in\mathcal{M}}\big\{F_\mu J(x)\big\} \quad \forall x \in X.$$

Now, let $B(X)$ denote the set of all bounded real-valued function on X. The pair $(B(X), \|\cdot\|_\theta)$, where

$$\|J\|_\theta = \sup_{x\in X}\big|J(x)\big|^2, \quad J \in B(X) \tag{6.1}$$

is a complete extended b-metric space.

Our goal is to find an optimal cost function $J^* \in B(X)$ such that

$$J(x) = \inf_{u \in U(x)} \{H(x,u,J)\} \quad \forall x \in X. \tag{6.2}$$

This is the so-called Bellman Equation. The purpose of this section is to find the unique fixed point of F within $B(X)$, which is the optimal cost function and a unique solution to the Bellman Equation (6.2), by applying Theorem 3.7. We require the following assumptions

A1 (Well posedness). For all $J \in B(X)$, and $\forall \mu \in \mathcal{M}$, we have that $F_\mu J \in B(X)$ and $FJ \in B(X)$.

A2 (Monotonicity). If $J, J' \in \mathcal{R}(x)$, and $J \le J'$, then

$$H(x,u,J) \le H(x,u,J') \quad \forall x \in X, u \in U(x).$$

A3 (Attainability). For all $J \in B(X)$, there exists $\mu \in \mathcal{M}$, such that: $F_\mu J = FJ$.

Theorem 6.1 *Assume that the data of the Bellman Equation* (6.2) *satisfies the following assumptions*:

i) F_μ *and* F *are monotone*;

ii) $F_\mu : B(X) \to B(X)$ *is a Ćirić operator.*

Then, the Bellman Equation (6.2) *has a unique solution in* $B(X)$.

Proof Let $B(X)$ be an extended b-metric space endowed with the extended b-metric $\|J\| = \sup_{x\in X}\{|J(x)|^2\}$. Note that X is a complete extended b-metric space with $\theta : X \times X \to [1,\infty)$ defined by $\theta(J,J') := 2|J(x)| + 3|J'(x)| + 1$.

Let us define the operator $F : B(X) \to B(X)$ as follows

$$FJ(x) = \inf_{u \in U(x)} \{H(x,u,J)\} \quad \forall x \in X.$$

First, note that from the common inequality $(a-b)^2 \ge 0$, we readily obtain

$$\frac{1}{4}(a^2 + b^2) \ge \frac{1}{2}ab. \tag{6.3}$$

We shall prove that the operator F satisfies all the conditions of Theorem 3.7. We have the following estimation

$$\begin{aligned}
|FJ(x) - FJ'(x)|^2 &\le |H(x,u,J) - H(x,u,J'|^2 \\
&\le \gamma^2 \Big| \max\Big\{ |J(x) - J'(x)|, |J(x) - F_\mu J(x)|, |J'(x) - F_\mu J'(x)|, \\
&\quad \frac{1}{2}(|J(x) - F_\mu J'(x)| + |J'(x) - F_\mu J(x)|) \Big\} \Big|^2 \, ds \\
&\le \gamma^2 \Big| \max\Big\{ |J(x) - J'(x)|^2, |J(x) - F_\mu J(x)|^2, |J'(x) - F_\mu J'(x)|^2, \\
&\quad |\frac{1}{2}(|J(x) - F_\mu J'(x)| + |J'(x) - F_\mu J(x)|)^2 \Big\} \, ds \\
&\le \gamma^2 \max\Big\{ |J(x) - J'(x)|^2 |J(x) - F_\mu J(x)|^2, |J'(x) - F_\mu J'(x)|^2,
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{4}\Big[\big(J(x)-F_\mu J'(x)\big)^2+\big(J'(x)-F_\mu J(x)\big)^2\\
&\quad+2\big(J(x)-F_\mu J'(x)\big)\big(J'(x)-F_\mu J(x)\big)\Big]\Big\}\\
&\le \gamma^2\max\Big\{\big|J(x)-J'(x)\big|^2,\big|J(x)-F_\mu J(x)\big|^2,\big|J'(x)-F_\mu J'(x)\big|^2,\\
&\quad\frac{1}{4}\Big[\big(J(x)-F_\mu J'(x)\big)^2+\big(J'(x)-F_\mu J(x)\big)^2\Big]\\
&\quad+\frac{1}{2}\Big[\big(J(x)-F_\mu J'(x)\big)\big(J'(x)-F_\mu J(x)\big)\Big]\Big\}.
\end{aligned}
$$

Moreover, from the above, by using $F_\mu J(x)\ge FJ(x)$, and (6.3), we obtain

$$
\begin{aligned}
\big|FJ(x)-FJ'(x)\big|^2 &\le \gamma^2\max\Big\{\big|J(x)-J'(x)\big|^2,\big|J(x)-F_\mu J(x)\big|^2,\big|J'(x)-F_\mu J'(x)\big|^2,\\
&\quad\frac{1}{2}\Big[\big|J(x)-F_\mu J'(x)\big|^2+\big|J'(x)-F_\mu J(x)\big|^2\Big]\Big\}\\
&\le \gamma^2\max\Big\{\big\|J-J'\big\|_\theta,\|J-F_\mu J\|_\theta,\big\|J'-F_\mu J'\big\|_\theta,\\
&\quad\frac{1}{2}\Big(\big\|J-F_\mu J'\big\|_\theta+\big\|J'-F_\mu J\big\|_\theta\Big)\Big\}\\
&\le \gamma^2\max\Big\{\big\|J-J'\big\|_\theta,\|J-FJ\|_\theta,\big\|J'-FJ'\big\|_\theta,\\
&\quad\frac{1}{2}\Big(\big\|J-FJ'\big\|_\theta+\big\|J'-FJ\big\|_\theta\Big)\Big\}\\
&\le \gamma^2\max\Big\{\big\|J-J'\big\|_\theta,\|J-TJ\|_\theta,\big\|J'-TJ'\big\|_\theta,\\
&\quad\frac{1}{2}\Big(\big\|J-TJ'\big\|_\theta+\big\|J'-TJ\big\|_\theta\Big)\Big\}\\
&\le \gamma^2 M\big(J,J'\big),
\end{aligned}
$$

for any $x\in X$. For α such that $0<\alpha=\gamma^2<1$ we have

$$
\big\|FJ-FJ'\big\|_\theta=\| FJ-FJ'\|_\infty\le \alpha M\big(J,J'\big)
$$

$\forall J,J'\in B(X)$.

Since $\lim_{n,m\to\infty}\theta(J_n,J'_m)=1$, the conclusion follows from Theorem 3.7. □

7 Conclusions and prospective research work

In this article we proved a common fixed-point theorem for generalized contractive mappings of Ćirić type in extended b-metric spaces and illustrated them with a simple example, as well as with three diverse classes of applications. More precisely, with the result we proved the existence of solutions to Volterra equations, to a system of fractional differential equations, and to the Bellman equations. The relevance of our main result of this article is twofold: It handles classes of operators that fail to be continuous, as well as spaces endowed with a metric that satisfies only a relaxed triangle inequality. This is a great advantage relative to other results of this type since it allows us to enlarge the class of applications.

In what concerns the Bellman Equation, Value Iterative methods associated with Reinforcement Learning schemes for optimal control problems with state constraints is an important instance as the value function is merely lower semicontinuous. We envisage future work along two directions: investigation of ways of generalizing and strengthening the main result proved here, and exploit the possibilities of applying this and related results in solving significant applied problems arising in various areas, notably, in optimal control with state constraints as well as optimization problems formulated in spaces appropriated for the operation on data sets. In particular, we will seek to extend the application of appropriate fixed-point results to match the shapes under conditions that are weaker than those addressed so far in the literature by resorting to á Cirić contractive operator on an extended b-metric space.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization, A.B, F.L.P and L.G.; methodology, F.L.P; validation, F.L.P, A.B, L.G and S.B.; formal analysis, F.L.P; investigation, A.B.; resources, A.B; L.G, S.B writing—original draft preparation, L.G and A.B , S.B; writing—review and editing, L.G., A.B,S.B and F.L.P;supervision,F.L.P.; project administration, F.L.P . All authors have read and agreed to the published version of the manuscript

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